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RESEARCH NOTES

A NOTE ON GORDAN'S THEOREM OVER CONE DOMAINS

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ABSTRACT. This note presents a proof of Gordan's Theorem over general closed, convex cone domains which follows in a natural way appealing to the standard definitions of closed convex cones and their respective polar cones.

KEY WORDS AND PHRASES. Gordan's Theorem, closed convex cones, polar cones.
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1. INTRODUCTION

Solvability theorems or theorems of the alternative for problems involving linear system of equations has played a major role in mathematical programming, and linear analysis, (Craven [1], Luenberger [2] or Mangasarian [3]). Mangasarian [3] presents a rigorous development of the classical theorems of alternatives, see Chapter 2 in [3]. Many theoretical aspects in mathematical programming appeal to various theorems of the alternative in establishing certain optimality conditions and duality properties.

Ben-Israel [4], and Berman and Ben-Israel [5] extended the classical theorems of the alternative to problems with linear equations over polyhedral cone domains. Thus extending the classical formulations over nonnegative orthants as originally presented by Gordan [6]; i.e., problems involving a linear system over inequalities (≥ 0). In particular, Ben-Israel proves a certain theorem, (Theorem 2 in [4]), which is utilized to establish extensions of the classical theorems of the alternative of Gordan, Motzkin, and Slater to problems formulated with linear equations over polyhedral convex cone domains. Ben-Israel underscores the fact that Theorem 2 cannot be used to generalize these results to general (nonpolyhedral) closed convex cones; see page 134 in [4]; hence, restricting the extension of the theorems of the alternative to problems involving general convex cone domains. Berman and Ben-Israel (cor. 1.5 in [5]), however, were able to establish a generalized Gordan's Theorem over general closed convex cone domain by appealing to Mazur's Theorem (Bourbaki [7]) or the Hahn-Banach Theorem (Schaefer [8]).

Craven (pg 31-33 in [1]), presents several theorems of the alternative over general closed cone domains; however, a generalization of Gordan's Theorem is not explicitly given in this development. Therefore, the purpose of this note is to present a proof of Gordan's generalized theorem appealing only to the standard definitions of cones and their respective polar cones; hence, differing from the proof given by Berman and Ben-Israel in [5].

2. ALTERNATIVE PROOF

An alternative proof of Gordan's theorem over arbitrary convex cone domains is now presented over finite dimensional space. Consider the following definitions:

Definition 1. C is a cone in E^n if for any vector $y \varepsilon C$ and $k > 0$ we have that $ky \varepsilon C$.

Definition 2. A cone C is pointed if $C \cap (-C) = \{0\}$.

Definition 3. C^* will denote the polar cone of an arbitrary cone C in E^n; that is

$$C^* = \{y^* \varepsilon E^n | y^{*'} y \geq 0 \quad \text{for all} \quad y \varepsilon C\} . \tag{2.1}$$

Gordan's theorem over convex cone domains, Lemma 2 below, is established appealing to the following lemma:

Lemma 1. Let C be a closed convex cone in E^n. Then $\text{Int}(C^*) \neq \emptyset$ iff C is a pointed cone.

Proof: Berman and Ben-Israel ([5], Lemma 0).

Lemma 2. (Gordan's Theorem for Arbitrary Cone Domains). Let M be any given nonvacuous $m \times n$ matrix, with C any arbitrary pointed, closed convex cone in E^n, then exactly one of the following systems is consistent;

(i) $Mx = 0$ for some $x \in C$; $x \neq 0$

or

(ii) $M'y \in \text{Int}(-C^*)$, $y \in E^m$.

Proof: (Not (ii) implies (i)).

Let $S_1 = \{z | z = M'y, y \in E^m\}$; $S_2 = \{z | z \in \text{Int}(-C^*)\}$, then $S_1 \cap S_2 = \emptyset$ and, moreover, S_1 and S_2 are convex sets. Since S_1 and S_2 are two disjoint convex sets in E^n, then there exists a hyperplane v (nonzero), such that

$v'z_1 \geq v'z_2$ for all $z_1 \in S_1$; for all $z_2 \in \bar{S}_2$, (the closure of S_2).

Hence,

$$v'M'y \geq v'z_2 \quad \text{for all } y \in E^m; \quad \text{for all } z_2 \in \bar{S}_2 . \tag{2.2}$$

Assume $v \notin C$, then there exists $z_2^* \in \bar{S}_2$ such that

$$v'z_2^* > 0 .$$

However, for any given $y^* \in E^m$, there exists $\bar{z}_2^* = kz_2^* \in \bar{S}_2$, where $k > 0$, such that $v'\bar{z}_2^* > v'M'y^*$, which violates (2.2). Hence, it follows that $v \in C$.

Now letting $\bar{z} = 0$, then $v'M'y \geq v'\bar{z} = 0$, hence, $v'M'y \geq 0$. However, letting $y = -Mv$, we have that $-v'M'Mv \geq 0$. Therefore, $Mv = 0$, $v \in C$ $(v \neq 0)$ hence, (i) holds.

To show now that (ii) implies not (i).

Let y^* be such that $M'y^* \in \text{Int}(-C^*)$, and assume there exists $x^* \in C$ such that $Mx^* = 0$; $x^* \neq 0$; then necessarily $y^{*'}(Mx^*) = 0$. A contradiction, since $M'y^* \in \text{Int}(-C^*)$ with $x^* \in C$, $(x^* \neq 0)$, implies that $x^{*'}(M'y^*) < 0$. Hence, the result follows.

3. CONCLUDING REMARKS

Lemma 2 requires that the cone C be closed, convex, and pointed; hence, by Lemma 1 $\text{Int}(-C^*) \neq \emptyset$. Clearly, relaxing this requirement could result in the inconsistency of both (i) and (ii) in Lemma 2.

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