ON THE SPECTRUM OF A DISTRIBUTION FUNCTION
AND ON UNIQUE FACTORIZATION

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ABSTRACT. The spectrum of a distribution function is related to the quasi-analyticity

of a class of functions $\{C \, M_{(j)}\}$, where $M_{(j)}$ is a multisequence of positive numbers.

For a regular multisequence, a result on the uniqueness of characteristic function

decomposition is given.

KEY WORDS AND PHRASES. *Distribution, characteristic function, spectrum, quasi-analytic, regular, factorization.*

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1. INTRODUCTION.

A general principle formulated by N. Wiener, that a function and its Fourier

transform cannot both be very small at infinity, has been the origin of important but

quite different results in the fields of Complex Analysis and Probability Theory.   In

this note, we give some results connecting these two fields and an example to illus-

trate this connection.

Two fundamental theorems on the above principle are recalled below. They both

rely on the classical Paley-Wiener Theorems. Theorem 1.1 by Ingham and Levinson [1]

has been used by various authors to establish results in Probability Theory, concern-

ing the spectrum of a distribution, unique factorization of characteristic functions,

etc. (see e.g. [2], [3]). Theorem 1.2 by S. Mandelbrojt [4] has seen many applicat-

ions in Analysis, especially in the theory of quasi-analytic and non-quasi-analytic

functions (see e.g. [4], [5], [6]).

THEOREM 1.1. Let $G(u) \in L(-\infty,\infty)$ and suppose that

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(u)^{-iux} du.$$

Let $G(u) = 0(e^{-\theta(u)})$ where $\theta(u)$ is a non-decreasing positive function such that $\int_{1}^{\infty} \frac{\theta(u)}{u^2} du = \infty.$

Then $F(x)$ cannot be equivalent to zero in any interval unless it is equivalent to zero over $(-\infty,\infty)$.

THEOREM 1.2. If $C(u)$ is a non-decreasing function on $[0,\infty)$ with $\int_{1}^{\infty} \frac{C(u)}{u^2} du < \infty$, then $\forall \delta, \exists$ an entire function $F(z)$ not identically zero such that

$$|F(z)| \le e^{\delta|y| - C(|z|)}.$$

A number of definitions and other results to be used later are recalled below.

A point x is called a point of the spectrum of a probability distribution F if $\forall \delta > 0$, $F(x + \delta) > F(x - \delta)$. The characteristic function f of F is defined by $f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$ and the following theorem can be found in [2], under a slightly different form.

THEOREM 1.3. Let $\theta(t)$ be a non-negative non-decreasing function for $t > 0$ and let $f(t)$ be the characteristic function of a distribution F such that $f(t) = 0(e^{-\theta(|t|)})$ for large $|t|$. Then F has $(-\infty,\infty)$ as spectrum if and only if the integral $\int_{1}^{\infty} \frac{\theta(t)}{t^2} dt$ diverges.

The proof of the above result is essentially based on Theorem 1.1 and various properties of the Rademacher Series (see [2]).

2. MAIN RESULTS.

Let $\{M_{(j)}\}$ be a multisequence of positive numbers where $(j) = (j_1, \ldots, j_m)$, j_k being an integer ≥ 0, $1 \le k \le m$. We set $|(j)| = \sum_{k=1}^{m} j_k$ and denote by $C\{M_{(j)}\}$ the class of C^{∞} complex-valued functions defined on R^m such that

$$\left|\left| \frac{\partial^{|(j)|}}{\partial x_1^{j_1} \ldots \partial x_m^{j_m}} f \right|\right|_{\infty} \le \alpha_f \, B_f^{|(j)|} M_{(j)}, \quad \forall (j)$$

where α_f and B_f are positive constants which depend only on f. Following Lelong [7], we define the class $C\{M_{(j)}\}$ to be quasi-analytic if it does not contain functions with

compact support. Various properties of these classes have been obtained (see [7],[8], and [9]) and differ from the classical one-dimensional case.

THEOREM 2.1. Let F be a probability distribution function with characteristic function f and $\{M_{(j)}\}$ a multisequence of positive numbers such that .

$$\sup_{(j)} \{||t|^{|(j)|} f(t)||_\infty / M_{(j)}\} < \infty.$$

Then F has $(-\infty,\infty)$ as spectrum if and only if $C\{M_{(j)}\}$ is quasi-analytic.

PROOF. We begin by extending the ideas of Valiron points and half-planes to several dimensions.

Let $\{M_{(j)}\}$ be a multisequence of positive numbers where $(j) = (j_1, \ldots, j_m)$, j_k being an integer ≥ 0. We set $\mu_p = \inf_{|(j)|=p} \{M_{(j)}\}$ and suppose that $\{\mu_p\}$ grows more rapidly than any function of the form $r^p, r > 0$.

In R^{m+1}, let's consider the points with co-ordinates $x_1 = j_1, \ldots, x_m = j_m$ and $y = -\log M_{(j)}$, called the Valiron points of the sequence $\{M_{(j)}\}$, in the lower half-space of R^{m+1}. Since $\frac{r^p}{\mu_p} \to 0$, $r > 0$ and $a > 0$, there exist at most a finite number of indices p such that $p \log r - \log \mu_p > a$. A fortiori, $\mathbf{V}(r) = (r_1, \ldots, r_m)$ and $a > 0$; there are at most a finite number of Valiron points such that

$$\sum_{k=1}^{m} j_k \log r_k - \log M_{(j)} \geq a. \qquad (2.1)$$

Now fix (r) and fix $\mathbf{a} > 0$. Let's consider the hyperplane in R^{m+1} determined by the equation:

$$y = -\sum_{k=1}^{m} x_k \log r_k + a.$$

It intersects the y-axis at the point with ordinate a and the x_ℓ-axis at the point $\frac{a}{\log r_\ell}$ with $1 \leq \ell \leq m$. By (2.1), $\exists b = b(r_1, \ldots, r_m)$ such that no Valiron point is above $y = -\Sigma x_k \log r_k + b(r_1, \ldots, r_m)$ and, at the same time, at least one Valiron point is on the hyperplane itself. Hence, for Valiron points, we have

$$-\sum_{k=1}^{m} j_k \log r_k + b(r_1, \ldots, r_m) \geq -\log M_{(j)}$$

or

$$b(r_1, \ldots, r_m) \geq \sup_{(j)} \{\sum_{k=1}^{m} j_k \log r_k - \log M_{(j)}\};$$

in fact

$$b(r_1, \ldots, r_m) = \sup_{(j)} \{\log (\frac{r_1^{j_1}, \ldots, r_m^{j_m}}{M_{(j)}})\}$$

$$= \log T(r_1, \ldots, r_m)$$

where

$$T(r_1, \ldots, r_m) = \sup_{(j)} \{\frac{r_1^{j_1}, \ldots, r_m^{j_m}}{M_{(j)}}\}$$

For $(r) = (r_1, \ldots, r_m)$ given, the hyperplane $H((r))$ with equation

$y = -\sum_{k=1}^{m} x_k \log r_k + b(r_1, \ldots, r_m)$ is called the Valiron hyperplane of the sequence

$\{M_{(j)}\}$ at the point $(r) \in R^m$.

We set $r_1 = r_2 = \ldots = r_m = r$ and consider the Valiron hyperplanes defined on the

main diagonal of R_m, whose equations become $y = -\log r \sum_{k=1}^{m} x_k + b(r)$. Here

$b(r) = \log \lambda(r)$ where $\lambda(r) = T(r, r, \ldots, r)$. We can see from the above arguments

that $b(r)$ is a positive non-decreasing function of $r > 0$.

Suppose now that $C\{M_{(j)}\}$ is quasi-analytic. By [8] we know that $\int_1^\infty \frac{b(r)}{r^2} dr$

diverges.

Let $\sup_{(j)} \{||t^{|(j)|} f(t)||_\infty / M_{(j)}\} = \gamma < \infty.$

Then

$$|f(t)| \le \gamma \frac{M_{(j)}}{|t|^{|(j)|}}, \forall (j)$$

and hence

$$|f(t)| = 0(1/\gamma(|t|)) \quad \text{as} \quad t > \infty.$$

Using Theorem 1.3, F then has $(-\infty, \infty)$ as spectrum. Conversely, suppose that F has

$(-\infty, \infty)$ as spectrum. Since $|f(t)| = 0(e^{-b(|t|)})$ $t \to \infty$, the integral $\int_1^\infty \frac{b(t)}{t^2} dt$ diverges again by [8], this is a sufficient condition for the class $C\{M_{(j)}\}$ to be quasi-analytic.

For one dimension, we have the following example and corollary.

EXAMPLE. If we associate to the Cauchy distribution (with density

$$p(x) = \frac{\theta}{\pi[\theta^2 + (x - \mu)^2]} , \quad -\infty < x < \infty, \; \theta, \; \mu \text{ real}, \; \theta > 0) \text{ the sequence } M_j = (\frac{j}{\theta e^\theta})^j,$$

we can easily verify that the above theorem is satisfied.

COROLLARY. Let $\{M_j\}_{j=0}^{\infty}$ be a sequence of positive numbers such that $M_0 = 1$, $M_n^2 \leq M_{n-1} M_{n+1}$ with $n = 1, 2, 3, \ldots$, and F is a distribution fuction with characteristic function f such that $\sup_{j \geq 0} \{ ||t^j f(t)||_{\infty}/M_j \} < \infty$. Then F has $(-\infty, \infty)$ as spectrum if and only if $\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} = \infty$.

PROOF. By the classical Denjoy-Carleman Theorem (see [9]), if M_j is logarithmically convex, the divergence of the integral $\int_1^{\infty} \frac{b(t)}{t^2} dt$ is equivalent to the divergence of the series $\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n}$.

3. UNIQUE FACTORIZATION.

An interesting property of distribution functions is that the cancellation law does not hold in general, i.e. $F_1 * F_2 = F_1 * F_3$ does not imply $F_2 = F_3$. Gnedenko [10] has given examples of characteristic function f such that $f(t) = f_1(t) f_2(t) = f_1(t) f_3(t)$ with $f_2(t) \neq f_3(t)$.

For a distribution function F, we define $F*(x) = F(-x)$, $\forall x$ and, for a > 0, $(\Delta_a F)(x) = F(x + a) - F(x - a)$.

We now consider a more specific multisequence called regular. For $\{M_{(j)}\}$, let's associate the marginal multisequence $\{M*_{(j)}\}$ defined by

$$M*_{(j)} = M_{|(j)|, 0, \ldots}, \quad M_{0, |(j)|, 0, \ldots, 0}$$

$$\ldots M_{0, 0, \ldots, |(j)|}, \quad \forall(j)$$

DEFINITION. A multisequence $\{M_{(j)}\}$ is regular if:

1) $M_{(j)} \geq M*_{(j)} \ \forall(j)$, with $\sup_{(j)} \{ (\frac{M_{(j)}}{M*_{(j)}})^{1/|(j)|} \} < \infty$.

2) $\{C \ M*_{(j)}\}$ is quasi-analytic with each marginal sequence $\{M_{0, \ldots, 0, \ell, 0, \ldots, 0}\}$ of the form $M_{0, \ldots, \ell, \ldots, 0} = (\ell!)^{\alpha_k} [\nu(\ell)]$ where $\sum_{k=1}^{m} \alpha_k = 1$ and $\nu(x)$ is a continuously differentiable function (depending on the marginal sequence) with $\nu(0) = 1$, $\nu'(x) \geq 0$ and $\frac{x \nu'(x)}{\nu(x)} = 0(1)$ as $x \to \infty$.

REMARK. The class of regular multisequence, which is obviously not empty, was introduced first by Hirschman [11] for one dimension, to generalize the m-th logarithmic sequence.

THEOREM. Let $C\{M_{(j)}\}$ be a regular class of functions and f the characteristic function of F. Then there exists a continuous non-decreasing function $\psi(t)$ for $t > 0$ such that:

1) If $f(t) = 0(e^{-|t|/\Psi(|t|)})$, then F has $(-\infty, \infty)$ as spectrum

2) If $(\Delta_a F^*)(u) = 0(e^{-|t|/\Psi(|t|)})$ for some $a > 0$ and f admits a decomposition $f(t) = f_1(t) f_2(t)$, then f_2 is uniquely determined by f and f_1.

PROOF. We first show that, for a regular multisequence $\{M_{(j)}\}$, the function $b^*(r)$ associated with the marginal multisequence $\{M^*_{(j)}\}$ is such that $b^*(r) \sim \dfrac{r}{\psi(r)}$ as $r \to \infty$ where $\psi(r)$ is a continuous, non-decreasing function with $\psi(0) > 0$.

We have

$$\lambda^*(r) = \sup_{p \geq 0} \left\{ \frac{r^p}{M^*_p} \right\} \tag{3.1}$$

with

$$\mu^*_p = \inf_{|(j)|=p} \{M^*_{(j)}\} = (p!)^{\alpha_1}[\nu_1(p)]^p \ldots (p!)^{\alpha_m}$$

$$[\nu_m(p)]^p = p! \ [\nu(p)]^p \quad \text{where} \quad \nu = \overset{m}{\underset{k=1}{\pi}} \nu_k.$$

We also have $\nu(0) = 1$, $\nu'(x) \geq 0$, and $\dfrac{x \nu'(x)}{\nu(x)} = \mathbf{o}(1)$ as $x \to \infty$.

Following Hirschman [11], write

$$b^*(r) = \max_{p > 0} \{p \log r - p! - p \log \nu(p)\} \tag{3.2}$$

and let $p^*(r)$ be the integer for which the maximum is attained. If we now treat p as a continuous variable in $G(p) = p \log r - p! - p \log \nu(p)$, the maximum would be attained as s), the solution of $\dfrac{dG(p)}{dp} = 0$. Using Stirling's formula $p! \sim \sqrt{2\pi p} \ p^p \ e^{-p}$, we have $\dfrac{d}{dp} (\log \Gamma(p + 1)) = \log p + \mathbf{o}(1)$ and, therefore, $\dfrac{dG(p)}{dp} = \log r - \log p - \log \nu(p) + \mathbf{o}(1)$ by the properties of ν. Hence,

$$\log r = \log s(r) - \log \nu(s(r)) + \mathbf{o}(1) \tag{3.3}$$

and $r \sim s(r) \ \nu (s(r))$.

Putting (3.3) in (3.2) when $p = p^*(r)$, we can obtain

$$b^*(r) \sim p^*(r) \ [1 + \mathbf{o}(1)] \sim s(r)$$

and so $b^*(r) \sim \dfrac{r}{\nu(s(r))}$ for $r > 0$.

By the non-decreasing property of ν and s, there exists a continuous function

$\psi(r)$, non-decreasing with $\psi(0) > 0$ such that $b*(r) \sim \dfrac{r}{\psi(r)}$ for $r > 0$.

Now let $\xi = \sup \{(\dfrac{M_{(j)}}{M*_{(j)}})^{1/|(j)|}\} < \infty$ and $\eta = \dfrac{1}{\xi}$.

We have $\lambda*(\eta r) \leq \lambda*(r)$ by Theorem 2 of [8] while $M_{(j)} \geq M*_{(j)}$ implies $\lambda(r) \leq \lambda*(r)$.

On the other hand, $\dfrac{M_{(j)}}{\eta^{|(j)|}} = M*_{(j)} \xi^{|(j)|} \geq M_{(j)}$ $\forall(j)$ implies $\lambda*(\eta r) \leq \lambda(r)$ by (3.1).

Now, $M*_{(j)}$ being quasi-analytic, the integral $\displaystyle\int_1^\infty \dfrac{b*(r)}{r^2}$ dr diverges and, by the

above inequalities, implies the divergence of $\displaystyle\int_1^\infty \dfrac{b(r)}{r^2}$ dr. If $f(t) = \mathbf{O}(e^{-|t|/\psi(|t|)})$,

then a fortiori $f(t) = \mathbf{O}(1/\lambda(|t|))$ and by Theorem 1.3, F has $(-\infty,\infty)$ as its spectrum.

The second part of the theorem follows from a result by E. Lukacs ([3, Theorem 3])

combined with the regularity of $C\{M_{(j)}\}$. The proof, based on classical results of

Khintchine, the inversion formula, and Parseval's Theorem, follows closely Lukacs

(see [3]).

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