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ON THE HARDY-LITTLEWOOD MAXIMAL THEOREM

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ABSTRACT. The Hardy-Littlewood maximal theorem is extended to functions of class PL in the sense of E. F. Beckenbach and T. Radó, with a more precise expression of the absolute constant in the inequality. As applications we deduce some results on hyperbolic Hardy classes in terms of the non-Euclidean hyperbolic distance in the unit disk.

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1. INTRODUCTION.

Let $D = \{|z| < 1\}$, let $T = [0,2\pi)$, and let u be a function subharmonic in D. For a function g on T we denote

$$\|g\|_p = [\frac{1}{2\pi}\int_T |g|^p(t)dt\ ]^{1/p};$$

hereafter always $0 < p < \infty$ unless otherwise specified. Then, although u is not defined on T we customarily denote

$$\|u\|_p = \limsup_{r \to 1-0} \|u_r\|_p,$$

where $u_r(t) = u(re^{it})$, $t \in T$, $0 < r < 1$. For simplicity, $\|f\|_p = \| |f| \|_p$ for f holomorphic in D. Let $S(t,R)$ be the domain consisting of the interior of the convex hull of the circle $|z| = R < 1$ and the point e^{it} $(t \in T)$; hereafter always $0 < R < 1$. The maximal function $M_R(u)$ of u is defined on T by

$$M_R(u)(t) = \sup\{u(z);\ z \in S(t,R)\}.$$

Let H^p be the Hardy class consisting of all functions f holomorphic in D such that $\|f\|_p < \infty$. Each $f \in H^p$ has the radial limit $f^*(t) = \lim_{r \to 1-0} f(re^{it})$ at e^{it} for a.e. $t \in T$, and $f^* \in L^p(T)$. We then observe that $\|f\|_p = \|f^*\|_p$ [1, Theorem 2.6, p. 21].

In the present paper we introduce the Hardy-Littlewood number $a(p,R)$ of order (p,R) by

$$a(p,R) = \sup\{\|M_R(|f|)\|_p/\|f\|_p;\ f \in H^p,\ f \not\equiv 0\}.$$

The celebrated Hardy-Littlewood theorem [3, Theorem 27, p. 114] then reads that $a(p,R) < \infty$ for each pair (p,R). The main purpose of the present paper is to prove that $a^*(p,R) = a(p,R) = a(1,R)^{1/p}$, where $a^*(p,R)$ is defined in terms of functions of class PL [4, p. 9].

A function u defined in D is said to be of class PL, or $u \in PL$, if $u \geq 0$ and if $\log u$ is subharmonic in D; we regard $-\infty$ as a subharmonic function. For $u \in PL$, the function u^p is subharmonic in D, and for f holomorphic in D, the modulus $|f| \in PL$. Let PL^p be the family of all $u \in PL$ such that $\|u\|_p < \infty$. It will be observed that $u \in PL^p$ has the radial limit $u^*(t)$ at e^{it} for a.e. $t \in T$ and that $\|u\|_p = \|u^*\|_p$. Apparently, $|f| \in PL^p$ if $f \in H^p$. Set

$$a^*(p,R) = \sup\{\|M_R(u)\|_p/\|u\|_p;\ u \in PL^p,\ u \not\equiv 0\}.$$

Since $1 \in H^p$, it follows that $1 \leq a(p,R) \leq a^*(p,R)$. We first observe

THEOREM 1. $a^*(p,R) = a(p,R) = a(1,R)^{1/p}$.

REMARK. Let S^p be the family of subharmonic functions $u \geq 0$ in D such that $\|u\|_p < \infty$, where $p > 1$. Then

$$b(p,R) = \sup \left\{ \|M_R(u)\|_p / \|u\|_p;\ u \in S^p,\ u \not\equiv 0 \right\}$$

is finite for $p > 1$ by [3, Theorem 26, p. 113]. Obviously,

$$a^*(p,R) \leq b(p,R) \quad \text{for} \quad p > 1.$$

To propose an application to the hyperbolic Hardy class H^p_σ we let

$$\sigma(z,w) = \tanh^{-1}(|z - w|/|1 - \bar{z}w|)$$

be the non-Euclidean hyperbolic distance between z and w in D. Set $\sigma(z) \equiv \sigma(z,0) = \frac{1}{2}\log[(1 + |z|)/(1 - |z|)]$, $z \in D$. For f holomorphic and bounded, $|f| < 1$, in D, the hyperbolic counterpart of $|f|$ is $\sigma(f)$. We thus define H^p_σ as the family of such f with $\|\sigma(f)\|_p < \infty$. The subharmonicity of $\sigma(f)^p = \exp[p \log \sigma(f)]$ follows from that of $\log \sigma(f)$ (or, $\sigma(f) \in PL$) observed in [6]. Therefore $\sigma(f) \in PL^p$ for all $f \in H^p_\sigma$. A few modifications of the proof of [6, Theorem 4], with $H^1_h = H^1_\sigma$, show that H^p_σ is a semigroup with respect to the multiplication, and is convex. Since each $f \in H^p_\sigma$ is bounded, f has the radial limit $f^*(t)$ at e^{it} for a.e. $t \in T$. We then propose

THEOREM 2. For each $f \in H^p_\sigma$, the function $\sigma(f^*)$ is a member of $L^p(T)$, and

$$\int_T \sigma(f(re^{it}), f^*(t))^p dt \to 0 \quad \text{as} \quad r \to 1-0.$$

The inequality

$$\int_T \sup \left\{ \sigma(f)^p(z);\ z \in S(t,R) \right\} dt \leq a(1,R) \int_T \sigma(f^*)^p(t)\,dt$$

holds for all $f \in H^p_\sigma$.

The first assertion, a consequence of the second, is the hyperbolic counterpart of the F. Riesz theorem [1, Theorem 2.6].

2. PROOFS.

For the proof of Theorem 1 it suffices to show that

$$a^*(p,R) \leq a(1,R)^{1/p} \leq a(p,R).$$

Since $a(p,R) \leq a^*(p,R)$, the identities in Theorem 1 follow.

To prove that $a(1,R)^{1/p} \leq a(p,R)$ we let $f \in H^1$ with $f \not\equiv 0$. Then f admits an inner-outer factorization, $f = IF$, where I and F are an inner and an outer function, respectively, such that the radial limits satisfy $|I^*| = 1$ and $|F^*| = |f^*|$ a.e. on T. Since F is zero-free in D, $g = F^{1/p} \in H^p$, so that $|f^*| = |g^*|^p$ a.e. on T. Therefore,

$$\|M_R(|f|)\|_1 \leq \|M_R(|F|)\|_1 = \|M_R(|g|)\|_p^p \leq a(p,R)^p \|g\|_p^p = a(p,R)^p \|f\|_1,$$

whence $a(1,R) \leq a(p,R)^p$.

To prove that $a^*(p,R) \leq a(1,R)^{1/p}$, we let $v \in PL^p$ with $v \not\equiv 0$. Setting $u = p \log v$ and $\varphi(x) = e^x$, one finds that $v^p = \varphi(u)$. Since $\varphi(u)$ admits a harmonic majorant in D, there exists a positive harmonic majorant of u in D [5, p. 65]. The F. Riesz decomposition then yields that $u = u^\wedge - P$, where $P \geq 0$ is the Green potential in D, and

$$u^\wedge(z) = \frac{1}{2\pi} \int_T \frac{1 - |z|^2}{|e^{it} - z|^2} \, d\mu(t) \qquad (z \in D)$$

is the Poisson integral of the measure

$$d\mu(t) = u^*(t)dt + d\mu_s(t).$$

The radial limit u^* of u is of $L^1(T)$ and $d\mu_s(t)$ is singular with respect to dt. It follows from a general theorem [2, Theorem], applied to the present u and φ, that $d\mu_s(t) \leq 0$ a.e. on T and that $\varphi(u^*) \in L^1(T)$. Consequently,

$$u(z) \leq h(z) \equiv \frac{1}{2\pi} \int_T \frac{1 - |z|^2}{|e^{it} - z|^2} \, u^*(t)dt \qquad (z \in D),$$

and the Jensen inequality yields that

$$\varphi(u) \leq \varphi(h) \leq V,$$

where V is the Poisson integral of $\varphi(u^*)$. Set $f = e^{h+ik}$, where k is a conjugate of h in D. Then, $|f| = \varphi(h) \le V$, so that $f \in H^1$ with $|f^*| = \varphi(h^*) = \varphi(u^*) = v^{*p}$. On the other hand, $v^p = \varphi(u) \le \varphi(h) = |f|$ in D, whence

$$\|M_R(v)\|_p^p \le \|M_R(|f|)\|_1 \le a(1,R)\ \|f\|_1.$$

The Lebesgue dominated convergence theorem, together with

$$v_r^p(t) \le M_R(v)^p(t) \qquad (t \in T),$$

yields that $\|v_r\|_p^p \nearrow \|v\|_p^p = \|v^*\|_p^p = \|f\|_1$ as $r \to 1-0$. Therefore $a^*(p,R) \le a(1,R)^{1/p}$ follows from

$$\|M_R(v)\|_p^p \le a(1,R)\ \|v\|_p^p.$$

We next prove Theorem 2. Set

$$a_\sigma(p,R) = \sup\{\|M_R(\sigma(f))\|_p / \|\sigma(f)\|_p;\ f \in H_\sigma^p,\ f \not\equiv 0\}.$$

Since $\sigma(f) \in PL^p$ for all $f \in H_\sigma^p$, it follows that $a_\sigma(p,R) \le a^*(p,R) = a(1,R)^{1/p}$, so that

$$\|M_R(\sigma(f))\|_p \le a(1,R)^{1/p} \|\sigma(f)\|_p.$$

As is observed in the proof of Theorem 1, $\sigma(f)^* = \sigma(f^*)$ a.e. on T because $\sigma(f) \in PL^p$, and $\|\sigma(f)\|_p = \|\sigma(f^*)\|_p$. Thus, the second assertion holds with $\sigma(f^*) \in L^p(T)$. The Lebesgue dominated convergence theorem with the estimate

$$\sigma(f(re^{it}),f^*(t))^p \le 2^p \sigma(f)^p(re^{it}) + 2^p \sigma(f^*)^p(t) \le 2^{p+1} M_R(\sigma(f)^p)(t)$$

$$= 2^{p+1} M_R(\sigma(f))^p(t),$$

again yields that

$$\int_T \sigma(f(re^{it}),f^*(t))^p dt \to 0 \quad \text{as} \quad r \to 1-0.$$

REMARK. Since $\sigma(f) \equiv 1$ for $f \equiv (e^2 - 1)/(e^2 + 1) \in H_\sigma^p$, it follows that $1 \le a_\sigma(p,R)$.

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