

CLASSIFICATION OF INJECTIVE FACTORS: THE WORK OF ALAIN CONNES

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ABSTRACT: The fundamental results of A. Connes which determine a complete set of isomorphism classes for most injective factors are discussed in detail. After some introductory remarks which lay the foundation for the subsequent discussion, an historical survey of some of the principal lines of the investigation in the classification of factors is presented, culminating in the Connes-Takesaki structure theory of type III factors. After a discussion of injectivity for finite factors, the main result of the paper, the uniqueness of the injective II_1 factor, is deduced, and the structure of II_∞ and type III injective factors is then obtained as corollaries of the main result.

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I. INTRODUCTION.

From the beginning, one of the central problems of the theory of von Neumann algebras has been their classification according to isomorphism type. The attack on the problem was initiated by the founding fathers Murray and von Neumann ([1], [2], [3], [4]), who introduced the fundamental notion of types (I, II_1 , II_∞ , and III), and who claimed early victories with the characterization of the hyperfinite

II_1 factor and a complete classification of the factors of type I. As the years passed, there was vigorous development of the theory, but no results appeared that were as definitive as these early advances. However, in 1973 a new era dawned with the publication of the thesis [5] of Alain Connes. Basing his work on earlier results of Tomita and Takesaki [6], Connes initiated a program for the classification of factors that can be termed as nothing short of revolutionary. In [5], [7],[8], and [9], he obtained a classification of factors of type III and automorphisms of certain factors of type II_1 and II_∞ which culminated in the remarkable work of [10], in which appear the first major advances beyond the classical theory in the classification of factors for the non-type I case. The purpose of these notes is to give a rather detailed discussion of the most important of these results.

We assume the reader has a familiarity with the theory of von Neumann algebras on the level of [11], say. We recall some basic facts that play an important role in the sequel. A factor is a von Neumann algebra with a trivial center, i.e., the only elements of the algebra which commute with every element are scalar multiples of the identity. We will be concerned primarily with von Neumann algebras of finite type, and we will use the tracial characterization of this (see [12], Section III-8). Recall that a trace τ on the factor N is a positive linear functional of norm 1, i.e., a state on N , which satisfies

$$\tau(ab) = \tau(ba), \quad \forall a, b \in N. \quad (1.1)$$

A factor is said to be finite if it has a trace. The trace on a finite factor is uniquely determined among the states by (1.1), and it is automatically faithful and ultraweakly continuous (Theorem 2.4.6 of [13]). These facts will be used frequently.

We will also employ the standard representation of a finite factor N . Let τ be the (canonical) trace on N . Then the representation π_τ in the Gelfand-Naimark-Segal construction $\{\pi_\tau, H_\tau, \xi_\tau\}$ corresponding to τ is faithful and ultraweakly continuous, and the cyclic vector ξ_τ is separating for $\pi_\tau(N)$ (i.e., ξ_τ is cyclic for $\pi_\tau(N)'$, where $'$ denotes the commutant). We denote H_τ by $L^2(N, \tau)$, and the L^2 -norm of $x \in N$ is by definition $\|x\|_2 = \tau(x^*x)^{1/2}$. π_τ is

called the standard representation of N , and when we identify N with its image $\pi_\tau(N)$ in $H = L^2(N, \tau)$, we will say that N acts standardly in H . Tensor products of C^* -algebras and von Neumann algebras will play an important role in what follows. Let $A_i, i=1,2$ be C^* -algebras, which for convenience we will assume act on Hilbert spaces $H_i, i=1,2$. Form the algebraic tensor product $A_1 \otimes A_2$, which can be viewed in a natural way as a $*$ -subalgebra of $B(H_1 \otimes H_2)$ the algebra of bounded operators on $H_1 \otimes H_2$. A seminorm ρ on $A_1 \otimes A_2$ is called a C^* -subcross seminorm if

$$(i) \quad \rho(x^*x) = \rho(x)^2, \quad x \in A_1 \otimes A_2,$$

$$(ii) \quad \rho(a_1 \otimes a_2) \leq \|a_1\| \|a_2\|, \quad a_i \in A_i, \quad i = 1, 2.$$

ρ is a C^* -crossnorm if ρ is a norm satisfying (i), and (ii) with \leq replaced by $=$. If we take the supremum of all C^* -subcross seminorms defined on $A_1 \otimes A_2$ we will obtain a C^* -crossnorm ν which by definition will be the largest C^* -crossnorm definable on $A_1 \otimes A_2$. Another crossnorm can be defined on $A_1 \otimes A_2$ by recalling that $A_1 \otimes A_2$ acts on $H_1 \otimes H_2$, and so each element of $A_1 \otimes A_2$ has a norm $\| \cdot \|$ considered as a bounded operator on $H_1 \otimes H_2$. It can be shown that the operator norm $\| \cdot \|$ is the smallest C^* -crossnorm definable on $A_1 \otimes A_2$. Completing $A_1 \otimes A_2$ in the ν -norm and the $\| \cdot \|$ -norm will yield C^* -algebras which we denote respectively by $A_1 \otimes_{\max} A_2$ and $A_1 \otimes_{\min} A_2$, the so-called maximal and minimal C^* -tensor products of A_1 and A_2 . These algebras will in general be distinct. If we further assume that A_1 and A_2 are von Neumann algebras, we may also take the closure of $A_1 \otimes A_2$ relative to the weak operator topology on $B(H_1 \otimes H_2)$, and thus obtain a von Neumann subalgebra of $B(H_1 \otimes H_2)$, the (spatial) W^* -tensor product $A_1 \otimes A_2$ of A_1 and A_2 . An excellent source of information about tensor products is [14].

Suppose M_i and N_i are finite factors, $i = 1, 2$, and $M_i \cong N_i, i = 1, 2$ (\cong denotes isomorphism). We claim that $M_1 \otimes M_2 \cong N_1 \otimes N_2$. Let $\phi_i: M_i \rightarrow N_i$ be isomorphisms, and let τ_i, σ_i denote the canonical traces on M_i and N_i , respectively. By uniqueness of the trace,

$$\sigma_i \circ \phi_i = \tau_i, \quad i = 1, 2. \tag{1.2}$$

Now $\varphi_1 \otimes \varphi_2$ is an isomorphism of $M_1 \otimes M_2$ onto $N_1 \otimes N_2$, and by (1.2), $(\sigma_1 \otimes \sigma_2) \circ (\varphi_1 \otimes \varphi_2) = \tau_1 \otimes \tau_2$. Since $\sigma_1 \otimes \sigma_2$ and $\tau_1 \otimes \tau_2$ are the canonical traces on $N_1 \otimes N_2$ and $M_1 \otimes M_2$, respectively, they are faithful and normal. Thus by [15], Lemma 1, (see Lemma 9.2 infra) $\varphi_1 \otimes \varphi_2$ extends to an isomorphism of $M_1 \otimes M_2$ onto $N_1 \otimes N_2$.

Finally, we define the infinite tensor product of von Neumann algebras. Let $\{M_n\}$ be a sequence of von Neumann algebras, let φ_n be a normal state on M_n , and let $\varphi = \otimes_n \varphi_n$ denote the infinite tensor product state on the infinite C*-tensor product $\otimes_n M_n$ of the M_n 's ([13], Section 1.23). Let π_φ denote the Gelfand-Naimark-Segal representation of $\otimes_n M_n$ determined by φ . Then the von Neumann algebra $\pi_\varphi(\otimes_n M_n)''$ is called the (W*-) infinite tensor product of $\{M_n\}$ relative to φ .

2. AN HISTORICAL PERSPECTIVE ON CONNES' WORK

We will preface our discussion of [10] by a rather brief overview of the main lines of work which led up to it. Because of the limitations of space and time, we have concentrated on emphasizing only work which deals directly with the classification of factors, and have reluctantly suppressed discussion of many other interesting and important developments in the structure theory of von Neumann algebras.

Our story begins, as all stories about von Neumann algebras do, with the work of Murray and von Neumann [1], [2], [3], and [4]. The classification of factors was the motivating problem for this work, and all other progress on the problem was based on this pioneering effort. In [1], Murray and von Neumann introduced the fundamental notion of types (I, II₁, II_∞, and III), and in [1] and [2], they obtained a complete classification of the factors of type I: if M is a factor of type I, then there exists a Hilbert space H such that M is isomorphic to $B(H)$, the algebra of all bounded operators on H . For our purposes, however, the most important advance that Murray and von Neumann made was their famous characterization of the hyperfinite II₁ factor. Since this result will play a key role in the sequel, we will describe it in detail.

Let M be a von Neumann algebra acting on a separable Hilbert space (from now until the end of this section, all von Neumann algebras will be assumed acting on a separable Hilbert space without explicit mention). M is said to be hyperfinite if there is a sequence $\{M_n\}$ of finite-dimensional von Neumann subalgebras of M , totally ordered by inclusion, whose union is dense in M relative to the weak operator topology. If M is a finite factor, hyperfiniteness of M is equivalent to the following condition (condition C of Murray-von Neumann): for each finite subset $\{x_1, \dots, x_n\}$ of M and $\varepsilon > 0$, there is a finite-dimensional subfactor C of M and $v_1, \dots, v_n \in C$ such that $\|x_i - v_i\|_2 < \varepsilon$, $i = 1, \dots, n$. The following theorem will be used in a crucial way in our later discussion:

2.1. THEOREM. ([4], Theorem XIV). Let M_1 and M_2 be factors of type II_1 . If M_1 and M_2 both satisfy condition C, then M_1 and M_2 are isomorphic.

We will denote by R the hyperfinite II_1 factor, unique up to isomorphism by Theorem 2.1.

Another important technique contained in the Murray-von Neumann papers is the so-called group-measure space construction of factors. This technique was used to give the first examples of type II_1 and II_∞ factors in [2], and the first example of a factor of type III in [3]. We will not go into the details of this construction right now: it will emerge later as a special case of the Connes-Takesaki crossed product construction.

The next major advance in the structure theory of von Neumann algebras occurred in 1949 with the publication of von Neumann's reduction theory [16]. This paper introduced and used the concept of direct integral of Hilbert spaces to decompose an arbitrary separably acting von Neumann algebra into a direct integral of factors, and thereby reduced the study of von Neumann algebras, at least in principle, to the study of factors.

This then was essentially the state of the art in the classification of von Neumann algebras at the start of the 1950's. Most of the work in the area consisted in a refining and strengthening of the tools bequeathed by Murray and

von Neumann. The theory at that time suffered from a dearth of examples of nonisomorphic factors, and consequently one of the main lines of work was the construction of such examples. Von Neumann had constructed one example of a type III factor in [3], but did not give another nonisomorphic one. In 1956, Pukansky [17] constructed a pair of nonisomorphic type III factors using refinements of the group-measure space construction, and there the matter lay until 1963 when J.T. Schwartz [18] found a new invariant, his well-known Property P: a von Neumann algebra M acting on H has Property P, if for each $T \in B(H)$, the weakly closed convex hull of $\{U T U^* : U \text{ a unitary operator in } M\}$ has a nonempty intersection with M' . Schwartz used this property to distinguish two nonisomorphic, hyperfinite II_1 factors, and also to construct another type III factor different from Pukansky's examples [19]. He showed that all hyperfinite factors have property P, and conjectured the converse. This converse was one of the many things established by Connes in [10].

Since it had been a fairly difficult task to obtain such a small number of examples of nonisomorphic factors, there gradually emerged a hope that a somewhat complete classification of all factors just might be possible. In 1967, that hope began to fade when R.T. Powers [20] constructed the first uncountable family of nonisomorphic factors. Let $\lambda \in [0,1]$ and let M_2 denote the 2×2 complex matrices. Let φ_λ denote the state defined on M_2 defined by

$$\varphi_\lambda : \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \rightarrow \frac{\lambda}{1+\lambda} \alpha + \frac{1}{1+\lambda} \delta$$

The Powers factor R_λ is defined as the infinite tensor product of countably many copies of M_2 relative to the product state $\otimes_n \varphi_n$, where each φ_n is φ_λ . We note that R_1 is the hyperfinite II_1 factor. R_λ is a hyperfinite factor of type III for each $\lambda \in (0,1)$, and Powers showed that R_{λ_1} is not isomorphic to R_{λ_2} if λ_1 and λ_2 are distinct elements of $(0,1)$. We will see this class of factors again.

After Powers' examples appeared, the flood gates opened, and nonisomorphic

factors soon poured out. In 1969, McDuff [21] exhibited an uncountable family of II_1 factors, and in 1970, Sakai [22], [23] found uncountable families of II_∞ and nonhyperfinite type III factors. This work dashed forever the hope of a complete classification of all factors. Attention began to focus on particular classes of factors for which there did seem to be a possibility of a satisfactory classification with results that were to shortly revolutionize the theory.

Let $\{M_n\}$ be a sequence of finite factors, with M_n of type I_{m_n} , $m_n < \infty$, for each n . Assume that M_n acts on H_n , and let x_n be a fixed unit vector in H_n . Then the functional ω_{x_n} defined on M_n by $\omega_{x_n} : T \rightarrow (Tx_n, x_n)$, $T \in M_n$ is a state on M_n . Form the infinite tensor product $\bigotimes_n M_n$ relative to the product state $\bigotimes_n \omega_{x_n}$. The resulting factor is an example of what Araki and Woods [24] dubbed an ITPFI factor (for "infinite tensor product of finite type I's"). In this paper, which appeared in 1968 and was motivated in part by Powers' work, these authors introduced an important new invariant for the ITPFI factors, the asymptotic ratio set, and succeeded in obtaining an almost complete classification of this class of factors. Both for its historical importance and for its foreshadowing of even bigger things to come, we will briefly describe the asymptotic ratio set and the Araki-Woods classification.

Let M be a factor. The asymptotic ratio set $r_\infty(M)$ of M is the subset of $[0, \infty)$ defined as follows:

$$r_\infty(M) = \{x \in [0, 1]: M \text{ is isomorphic to } M \otimes R_x\} \\ \cup \{x \in (1, \infty): M \text{ is isomorphic to } M \otimes R_{1/x}\}$$

where R_λ is the Powers factor defined previously. This is not the definition of $r_\infty(M)$ first given by Araki and Woods for the ITPFI factors. The first definition ([24], Definition 3.2) was expressed in terms of a complicated limiting procedure involving the ratios of the eigen-values of the density matrices of the states occurring in the infinite tensor product decomposition of the ITPFI factors, and Araki and Woods later showed that the definition given above was equivalent to the original one ([24], proof of Theorem 5.9). If M is an ITPFI factor, they deduced that $r_\infty(M)$ must have one of the following forms:

$$\begin{aligned}
S_0 & : \{0\} , \\
S_1 & : \{1\} , \\
S_x & : \{0, x^n : n = 0, \pm 1, \pm 2, \dots\} , \quad x \in (0, 1) , \\
S_{01} & : \{0, 1\} , \\
S_\infty & : [0, \infty) .
\end{aligned}$$

Araki and Woods showed that S_∞ corresponds to only one isomorphism class of ITPFI factors, and it is of type III. We will denote this factor by R_∞ . The invariants S_x , $x \in [0, 1]$ were also shown to come from only one isomorphism class, the one determined by the Powers factor R_x . Thus all isomorphism classes of ITPFI factors, with the exception of those in the S_{01} class, were completely determined. The Araki-Woods classification scheme was the first significant advance in the classification of factors beyond the initial results of Murray and von Neumann, and it was to have a great influence on the young Alain Connes.

The second major development which Connes would put to good use came from Japan. A problem of interest at the time concerned the commutant of a tensor product of von Neumann algebras: if M_1 and M_2 are von Neumann algebras, does $(M_1 \otimes M_2)'$ equal $M_1' \otimes M_2'$? In 1967, M. Tomita [25], [26] answered this question affirmatively by a new and original analysis of the spatial relationship between a von Neumann algebra and its commutant. The exposition of [25] and [26] was somewhat obscure, however, and in 1970, M. Takesaki published his seminal monograph [6] which explained and extended Tomita's earlier work. Let M be a von Neumann algebra with a vector which is both cyclic and separating for M . Takesaki associated a closed, densely defined, self-adjoint operator Δ with M , the modular operator, which has two very useful properties. The first is that $\{\Delta^{it} : t \in (-\infty, \infty)\}$ forms a one-parameter unitary group for which $\Delta^{-it} M \Delta^{it} = M$, the modular automorphism group of M , and the second is that Δ induces a conjugate-linear, involutive isometry J of the underlying Hilbert space for which $JMJ = M'$. This shows in particular that M and M' are anti-isomorphic, and is the key to Tomita's proof of the commutation theorem for tensor products. In actuality, the existence of a cyclic and separating vector is not necessary for the definition

of the modular operator, and in fact if φ is any faithful, normal, positive linear functional on M , then the modular operator Δ_φ and the modular automorphism group $\{\Delta_\varphi^{it}\}$ corresponding to φ can be constructed in H_φ relative to $\pi_\varphi(M)$, where $\{\pi_\varphi, H_\varphi\}$ is the representation of M arising from the GNS construction induced by φ . The construction of the modular operator and the verification of its main properties are quite technical, and for that reason we will not go into the details. We instead refer the reader to Rieffel and van Daele's excellent development of the Tomita-Takesaki theory [27], and, of course, also to the original memoir [6].

The stage was now set, and in 1973 Connes' thesis [5] appeared. This work contained a classification scheme for factors of type III which was to have a profound influence on all subsequent work in this area. Taking his cue from the Araki-Woods classification scheme and the Tomita-Takesaki theory, Connes introduced the modular spectrum as an isomorphism invariant for type III von Neumann algebras. Let M be a separable acting type III von Neumann algebra. The modular spectrum $S(M)$ of M is the intersection of the Arveson spectra (see [29]) of the modular operators Δ_φ corresponding to all faithful, normal, positive linear functionals φ on M . Connes and van Daele proved the remarkable fact that $S(M) \setminus \{0\}$ forms a subgroup of $(0, \infty)$ and that $S(M)$ is a closed set which is an isomorphism invariant of M . Connes then divided the type III von Neumann algebras into subtypes as follows: M is said to be

- (i) of type III_0 , if $S(M) = \{0, 1\}$;
- (ii) of type III_λ , if $S(M) = \{0, \lambda^n; n=0, \pm 1, \pm 2, \dots\}$, $\lambda \in (0, 1)$;
- (iii) of type III_1 , if $S(M) = [0, \infty)$.

This is a direct generalization of the Araki-Woods classification; for a type III ITPFI factor M , $r_\infty(M) = S(M)$, and so

- M is of class $S_{01} \Leftrightarrow M$ is of type III_0 ;
- M is of class $S_\lambda \Leftrightarrow M$ is of type III_λ , $\lambda \in (0, 1)$;
- M is of class $S_\infty \Leftrightarrow M$ is of type III_1 .

Connes then proceeded to prove a fundamental structure theorem for the factors of type III_λ , $\lambda \in [0,1)$ in terms of discrete crossed products of von Neumann algebras. The case of type III_1 was not treated, but Connes was not alone in this work. In the same year as Connes' thesis appeared, Takesaki also offered [28], which solved the III_1 case by the introduction of continuous crossed products, and which also developed a duality theory for crossed products that was to be very influential. We now proceed to describe the structure theorems of Connes and Takesaki.

Let G be a locally compact abelian group, M a von Neumann algebra. A continuous action α of G on M is a homomorphism α of G into $\text{Aut}(M)$, the group of all $*$ -automorphisms of M , such that for each $x \in M$, the mapping $g \rightarrow \alpha_g(x)$ is $*$ -strongly continuous. If $G = \mathbb{R}$ (as will be the case in the sequel), we call α a one-parameter action on M .

Suppose M acts on a Hilbert space H . Let λ denote Haar measure on G , and let $L^2(G;H)$ denote the Hilbert space of all H -valued, λ -square integrable functions on G . We define the representations π_α of M and λ of G on $L^2(G;H)$ as follows:

$$\begin{aligned} (\pi_\alpha(x)\xi)(h) &= \alpha_h^{-1}(x)\xi(h); \quad x \in M, \quad h \in G, \quad \xi \in L^2(G;H); \\ (\lambda(g)\xi)(h) &= \xi(g^{-1}h); \quad g, h \in G, \quad \xi \in L^2(G;H). \end{aligned}$$

The crossed product $W^*(M,\alpha)$ of M by α is the von Neumann subalgebra of $B(L^2(G;H))$ generated by $\{\pi_\alpha(x) : x \in M\}$ and $\{\lambda(g) : g \in G\}$. If $\theta \in \text{Aut}(M)$, the crossed product $W^*(M,\theta)$ of M by θ is the crossed product of M by the discrete action $n \rightarrow \theta^n$, $n=0, \pm 1, \pm 2, \dots$. We will often refer to $W^*(M,\theta)$ as a discrete crossed product or a discrete decomposition.

The oldest, and in many ways the most important, example of a crossed product is the classical group-measure space construction of Murray and von Neumann. Let (X,μ) be a σ -finite measure space with positive measure μ , and call a bijective mapping T of X onto X an automorphism of X if T and T^{-1} are

measurable and T maps sets of μ -measure zero to sets of μ -measure zero. An automorphism T of X induces an algebra automorphism α_T of $L^\infty(X, \mu)$ in the natural way:

$$\alpha_T : f \rightarrow f \circ T ; \quad f \in L^\infty(X, \mu).$$

Let $H = L^2(X, \mu)$ denote the Hilbert space of all μ -square integrable functions on X . $L^\infty(X, \mu)$ acts by pointwise multiplication on H and thereby forms a maximal abelian von Neumann subalgebra of $B(H)$, and α_T is a $*$ -automorphism of this von Neumann algebra. The von Neumann algebra given by the group-measure space construction is simply the discrete crossed product $W^*(L^\infty(X, \mu), \alpha_T)$. $W^*(L^\infty(X, \mu), \alpha_T)$ is hyperfinite, and if T is ergodic, it is a factor. Two automorphisms T_1 and T_2 of X are weakly equivalent if there exists an automorphism U of X such that $\{T_1^n(x) : n \in \mathbb{Z}\} = \{(UT_2^n U^{-1})(x) : n \in \mathbb{Z}\}$ for μ -almost all x . In a famous paper which generalized the work of Araki and Woods, Krieger [30] proved that if T_1 and T_2 are ergodic automorphisms of X then $W^*(L^\infty(X, \mu), \alpha_{T_1})$ is isomorphic to $W^*(L^\infty(X, \mu), \alpha_{T_2})$ if and only if T_1 and T_2 are weakly equivalent. For this reason, Connes calls a discrete crossed product of an abelian von Neumann algebra by an ergodic automorphism of the algebra a Krieger factor, and so therefore will we. (Incidentally, Connes cites the work of Krieger (along with Araki-Woods [24]) in the introduction to [5] as being one of the primary motivations for developing his classification scheme for type III factors).

We are now in a position to state what may appropriately be called the first, second, and third fundamental structure theorems for type III factors.

2.2. THEOREM. (Takesaki, [28]). Let M be a factor of type III. There exists a von Neumann algebra N of type II_∞ , a semifinite, faithful, normal trace τ on N , and a one-parameter action $\alpha = \alpha_t$ on N such that

$$(i) \quad \tau \circ \alpha_t = e^{-t} \tau ;$$

(ii) M is isomorphic to $W^*(N, \alpha)$.

2.3. THEOREM. (Connes, [5]). Let M be a factor of type III_λ , $\lambda \in (0,1)$. There exists a factor N of type II_∞ , an automorphism θ of N , and a faithful normal, semifinite trace τ of N such that

- (i) $\tau\theta = \lambda \tau$;
- (ii) M is isomorphic to $W^*(N,\theta)$.

Furthermore, any pair (N,θ) satisfying (i) gives rise to a crossed product factor of type III_λ , and any two such pairs (N_1,θ_1) , (N_2,θ_2) give isomorphic factors if and only if they correspond to the same λ and there is an isomorphism π of N_1 onto N_2 such that $p(\pi \theta_1 \pi^{-1}) = p(\theta_2)$, where p is the canonical quotient map of $\text{Aut}(N_2)$ onto the quotient of $\text{Aut}(N_2)$ by its subgroup of inner automorphisms.

2.4. THEOREM. (Connes, [5]). Let M be a factor of type III_0 . There exists a von Neumann algebra N of type II_∞ with a diffuse center, an automorphism θ of N which is ergodic on the center of N , and a faithful, normal, semifinite trace τ on N such that

- (i) for some $\lambda_0 < 1$, $\tau(\theta(x)) \leq \lambda_0 \tau(x)$, for all positive elements x of N ;
- (ii) M is isomorphic to $W^*(N,\theta)$.

Any pair (N,θ) satisfying the above conditions gives rise to a factor of type III_0 .

These theorems in principle reduce the study of factors of type III to the study of von Neumann algebras of type II_∞ and their automorphisms, and Connes lost no time in beginning such a study.

If M is a II_∞ factor acting on a separable Hilbert space H , a theorem of Murray and von Neumann ([4], Theorem IX) allows one to write M as the tensor product of a II_1 factor N and $B(H)$. Thus the study of automorphisms of II_∞ factors can be effectively reduced to the study of automorphisms of II_1 factors. Now the hyperfinite II_1 factor is in many ways the simplest of all the II_1 factors, and hence any program which aspires to a classification of automorphisms of II_1 factors must first of all handle the case of the hyperfinite II_1

factor R . In [9], this is precisely what Connes did. In this deep and penetrating study, he determined all outer conjugacy classes of automorphisms of R , and in particular showed that the quotient of $\text{Aut}(R)$ by its subgroup of inner automorphisms is a simple group with only countably many conjugacy classes. By our previous comments, all automorphisms of the II_∞ factor $R_{0,1} = \mathbb{R} \otimes B(H)$ can now be determined.

The factor $R_{0,1}$ is also of interest for another reason. An old question of Murray-von Neumann asked whether all hyperfinite II_∞ factors are isomorphic to $R_{0,1}$? This question arose naturally from their work on the hyperfinite II_1 factor, and for a long time was viewed as one of the most important open questions in the theory of von Neumann algebras. In [10] it received an affirmative answer. The key to this answer lies in the concept of injectivity, an idea introduced by Tomiyama in [31] and exploited by him and others in the study of tensor products of von Neumann algebras (the terminology is due to Effros and Lance [14]). The great achievement of [10] was to identify injectivity and hyperfiniteness for separably acting factors, first in the II_1 case, then for the II_∞ case, and finally for the type III case using the fundamental structure Theorems 2.2, 2.3, and 2.4. It is to a detailed discussion of these ideas that we now turn.

A few words are in order concerning the organization of the remainder of the paper. The main theorem occurs in Section 8, and asserts that all injective II_1 factors are isomorphic. In Section 3, we define injectivity for von Neumann algebras and give a tracial characterization of injectivity for finite factors. Sections 4 and 5 are concerned with establishing a certain type of finite dimensional approximation property for standardly acting, injective, finite factors which plays a central role in the proof of the main theorem. Section 6 discusses semi-discreteness for injective finite factors. Automorphisms of factors are briefly treated in Section 7, and several important results of Connes on automorphisms of II_1 factors are stated for use in the proof of the main theorem. Section 8 commences with the proof of the main theorem, Section 9 provides some necessary lemmas on embeddings in ultraproducts, and proof of the main theorem is completed

in Section 10. The eleventh and final section uses the results of Sections 2 and 8, together with some results from [7] and [3], to obtain the classification of injective factors of type II_∞ and III_λ , $\lambda \in [0,1)$.

3. INJECTIVE VON NEUMANN ALGEBRAS AND THE HYPERTRACE.

A von Neumann algebra N acting on a Hilbert space H is said to be injective if there is a projection E of $B(H)$ onto N with $\|E\| = 1$. It follows easily that $E \geq 0$, and by a well-known theorem of Tomiyama ([31]), E is a conditional expectation, i.e.,

$$E(axb) = aE(x)b, \quad a, b \in N, \quad x \in B(H).$$

Suppose now that N is an injective finite factor acting on H . Let τ denote a tracial state on N , and set $\psi = \tau \circ E$, where E is a projection of norm 1 of $B(H)$ onto N . Then ψ is a state on $B(H)$ such that for all $x \in B(H)$, $a \in N$,

$$\begin{aligned} \psi(xa) &= \tau(E(xa)) = \tau(E(x)a) \\ &= \tau(aE(x)) = \tau(E(ax)) \\ &= \psi(ax). \end{aligned} \tag{3.1}$$

A state ψ of $B(H)$ satisfying (3.1) is called a hypertrace of N .

Conversely, suppose N is a finite factor acting standardly on H with hypertrace ψ . We will eventually show that this forces N to be injective, but before doing this we need to recall some facts about the Hilbert space $L^2(N, \tau)$ of Segal ([32]). An element x of $L^2(N, \tau)$ is a closed, densely defined operator on H affiliated with N in the sense of Definition 2.1 of [32]. A positive operator T affiliated with N is called integrable when its spectral measure $\mu(E) = \tau(\chi_E(T))$ (E a Borel subset of $(0, +\infty)$) satisfies $\int_0^\infty \lambda d\mu(\lambda) < \infty$, and we set $\tau(T) = \int_0^\infty \lambda d\mu(\lambda)$. $L^2(N, \tau)$ consists of all integrable x satisfying $\tau(x^*x) < \infty$, and the norm of $L^2(N, \tau)$ is given by $\|x\| = \tau(x^*x)^{1/2}$. N is contained in $L^2(N, \tau)$ as a dense submanifold, and for $a \in N$, $\|a\| = \tau(a^*a)^{1/2} = \|a\|_2$.

If we restrict the hypertrace ψ to N , a tracial state on N results and so $\psi|_N = \tau$. The Schwartz inequality for positive linear functionals therefore

yields for each $A \in B(H)$,

$$\begin{aligned} |\psi(aA)| &\leq \psi(A^*A)^{\frac{1}{2}} \psi(a^*a)^{\frac{1}{2}} = \psi(A^*A)^{\frac{1}{2}} \tau(a^*a)^{\frac{1}{2}} \\ &\leq \|A\| \tau(a^*a)^{\frac{1}{2}}, \quad \forall a \in N. \end{aligned} \tag{3.2}$$

Thus each $A \in B(H)$ determines a bounded linear functional $\phi_A : a \rightarrow \psi(aA)$ on $L^2(N, \tau)$, and hence a unique $E(A)$ in $L^2(N, \tau)$ such that

$$\tau(aE(A)) = (a, E(A)^*)_{L^2(N, \tau)} = \psi(aA), \quad \forall a \in N. \tag{3.3}$$

Claim: If $A \geq 0$, then ϕ_A is positive on N . Let $a \in N$, $T \in B(H)$. Then

$$\begin{aligned} \psi(a^*aT^*T) &= \psi(aT^*Ta^*) \quad (\psi \text{ is a hypertrace}) \\ &= \psi(aT^*(aT^*)^*) \geq 0. \end{aligned}$$

This verifies the claim.

We can now show that $E(A) \in N$ for $A \geq 0$. Since $E(A)$ is affiliated with N , it suffices by the double commutant theorem to show that $E(A)$ extends to a bounded operator on $H = L^2(N, \tau)$.

Let $a, b \in N$, $\|a\|_2, \|b\|_2 \leq 1$. Then

$$\begin{aligned} |(E(A)a, b)_H| &= |\tau(b^*E(A)a)| \\ &= |\tau(ab^*E(A))|, \text{ by [32], Corollary 11.2} \\ &= |\psi(ab^*A)| = |\phi_A(ab^*)|, \text{ by (3.3)} \\ &\leq \phi_A(a^*a)^{\frac{1}{2}} \phi_A(bb^*)^{\frac{1}{2}}, \text{ since } \phi_A \geq 0 \\ &\leq \|A\| \tau(a^*a)^{\frac{1}{4}} \tau(bb^*)^{\frac{1}{4}}, \text{ by (3.2)} \\ &= \|A\| \|a\|_2^{\frac{1}{2}} \|b\|_2^{\frac{1}{2}} \\ &\leq \|A\|. \end{aligned}$$

Since N is dense in $H = L^2(N, \tau)$, it follows that $E(A)$ extends to a bounded operator on H with $\|E(A)\| \leq \|A\|$.

We conclude that $E(A) \in N$, for all $A \in B(H)$. It is now straightforward to deduce from (3.3) that $A \rightarrow E(A)$ is a projection of norm 1 of $B(H)$ onto N .

The following proposition now obtains:

3.1. PROPOSITION. Let N be a finite factor. N is injective if and only if N admits a hypertrace in its standard representation.

4. THE METHOD OF DAY FOR INJECTIVE FINITE FACTORS.

Let N be a von Neumann algebra, ϕ a linear functional on N , u a unitary

in N . We denote by $\varphi \circ \text{ad } u$ the linear functional

$$x \rightarrow \varphi(uxu^*), \quad x \in N.$$

4.1. PROPOSITION. Let N be an injective finite factor acting standardly in H . Then for each finite set $\{u_1, \dots, u_n\}$ of unitaries in N and $\varepsilon > 0$, there exists a normal state φ on $B(H)$ such that

$$\|\varphi - (\varphi \circ \text{ad } u_j)\| < \varepsilon, \quad j = 1, \dots, n. \quad (4.1)$$

PROOF. The proof uses a separation argument first employed by Day [33] in the context of countable amenable discrete groups.

We consider the Banach space $(B(H)_*)^n$ formed by taking the Banach space direct sum of n copies of the predual of $B(H)$. Consider the set

$$C = \{(\varphi - (\varphi \circ \text{ad } u_1), \dots, \varphi - (\varphi \circ \text{ad } u_n)) : \varphi \in B(H)_*\}.$$

To verify (4.1), it suffices to show that $0 \in$ norm closure of $C = C^-$.

Suppose $0 \notin C^-$. Then there is a norm-continuous linear functional f on $(B(H)_*)^n$ and $\alpha > 0$ such that

$$\text{Re } f(\varphi - (\varphi \circ \text{ad } u_1), \dots, \varphi - (\varphi \circ \text{ad } u_n)) \geq \alpha, \quad \forall \varphi \in B(H)_*.$$

Now there exist $x_1, \dots, x_n \in B(H)$ such that $f(g) = \sum_1^n g_i(x_i)$ for $g = (g_1, \dots, g_n) \in (B(H)_*)^n$. Hence

$$\text{Re } \sum_{j=1}^n (\varphi(x_j) - \varphi(u_j x_j u_j^*)) \geq \alpha, \quad \forall \varphi \in B(H)_*. \quad (4.2)$$

By Proposition 3.1, N has a hypertrace ψ . By the bipolar theorem ([34, Theorem 2.14]), ψ is the $\sigma(B(H)_*, B(H))$ -limit of a net $(\varphi_\alpha)_{\alpha \in A}$ of normal states on $B(H)$. Since (4.2) holds for each φ_α , we hence conclude that (4.2) holds for ψ . But $\psi(x_j) = \psi(u_j x_j u_j^*)$, $j = 1, \dots, n$. We conclude that $0 \in C^-$.

QED

5. AN ANALOG OF FÖLNER'S CONDITION FOR INJECTIVE FACTORS.

As mentioned in the previous section, Day used the technique of Proposition 4.1 to show that when G is a countable, amenable, discrete group, there exists

a sequence $\{ F_n \}$ of normalized positive functions in $\ell^1(G)$ such that $\|gF_n - F_n\|_1 \rightarrow 0$, for all $g \in G$. Somewhat earlier Følner [35] had given a stronger result by finding a sequence $\{ S_n \}$ of finite subsets of G such that $S_n \neq \emptyset$, for all n , and

$$\frac{(gS_n \Delta S_n)^{\#}}{S_n^{\#}} \rightarrow 0, \forall g \in G,$$

where $A^{\#}$ = cardinality of $A \subseteq G$ and Δ denotes symmetric difference.

Let N be an injective finite factor acting standardly in H . The next proposition gives an analog of Følner's result for N , and will play a crucial role in the sequel. Before giving its statement and proof, we need a rather technical lemma concerning certain approximations for positive Hilbert-Schmidt operators.

For its proof, consult [10], Theorem 1.22 or [37], Section 2.5.

LEMMA. Let H be a Hilbert space. Let χ_a denote the characteristic function of the interval $(a, +\infty)$, $a \geq 0$. Let h_1, \dots, h_n be positive Hilbert-Schmidt operators on H such that

$$\|h_j - h_1\|_{HS} < \varepsilon \|h_1\|_{HS}, \quad j = 1, \dots, n,$$

for some $\varepsilon > 0$, where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Then there exists $a > 0$ such that $\chi_a(h_1) \neq 0$ and

$$\sum_{j=1}^n \|\chi_a(h_j) - \chi_a(h_1)\|_{HS}^2 < 3n\varepsilon \|\chi_a(h_1)\|_{HS}^2.$$

5.1. PROPOSITION. Let N be an injective finite factor acting standardly in H . For each finite subset $\{x_1, \dots, x_n\} \subseteq N$ and $\varepsilon > 0$, there exists a finite-rank projection $E \in B(H)$ such that $E \neq 0$ and

$$\|[E, x_j]\|_{HS} < \varepsilon \|E\|_{HS}, \quad j = 1, \dots, n. \tag{5.1}$$

(For operators A and B , we denote $AB-BA$ by $[A, B]$.)

PROOF. By Proposition 4.1, for any set $\{u_1, \dots, u_n\}$ of unitaries of N , we can find a normal state φ of $B(H)$ such that

$$\|\varphi - (\varphi \circ \text{ad } u_j)\| < \varepsilon, \quad j = 1, \dots, n. \quad (5.2)$$

Let ρ denote the density matrix of φ , i.e.,

$$\varphi(\cdot) = \text{Tr}(\rho \cdot), \quad \rho \geq 0, \quad \text{Tr}(\rho) = 1,$$

where Tr = canonical trace on $B(H)$. For $x \in B(H)$, we have

$$\varphi \circ \text{ad } u_j(x) = \varphi(u_j x u_j^*) = \text{Tr}(\rho u_j x u_j^*) = \text{Tr}((u_j^* \rho u_j)x).$$

Thus $\varphi \circ \text{ad } u_j - \varphi$ has density matrix $u_j^* \rho u_j - \rho$ and since the norm of a normal functional on H is the trace norm of its density matrix, we conclude from (5.2) that

$$\|u_j^* \rho u_j - \rho\|_{\text{Tr}} < \varepsilon, \quad j = 1, \dots, n. \quad (5.3)$$

For $j = 1, \dots, n$, set $h_j = u_j^* h u_j$, where $h = \rho^{\frac{1}{2}}$ is a positive Hilbert-Schmidt operator. By (5.3) and the Powers-Størmer inequality [36],

$$\begin{aligned} \|h_j - h\|_{\text{HS}} &\leq \|h_j^2 - h^2\|_{\text{Tr}} = \|u_j^* \rho u_j - \rho\|_{\text{Tr}} \\ &< \varepsilon, \quad j = 1, \dots, n, \\ \|h\|_{\text{HS}} &= \|h^2\|_{\text{Tr}}^{1/2} = \|\rho\|_{\text{Tr}}^{1/2} = 1. \end{aligned}$$

Thus by the lemma, there exists $a > 0$ such that $\chi_a(h) \neq 0$ and

$$\|\chi_a(h_j) - \chi_a(h)\|_{\text{HS}} < (3(n+1)\varepsilon^{\frac{1}{2}})^{\frac{1}{2}} \|\chi_a(h)\|_{\text{HS}}, \quad j = 1, \dots, n. \quad (5.4)$$

Let $E = \chi_a(h)$. Then $\chi_a(h_j) = u_j^* E u_j$ and $\text{Tr } E < \infty$. Thus E has finite rank, and by (5.4)

$$\|E - u_j^* E u_j\|_{\text{HS}} < (3(n+1)\varepsilon^{\frac{1}{2}})^{\frac{1}{2}} \|E\|_{\text{HS}}, \quad k = 1, \dots, n. \quad (5.5)$$

Claim: $\|[E, u_j]\|_{\text{HS}} = \|E - u_j^* E u_j\|_{\text{HS}}$. Let $\{e_\alpha\}$ be an orthonormal basis for H , and let $A_j = u_j^* [E, u_j]$. Then

$$\begin{aligned} \|E - u_j^* E u_j\|_{\text{HS}}^2 &= \|A_j\|_{\text{HS}}^2 = \sum_{\alpha} (A_j A_j^* e_\alpha, e_\alpha)^2 \\ &= \sum_{\alpha} ([E, u_j] [E, u_j]^* u_j e_\alpha, u_j e_\alpha)^2 \end{aligned}$$

$$= \|[E, u_j]\|_{HS}^2, \quad j = 1, \dots, n.$$

(5.1) now follows by the claim, (5.5), the arbitrariness of ε , and the fact that N is the span of its unitary group. QED

6. SEMIDISCRETENESS AND PURIFICATION OF FACTOR STATES.

Let A be a C^* -algebra acting on a Hilbert space H . Let H^C denote the conjugate Hilbert space with $\xi \rightarrow \xi^C$ a conjugate-linear isometry of H onto H^C . For $x \in B(H)$, denote by x^C the element in $B(H^C)$ such that $x^C \xi^C = (x\xi)^C$, $\xi \in H$. Let $A^C = \{x^C : x \in A\}$. A^C is a C^* -subalgebra of $B(H^C)$.

Let φ be a factor state on A . Assume for simplicity that in the Gelfand-Naimark-Segal construction $(\pi_\varphi, H_\varphi, \xi_\varphi)$ corresponding to φ , ξ_φ is separating for $\pi_\varphi(A)''$. Let $A \otimes A^C$ denote the algebraic tensor product of A and A^C , and define $\tilde{\varphi}: A \otimes A^C \rightarrow \mathbb{C}$ by

$$\tilde{\varphi}(a \otimes b^C) = (\pi_\varphi(a) \xi_\varphi, \Delta_\varphi^{-\frac{1}{2}} \pi_\varphi(b) \xi_\varphi)_{H_\varphi}, \quad a \in A, \quad b^C \in A^C$$

where Δ_φ is the modular operator corresponding to φ , the existence of which follows from Tomita-Takesaki theory. $\tilde{\varphi}$ always extends to a state on $A \otimes_{\max} A^C$; if $\tilde{\varphi}$ extends to a state on $A \otimes_{\min} A^C$, the spatial tensor product on $H \otimes H^C$, we say that φ admits a purification.

Let N be a factor acting on H . An early result of Murray and von Neumann [1] asserts that the homomorphism η defined on $N \otimes N'$ by $\eta: a \otimes a' \rightarrow aa'$ is an isomorphism. N is semidiscrete (in the sense of Effros and Lance, [14]) if this isomorphism extends to an isomorphism of $N \otimes_{\min} N'$, i.e., if

$$\left\| \sum_{j=1}^n a_j \otimes b_j \right\|_{N \otimes N'} = \left\| \sum_{j=1}^n a_j b_j \right\|_H, \quad \forall a_1, \dots, a_n \in N, \\ b_1, \dots, b_n \in N'.$$

Connes characterizes semidiscreteness in terms of purification of states as follows ([37], Section 2.8):

6.1. PROPOSITION. Let N be a factor on H . Then N is semidiscrete if and only if N has a faithful normal state which admits a purification.

Now, let N be an injective finite factor, with τ denoting the canonical trace. τ is normal and faithful. We claim that τ admits a purification. To

see this notice first that the map S induced by the involution and the canonical trace vector 1 on $\pi_\tau(N)1$, viz., $S:a(1) \rightarrow a^*(1)$, is isometric, since

$$\|a(1)\|_{H_\tau}^2 = \tau(a^*a) = \tau(aa^*) = \|a^*(1)\|_{H_\tau}^2$$

Thus, $\Delta_\tau = (S^*S)^{\frac{1}{2}} = I$, whence for all $a \otimes b^c \in N \otimes N^c$,

$$\tilde{\tau}(a \otimes b^c) = (\pi_\tau(a)1_\tau, \pi_\tau(b)1_\tau)_{H_\tau} = \tau(b^*a).$$

But by Proposition 5.1 and ([37], Section 2.7), we have the following inequality:

$$\begin{aligned} |\tau(\sum_{j=1}^n b_j^* a_j)| &= |\tau(\sum_{j=1}^n a_j b_j^*)| \\ &\leq \|\sum_{j=1}^n a_j \otimes b_j^c\|_{\otimes_{\min} C}, \quad \forall a_j, b_j \in N. \end{aligned}$$

It follows that $\tilde{\tau}$ is bounded on $N \otimes N^c$ relative to the spatial tensor product norm, and therefore has an extension to $N \otimes_{\min} N^c$. We can now deduce from Proposition 6.1:

6.2. PROPOSITION. Injective finite factors are semidiscrete.

7. AUTOMORPHISMS OF FACTORS.

Let M be a von Neumann algebra. We set

$\text{Aut}(M)$ = automorphism group of M (by automorphism, we will always mean $*$ -automorphism);

$\text{Int}(M)$ = inner automorphisms of M , i.e., the set of all automorphisms of the form $\text{adu}: x \rightarrow u^*xu$, $x \in M$, u a unitary operator in M . This is a normal subgroup of $\text{Aut}(M)$;

$\text{Out}(M) = \text{Aut}(M)/\text{Int}(M)$;

$p:\text{Aut}(M) \rightarrow \text{Out}(M)$ = canonical quotient map.

The weak topology on $\text{Aut}(M)$ is the topology of point-norm convergence in the predual M_* of M for the action $\theta(\varphi) = \varphi(\theta^{-1})$, i.e., a net $\{\theta_\alpha\}$ of automorphisms converges weakly to $\theta \in \text{Aut}(M)$ if and only if

$$\lim_\alpha \|\varphi \circ \theta_\alpha^{-1} - \varphi \circ \theta^{-1}\| = 0, \quad \forall \varphi \in M_*.$$

We set $\overline{\text{Int } M}$ = closure of $\text{Int}(M)$ in the topology just described.

Let f be a fixed linear functional on M , and let $a \in M$. The linear

functionals a_f and f_a on M are defined respectively by $a_f: x \rightarrow f(xa)$, $f_a: x \rightarrow f(ax)$, $x \in M$. We set $[a, f] = a_f - f_a$. A bounded sequence $\{x_n\} \subseteq M$ is centralizing if $\|[x_n, \varphi]\| \rightarrow 0$, for all $\varphi \in M_*$. An automorphism θ of M is centrally trivial if $\theta(x_n) - x_n \rightarrow 0$, $*$ -strongly, for every centralizing sequence $\{x_n\}$ of M . We set

$Ct(M)$ = group of all centrally trivial automorphisms of M .

$Ct(M)$ is a normal subgroup of $Aut(M)$.

7.1. LEMMA. $Int(M) \subseteq Ct(M)$.

PROOF. We notice first that

(*) if $\{a_n\}$ is a bounded sequence in M and $\|a_n \varphi\|, \|a_n * \varphi\|$ both tend to 0 for all $\varphi \in M_*$, then $a_n \rightarrow 0$ $*$ -strongly.

Let $\theta = ad u \in Int(M)$, and let $\{x_n\}$ be a centralizing sequence. We must show that $\theta(x_n) - x_n \rightarrow 0$ $*$ -strongly. Now $\theta(x_n) - x_n = [u, x_n]u^*$, and setting $a_n = [u, x_n]u^*$, it suffices by (*) to show that

$$\lim_n \|[u, x_n] \varphi\| = \lim_n \|[u, x_n] * \varphi\| = 0, \quad \forall \varphi \in M_* \quad (7.1)$$

But $[u, x_n] * \varphi = -[u^*, x_n^*] \varphi$. Thus to verify (7.1), it suffices to show that for any unitary $u \in M$, any centralizing sequence $\{x_n\}$, and any $\varphi \in M_*$, we have

$$\lim_n \|[u, x_n] \varphi\| = 0. \quad (7.2)$$

Let $\psi \in M_*$. One verifies by direct computation that for $y \in M$,

$$[ux_n u^*, \psi](y) = [x_n, u^* \psi u](u^* y u) \quad (7.3)$$

For $\varphi \in M_*$, $y \in M$.

$$\begin{aligned} [u, x_n] \varphi(y) &= \varphi(y u x_n - y x_n u) = u \varphi(y u x_n u^* - y x_n) \\ &= u \varphi(y u x_n u^* - u x_n u^* y) + u \varphi(u x_n u^* y - y x_n) \\ &= [u x_n u^*, u \varphi](y) + u \varphi(u x_n u^* y - y x_n) \\ &= [x_n, \varphi u](u^* y u) + u \varphi(u x_n u^* y - y x_n), \text{ by (7.3)}. \end{aligned} \quad (7.4)$$

Now,

$$\begin{aligned} u x_n u^* y - y x_n &= u x_n u^* y - y u^* u x_n \\ &= (u x_n u^* y u^* - y u^* u x_n u^*) u \end{aligned}$$

$$= [u x_n u^*, y u^*] u .$$

Therefore,

$$\begin{aligned} u\varphi(u x_n u^* y - y x_n) &= u^2 \varphi([u x_n u^*, y u^*]) \\ &= -u^2 \varphi([y u^*, u x_n u^*]) \\ &= [u x_n u^*, -u^2 \varphi] (y u^*) \\ &= -[x_n, u\varphi u] (u^* y) , \text{ by (7.3) .} \end{aligned} \tag{7.5}$$

Thus by (7.4) and (7.5) ,

$$[u, x_n] \varphi(y) = [x_n, u\varphi u] (u^* y u) - [x_n, u\varphi u] (u^* y) . \tag{7.6}$$

(7.2) now follows from (7.6) and the fact that $\{x_n\}$ is centralizing. QED

7.2. LEMMA. Let M be a von Neumann algebra with separable predual.

Then for each $\theta \in \text{Ct}(M)$, $\alpha \in \overline{\text{Int } M}$,

$$p(\theta)p(\alpha) = p(\alpha)p(\theta) .$$

PROOF. For $\varphi \in M_*$, denote the seminorm $x \rightarrow \varphi(x^*x)^{\frac{1}{2}}$ by $\|\cdot\|_\varphi$. Since θ is centrally trivial, for each positive integer n there is a neighborhood V_n of 1 in $\text{Aut}(M)$ such that for all unitaries $u \in M$ with $\text{ad } u \in V_n$,

$$\|\theta(u) - u\|_{\varphi \circ \alpha^{-1}} < 2^{-n} ,$$

$$\|\theta(u) - u\|_{\varphi \circ (\theta \alpha^{-1} \theta^{-1})} < 2^{-n} .$$

Let $\{W_n\}$ be a decreasing basis of neighborhoods of α in $\text{Aut}(M)$ such that $W_n W_n^{-1} \subseteq V_n$ and $\|\varphi \circ \beta^{-1} - \varphi \circ \alpha^{-1}\| < 2^{-2n}$. Let u_n be a unitary in M such that $\text{ad } u_n \in W_n$. Then $\theta \alpha \theta^{-1} = \lim_n \text{ad } \theta(u_n)$, so we must show that $u_n^* \theta(u_n)$ converges $*$ -strongly to a unitary in M .

Let $v_n = u_{n+1} u_n^*$, so that $v_n \in W_n W_n^{-1}$. Then

$$\|\theta(v_n) - v_n\|_{\varphi \circ \alpha^{-1}} < 2^{-n} , \text{ and therefore}$$

$$\|\theta(v_n^*) v_n^{-1}\|_{\varphi \circ \text{ad } u_n^*} < 2 \cdot 2^{-n} + 2^{-n} = 3 \cdot 2^{-n} ,$$

since $\text{ad } u_n \in W_n$. Thus

$$\|\theta(v_n^*)v_n u_n - u_n\|_\varphi < 3 \cdot 2^{-n}$$

and

$$\|\theta(u_{n+1}^*)u_{n+1} - \theta(u_n^*)u_n\|_\varphi = \|\theta(u_n^*)\theta(v_n^*)v_n u_n - \theta(u_n^*)u_n\|_\varphi < 3 \cdot 2^{-n}.$$

Also, because

$$\|\theta(v_{n+1}) - v_{n+1}\|_{\varphi \circ (\theta\alpha^{-1}\theta^{-1})} < 2^{-n},$$

and

$$\|\varphi \circ (\text{ad } \theta(u_n^{-1})) - \varphi \circ (\theta\alpha^{-1}\theta^{-1})\| < 2^{-n},$$

we obtain

$$\|u_{n+1}^* \theta(u_{n+1}) - u_n^* \theta(u_n)\|_\varphi < 3 \cdot 2^{-n}.$$

This shows that $u_n^* \theta(u_n)$ converges $*$ -strongly to a unitary $u \in M$ such that $\text{ad } u = \alpha^{-1}\theta\alpha\theta^{-1}$. QED

The relation between $\text{Int}(M)$, $\overline{\text{Int}(M)}$, and $\text{Ct}(M)$ has a strong bearing on the structure of M . By Lemma 7.2, one always has $p(\text{Ct}(M)) \subseteq \text{commutant of } p(\overline{\text{Int } M})$.

On the other hand, Connes proves the following remarkable theorem:

7.3. THEOREM ([9], Theorem 2.2.1). Let M be a factor with separable predual, and let R denote the hyperfinite factor of type II_1 . The following are equivalent:

- (a) M is isomorphic to $M \otimes R$.
- (b) $p(\overline{\text{Int } M})$ is nonabelian.
- (c) $\overline{\text{Int } M} \not\subseteq \text{Ct}(M)$.

Moreover, if (a) holds, then $p(\text{Ct}(M)) = \text{commutant of } p(\overline{\text{Int } M})$.

We now list several theorems which allow us to manipulate $\overline{\text{Int } N}$ and $\text{Ct}(N)$ for a factor N of type II_1 . Regretfully we must omit all proofs; for details, consult the indicated references (all results are due to Connes).

Let N be a finite factor, τ the canonical trace on N . Recall that N has property Γ of Murray and von Neumann ([4]) if for each finite subset

$\{x_1, \dots, x_n\}$ of N and $\varepsilon > 0$, there is a unitary $u \in M$ with $\tau(u) = 0$ and $\| [u, x_j] \|_2 < \varepsilon$, $j = 1, \dots, n$.

7.4. THEOREM. ([7], Corollary 3.8). Let N be a II_1 factor with separable predual. Then

$\text{Int}(N) \neq \overline{\text{Int}} N$ if and only if N has property Γ .

7.5. THEOREM. ([10], Theorem 2.1). Let N be a II_1 factor acting standardly on H . Let K denote the compact operators on H , $C^*(N, N')$ the C^* -algebra generated by N and N' . Then

N has property Γ if and only if $C^*(N, N') \cap K = (0)$.

Since N acts standardly on H , $C^*(N, N')$ is irreducible. The next theorem gives a complete characterization of $\overline{\text{Int}} N$ for II_1 factors.

7.6. THEOREM ([10], Theorem 3.1). Let N be a II_1 factor acting standardly on H . Then $\theta \in \overline{\text{Int}} N$ if and only if θ extends to an automorphism $\alpha \in \text{Aut}(C^*(N, N'))$ such that

$$\alpha|_N = \theta, \quad \alpha|_{N'} = \text{Identity on } N'.$$

7.7. THEOREM. ([10], Corollary 4.4). Let N_1, N_2 be II_1 factors. Then for $\theta_i \in \text{Aut}(N_i)$, $i = 1, 2$,

$$\theta_1 \otimes \theta_2 \in \text{Ct}(N_1 \otimes N_2) \text{ if and only if } \theta_i \in \text{Ct}(N_i), i = 1, 2.$$

8. UNIQUENESS OF THE INJECTIVE II_1 FACTOR.

In this section, we begin the proof of the main theorem. The proof will end in Section 10.

8.1 THEOREM. ([10], Theorem 5.1). All injective factors of type II_1 acting on a separable Hilbert space are isomorphic.

PROOF. Let N denote an injective II_1 factor acting on a separable Hilbert space, and acting standardly on $H = L^2(N, \tau)$. We will show that N is isomorphic to the hyperfinite II_1 factor R .

By Proposition 6.2, N is semidiscrete, i.e., the mapping $\eta: N \otimes N' \rightarrow B(H)$ given by $\eta: \sum_{i=1}^n a_i \otimes b_i \rightarrow \sum_{i=1}^n a_i b_i$ is isometric as a mapping from $B(H \otimes H) \rightarrow B(H)$.

Since

$$C^*(N, N') = \text{norm closure of } \left\{ \sum_1^n a_i b_i : a_i \in N, b_i \in N', n \in \mathbb{Z}_+ \right\},$$

it follows that η extends to an isomorphism of $N \otimes N'$ onto $C^*(N, N')$.

We claim that N and N' are simple, i.e., they contain no closed, two-sided ideals. Since N is a finite factor, by [38], Theorem 6.2, N contains no nonzero maximal ideals. Since every two-sided ideal of N is contained in a maximal one, we conclude that N is simple. Since N is conjugate-linearly isomorphic to N' ([13], Proposition 2.9.2), it follows that N' is also simple.

By a theorem of Takesaki ([39]), we conclude that $N \otimes N'$ is simple so that $C^*(N, N')$ is simple. Hence if K denotes the (closed, two-sided) ideal of compact operators on H , then $C^*(N, N') \cup K$ is either (0) or $C^*(N, N')$. Since $I \in C^*(N, N')$ and H is infinite-dimensional, $C^*(N, N') \cap K \neq C^*(N, N')$, so $C^*(N, N') \cap K = (0)$. Thus by Theorems 7.4 and 7.5,

$$\text{Int}(N) \neq \overline{\text{Int } N}. \tag{8.1}$$

Let σ_F denote the flip automorphism of $N \otimes N$, i.e.,

$$\sigma_F(x \otimes y) = y \otimes x, \quad x, y \in N.$$

We want to show that $\sigma_F \in \overline{\text{Int}(N \otimes N)}$.

To do this, let M be a standardly acting semidiscrete II_1 factor, and let $\theta \in \text{Aut}(M)$. Then for all $a_1, \dots, a_n \in M, b_1, \dots, b_n \in M'$,

$$\begin{aligned} \left\| \sum_1^n a_i b_i \right\| &= \left\| \sum_1^n a_i \otimes b_i \right\| \quad (M \text{ is semidiscrete}) \\ &= \left\| \sum_1^n \theta(a_i) \otimes b_i \right\|, \quad \text{since } \theta \otimes \text{id} \text{ extends to} \\ &\quad \text{an automorphism of } M \otimes_{\min} M' \\ &= \left\| \sum_1^n \theta(a_i) b_i \right\|. \quad (\text{semidiscreteness again}) \end{aligned}$$

This shows that θ extends to an automorphism α of $C^*(M, M')$ such that $\alpha|_M = \theta, \alpha|_{M'} = \text{identity on } M'$. Thus by Theorem 7.6, $\theta \in \overline{\text{Int } M}$.

We have hence shown that

$$\text{Aut } M = \overline{\text{Int } M}.$$

Now by a theorem of Effros and Lance ([14], Proposition 5.6) $N \otimes N$ inherits semidiscreteness from N , and standard facts about tensor products imply that $N \otimes N$ is a standardly acting II_1 factor. We hence conclude that $\text{Aut}(N \otimes N) = \overline{\text{Int}}(N \otimes N)$, and so $\sigma_F \in \overline{\text{Int}}(N \otimes N)$.

Since N acts on a separable Hilbert space, so does $N \otimes N$, and therefore $N \otimes N$ has separable predual. By Lemma 7.2, $p(\text{Ct}(N \otimes N)) \subset \text{commutant of } p(\overline{\text{Int}}(N \otimes N))$, whence by Theorem 7.7,

$$p(\theta \otimes 1)p(\sigma_F) = p(\sigma_F)p(\theta \otimes 1), \quad \forall \theta \in \text{Ct}(N).$$

Since $\sigma_F(\theta \otimes 1)\sigma_F^{-1} = 1 \otimes \theta$, we get

$$p(1 \otimes \theta) = p(\sigma_F(\theta \otimes 1)\sigma_F^{-1}) = p(\theta \otimes 1),$$

whence

$$\theta \otimes \theta^{-1} \in \text{Int}(N \otimes N). \quad (8.2)$$

Now a theorem of R. Kallman ([40], Corollary 1.14) asserts that if M_1 and M_2 are von Neumann algebras, $\theta_i \in \text{Aut}(M_i)$, $i = 1, 2$, then $\theta_1 \otimes \theta_2 \notin \text{Int}(M_1 \otimes M_2)$ if and only if either $\theta_1 \notin \text{Int}(M_1)$ or $\theta_2 \notin \text{Int}(M_2)$. Thus by (8.2), $\theta \in \text{Int}(N)$. Recalling Lemma 7.1, we have thus shown that

$$\text{Ct}(N) = \text{Int}(N). \quad (8.3)$$

We conclude from (8.1) and (8.3) that

$$\text{Ct}(N) = \text{Int}(N) \subset \overline{\text{Int}} N, \\ \neq$$

so by Theorem 7.3, N is isomorphic to $N \otimes R$.

This completes the first major step of the proof of Theorem 8.1. In order to fully exploit this isomorphism, we must relate N more closely to R (after all, we are trying to show that N and R are in fact the same). This is done by embedding N in the ultraproduct R^ω , ω a free ultrafilter. We take up the details in the next section and it is there that Proposition 5.1 plays a crucial role.

9. EMBEDDINGS OF N IN ULTRAPRODUCTS.

Let Z_+ denote the positive integers with the discrete topology. Let

$C_b(Z_+) = \ell^\infty$ denote the C*-algebra of bounded sequences. We identify the Stone-Ćech compactification βZ_+ of Z_+ with M_{ℓ^∞} , the maximal ideal space of ℓ^∞ . Points $n \in Z_+$ correspond to the homomorphism of evaluation at n , and free ultrafilters $\omega \in \beta Z_+ \setminus Z_+$ correspond to homomorphisms not of this form.

Suppose $\omega \in \beta Z_+ \setminus Z_+$ corresponds to $\phi \in M_{\ell^\infty}$. Then for a sequence $\{a_n\} \in \ell^\infty$, we write $\lim_{n \rightarrow \omega} a_n = a$ if $\phi(\{a_n\}) = a$.

Now, let $(N_k)_{k \in Z_+}$ be a sequence of factors with finite normalized traces τ_k , and let ω be a free ultrafilter on Z_+ . On $\bigoplus_1^\infty N_k$, the C*-direct sum of the N_k 's, we define the trace

$$\tau_\omega : (x_k) \rightarrow \lim_{k \rightarrow \omega} \tau_k(x_k).$$

Let $Z_\omega = \text{kernel of } \tau_\omega = \{x \in \bigoplus_1^\infty N_k : \tau_\omega(x^*x) = 0\}$. Z_ω is a closed, two-sided ideal in $\bigoplus_1^\infty N_k$. Let

$$\prod_\omega N_k = \bigoplus_1^\infty N_k / Z_\omega.$$

Then ([41], p. 451) $\prod_\omega N_k$ is a von Neumann algebra with finite normalized trace τ_ω , called the ultraproduct of the N_k 's corresponding to ω .

Denote by M^ω the ultraproduct formed by countably many copies of the factor M . We proceed to construct an embedding of N into R^ω .

Let F_n denote the free group on the generators g_1, \dots, g_n . If $m \in F_n$, we define the length of m as the sum of the absolute values of all exponents appearing in a reduced presentation of m . Let $F_k = \text{set of all } m \in F_n \text{ of length } \leq k$.

Each n -tuple $u = (u_1, \dots, u_n)$ of unitaries defines a unitary representation u of F_n as follows: if $m \in F_n$, $u(m)$ is the element formed by replacing each g_i occurring in m by u_i , $i = 1, \dots, n$.

9.1. LEMMA. Let u_1, \dots, u_n be unitaries in N . For each $\varepsilon > 0$, there is a finite-dimensional factor Q and unitary operators $v_1, \dots, v_n \in Q$ such that

$$|\tau(u(m)) - \tau_q(v(m))| < \varepsilon, \quad m \in F_k,$$

where $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$, and τ_q is the normalized trace on Q .

PROOF. We first strengthen Proposition 5.1 as follows: given

$x_1, \dots, x_n \in N$ and $\varepsilon > 0$, there exists a finite-rank projection $E \neq 0$ on H such that for $j = 1, \dots, n$,

$$\|[E, x_j]\|_{HS} < \varepsilon \|E\|_{HS},$$

$$\left| \tau(x_j) - \frac{\langle x_j E, E \rangle_{HS}}{\|E\|_{HS}^2} \right| < \varepsilon.$$

To see this, we first claim that there exist unitaries $\{u_1, \dots, u_m\} \subseteq N$ and $\delta > 0$ such that for any state ϕ on N with $\|[u_j, \phi]\| \leq \delta$, $j = 1, \dots, m$ one has $|\tau(x_j) - \phi(x_j)| < \varepsilon$, $j = 1, \dots, n$. For suppose not. Then for each finite subset σ of unitaries of N and $\delta > 0$, there is a state $\phi_{\sigma, \delta}$ on N such that

$$(i) \quad \|[u, \phi_{\sigma, \delta}]\| \leq \delta, \quad \forall u \in \sigma,$$

$$(ii) \quad |\phi_{\sigma, \delta}(x_j) - \tau(x_j)| \geq \varepsilon, \quad \text{for some } j \in \{1, \dots, n\}.$$

If we partially order the set of (σ, δ) 's by setting $(\sigma_1, \delta_1) \leq (\sigma_2, \delta_2)$ if $\sigma_1 \subset \sigma_2$ and $\delta_1 \leq \delta_2$, then $\{\phi_{\sigma, \delta}\}$ is a net and so by weak *-compactness of the state space of N , $\{\phi_{\sigma, \delta}\}$ weak *-accumulates at a state ϕ of N . By (i), ϕ is unitarily invariant, and so $\phi = \tau$. But since $\phi_{\sigma, \delta} \rightarrow \phi = \tau$ (weak*), we may find $\phi_{\sigma, \delta}$ such that

$$|\phi_{\sigma, \delta}(x_j) - \tau(x_j)| < \varepsilon, \quad j = 1, \dots, n,$$

contradicting (ii). This verifies the claim.

Now use Proposition 5.1 to find a finite rank projection $E \neq 0$ such that for $i = 1, \dots, n$, $j = 1, \dots, m$,

$$(iii) \quad \|[x_i, E]\|_{HS} < \varepsilon \|E\|_{HS} \quad ,$$

$$(iv) \quad \|[u_j, E]\|_{HS} < \frac{\delta}{2} \|E\|_{HS} \quad .$$

We assert that the state ψ on N defined by

$$\psi(x) = \frac{(xE, E)_{HS}}{\|E\|_{HS}^2} = \frac{\text{Tr}(ExE)}{\text{Tr } E}$$

satisfies $\|[u_j, \psi]\| \leq \delta$, $j = 1, \dots, n$. This is so because

$$\begin{aligned} \psi_{u_j}(x) &= \psi(u_j x u_j^*) = \frac{\text{Tr}(Eu_j x u_j^* E)}{\text{Tr } E} \\ &= \frac{\text{Tr}(Eu_j x u_j^* E u_j u_j^*)}{\text{Tr } E} \\ &= \frac{\text{Tr}((u_j^* E u_j) x (u_j^* E u_j))}{\text{Tr } E} \\ &= \frac{(xE_j, E_j)_{HS}}{\text{Tr } E} \quad , \quad E_j = u_j^* E u_j \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} |\psi(x) - \psi_{u_j}(x)| &= |(xE, E)_{HS} - (xE_j, E_j)_{HS}| \cdot \|E\|_{HS}^{-2} \\ &\leq (|(xE - E_j), E)_{HS}| + |(xE_j, E - E_j)_{HS}| \cdot \|E\|_{HS}^{-2} \\ &\leq \|x\| \|E - E_j\|_{HS} \|E\|_{HS}^{-2} (\|E\|_{HS} + \|E_j\|_{HS}) \\ &\leq \|x\| \frac{\delta}{2} \|E\|_{HS}^{-1} \cdot 2\|E\|_{HS} \quad , \quad \text{by (iv)} \\ &= \delta \|x\| \quad . \end{aligned}$$

Since $[u_j, \psi](x) = (\psi - \psi_{u_j})(xu_j)$, $x \in N$, we conclude that

$\|[u_j, \psi]\| \leq \delta$, $j = 1, \dots, n$, as asserted. Thus, by the previous claim,

$$|\tau(x_j) - \frac{(x_j E, E)_{HS}}{\|E\|_{HS}^2}| < \varepsilon \quad , \quad j = 1, \dots, n \quad ,$$

which together with (iii) is what we are seeking.

Now, letting $\delta = \varepsilon/2k$, we may hence find a nonzero finite-rank projection E such that

$$\begin{aligned} \| [E, u_j] \|_{HS} &< \delta \|E\|_{HS}, \quad j = 1, \dots, n, \\ \left| \tau(u(m)) - \frac{(u(m)E, E)_{HS}}{\|E\|_{HS}^2} \right| &< \varepsilon, \quad m \in F_k. \end{aligned} \tag{9.1}$$

Let $E_j = u_j E u_j^*$; then $\|E_j - E\|_{HS} < \delta \|E\|_{HS}$, and, by [10], Lemma 1.4, there are unitary operators $w_j \in B(H)$ with $w_j E_j w_j^* = E$ and $\|w_j - I\|_{HS} < 3\delta \|E\|_{HS}$.

Let $v_j' = w_j u_j$. Then

$$v_j' E = E v_j', \quad \|v_j' - u_j\|_{HS} < \frac{\varepsilon}{k} \|E\|_{HS}, \quad j = 1, \dots, n. \tag{9.2}$$

Let $v_j = v_j' E = E v_j'$. Then v_j is a unitary operator in the finite-dimensional factor $Q = EB(H)E$. The normalized trace τ_Q on Q is given by

$$\tau_Q(x) = \frac{(xE, E)_{HS}}{\|E\|_{HS}^2}, \quad x \in Q.$$

Since for any unitaries $a, b \in B(H)$ we have

$$\|au_j b - av_j' b\|_{HS} < \frac{\varepsilon}{k} \|E\|_{HS},$$

we conclude by (9.2) and induction on the length of the word m that

$$\|u(m) - v'(m)\|_{HS} \leq (\text{length of } m) \frac{\varepsilon}{k} \|E\|_{HS} \tag{9.3}$$

By (9.2), E commutes with all v_j' , so that $(v'(m)E) = v(m)$ for each word m .

Thus by (9.3) and the Schwartz inequality for the Hilbert-Schmidt norm,

$$\left| (u(m)E, E)_{HS} - (v(m)E, E)_{HS} \right| < \varepsilon \|E\|_{HS}^2, \quad m \in F_k.$$

By (9.1) and the definition of τ_Q , this is what we seek. QED

We are now ready to construct an embedding of N into R^ω . The construction depends on the following extension-lemma of Pearcy and Ringrose

([15], Lemma 1), which we record for the convenience of the reader. (In what follows, $-WOT$ denotes closure in the weak operator topology).

9.2. LEMMA. Let M_i be a von Neumann algebra, ϕ_i a faithful, normal, positive linear functional on M_i , and A_i a $*$ -subalgebra of M_i such that $A_i^{-WOT} = M_i$, $i=1,2$. Suppose that π is an algebraic $*$ -isomorphism from A_1 onto A_2 and that $\phi_2 \circ \pi = \phi_1$ on A_1 . Then π can be extended to a $*$ -isomorphism $\tilde{\pi}$ of M_1 onto M_2 with $\phi_2 \circ \tilde{\pi} = \phi_1$ on M_1 .

9.3. LEMMA. For each $\omega \in \beta Z_+ \setminus Z_+$, there exists a trace-preserving $*$ -monomorphism of N into R^ω .

PROOF. Let F_∞ denote the free group on countably many generators $\{g_n\}$. Let τ_0 denote the normalized trace on R . Since N acts separably, it contains a strongly dense sequence, and so we may find a sequence $\{u_n\}$ of unitary operators generating N as a von Neumann algebra. Finally, let $F_k \subseteq F_\infty$ be the set of all words involving only g_1, \dots, g_k and with length $\leq k$.

Using Lemma 9.1 and the hyperfiniteness of R , choose for each k unitaries v_1^k, \dots, v_k^k in R such that

$$|\tau(u(m)) - \tau_0(v^k(m))| < \frac{1}{k}, \quad m \in F_k, \tag{9.4}$$

where $v^k = (v_1^k, \dots, v_k^k, I, I, \dots)$. For each $j \in Z_+$, let v_j be the unitary in R^ω represented by $(v_j^k)_k \in Z_+$. Then since each $m \in F_\infty$ belongs to F_k for all k sufficiently large, we get, with $v = (v_1, \dots, v_j, \dots)$,

$$\tau_{0,\omega}(v(m)) = \lim_{k \rightarrow \omega} \tau_0(v^k(m)) = \tau(u(m)), \quad m \in F_\infty. \tag{9.5}$$

Let $A = *$ -algebra generated by $\{u_n\}$. For $x = \sum_m \lambda_m u(m)$, let $\pi(x) = \sum_m \lambda_m v(m)$. If $\sum_m \lambda_m u(m) = 0$, then

$$\sum_m \bar{\lambda}_m \lambda_{m'}, \quad \tau(u(m) * u(m')) = 0,$$

whence by (9.5)

$$\sum_m \bar{\lambda}_m \lambda_{m'}, \quad \tau_{0,\omega}(v(m) * v(m')) = 0,$$

and so $\sum_m \lambda_m v(m) = 0$. Thus π is well-defined. By construction π is a

*-homomorphism of A into R^ω which preserves the trace, and is hence monomorphic. We now apply Lemma 9.2 to π to obtain the desired *-monomorphism of N into R^ω . QED

9.4. LEMMA. For any $\omega \in \beta Z_+ \setminus Z_+$ there is trace-preserving *-monomorphism θ of $N \otimes N$ into $(N \otimes R)^\omega$ such that

(a) For each $x \in N$, $\theta(x \otimes 1_N)$ is represented by the constant sequence $(x \otimes 1_R)_\nu \in Z_+$.

(b) For each $y \in N$, $\theta(1_N \otimes y)$ is represented by a sequence of the form $(1_N \otimes z_\nu)_\nu \in Z_+$, $z_\nu \in R$.

PROOF. Let π_1 be the unique isomorphism of N into $(N \otimes R)^\omega$ such that $\pi_1(x) = (x \otimes 1_R)_n \in Z_+$, $x \in N$, π the isomorphism of Lemma 9.3, and π_2 the isomorphism of N into $(N \otimes R)^\omega$ such that $\pi_2(y) = (1_N \otimes \pi(y))_n \in Z_+$, where $(\pi(y))_n \in Z_+$ is a representing sequence for $\pi(y)$ in R^ω , $y \in N$.

We claim that $x \otimes y \rightarrow \pi_1(x)\pi_2(y)$ extends to a trace-preserving *-monomorphism $\theta : N \otimes N \rightarrow (N \otimes R)^\omega$ satisfying (a) and (b).

With this definition, it is clear that θ satisfies (a) and (b), so we must show that θ is well-defined and can be extended as claimed.

Now π_1 and π_2 are *-isomorphisms, and so $\pi_1 \otimes \pi_2$ defines a *-isomorphism on $N \otimes N$. Since $\pi_1(N)$ and $\pi_2(N)$ are commuting subfactors \min of $(N \otimes R)^\omega$, $\pi_2(N) \subseteq \pi_1(N)'$, so if η denotes the η -isomorphism of Murray-von Neumann (Sec. 6) we have $\theta = \eta \circ (\pi_1 \otimes \pi_2)$ $N \otimes N$. θ is thus a well-defined *-monomorphism on $N \otimes N$. Since $(N \otimes N)^{-WOT} = N \otimes N$, it suffices by Lemma 9.2 to show that θ preserves the trace.

Let $\tau = \text{trace on } N$, $\tau_0 = \text{trace on } R$. Then

$$\begin{aligned} (\tau \otimes \tau_0)_\omega (\theta(x \otimes y)) &= (\tau \otimes \tau_0)_\omega [(x \otimes \pi(y))_n] \\ &= \lim_{n \rightarrow \omega} \tau(x)\tau_0(\pi(y)_n) \end{aligned}$$

$$\begin{aligned}
 &= \tau(x) \lim_{n \rightarrow \omega} \tau_0(\pi(y)_n) = \tau(x) \tau_{0,\omega}(\pi(y)) \\
 &= \tau(x)\tau(y) = \tau \otimes \tau(x \otimes y) \quad , \forall x, y \in N \quad ,
 \end{aligned}$$

i.e., θ preserves the trace.

QED

9.5. LEMMA. Let $x_1, \dots, x_n \in N$, $\varepsilon > 0$. Then there exist $z_1, \dots, z_n \in R$ and a unitary $X \in N \otimes R$ such that

$$\|x_j \otimes 1_R - X(1_N \otimes z_j)X^*\|_2 < \varepsilon, \quad j = 1, \dots, n. \quad (9.6)$$

PROOF. Let $\theta: N \otimes N \rightarrow (N \otimes R)^\omega$ be the isomorphism of Lemma 9.4. As we saw in Section 8, the flip automorphism $\sigma_F \in \text{Aut}(N \otimes N)$ is in $\overline{\text{Int}}(N \otimes N)$, so there is a unitary $v \in N \otimes N$ such that

$$\|x_j \otimes 1_N - v(1_N \otimes x_j)v^*\|_2 < \frac{\varepsilon}{2}, \quad j = 1, \dots, n. \quad (9.7)$$

Now θ preserves the trace, and hence the L^2 -norm. Thus by (9.7),

$$\|\theta(x_j \otimes 1) - \theta(v)\theta(1 \otimes x_j)\theta(v)^*\|_2 < \frac{\varepsilon}{2}, \quad j = 1, \dots, n. \quad (9.8)$$

Let $(X_\nu)_{\nu \in Z_+}$ be a representing sequence of unitary operators in $N \otimes R$ for $X = \theta(v) \in (N \otimes R)^\omega$ (the injectivity of θ is invoked here!). For each j , let $(z_j^{(\nu)})_{\nu \in Z_+}$ be a sequence in R such that $(1 \otimes z_j^{(\nu)})$ represents $\theta(1 \otimes x_j)$. Then by (9.8),

$$\lim_{\nu \rightarrow \omega} \|x_j \otimes 1_R - X_\nu(1_N \otimes z_j^{(\nu)})X_\nu^*\|_2 < \frac{\varepsilon}{2}, \quad j = 1, \dots, n. \quad (9.9)$$

Since the evaluations at points in Z_+ are weak $*$ -dense in M_ℓ^∞ , it follows from (9.9) that X_ν and $z_1^{(\nu)}, \dots, z_n^{(\nu)}$ satisfy (9.6) for suitable $\nu \in Z_+$.

10. THE PROOF OF THEOREM 8.1 COMPLETED.

Recall from Section 2 that a finite factor M satisfies condition C if for each finite subset $\{x_1, \dots, x_n\}$ of M and $\varepsilon > 0$, there exists a finite-dimensional subfactor C of M and v_1, \dots, v_n in C such that

$\|x_j - v_j\|_2 < \varepsilon$, $j = 1, \dots, n$. In order to show that N is isomorphic to R ,

it suffices by Theorem 2.1 to verify condition C for N . This is what we now proceed to do.

We saw that $N \cong N \otimes R$ in Section 8. Now $R \cong M_2 \otimes M_2 \otimes \dots$, where $M_2 = 2 \times 2$ matrices and the infinite tensor product is taken relative to the product state $\phi = \otimes_n \phi_n$, where each ϕ_n is the normalized trace on M_2 . It follows that $R \cong R \otimes R \otimes \dots$, where the infinite tensor product is taken relative to the infinite tensor product of the canonical trace on R (see [13], pp. 205-206). Hence we may identify N with $N \otimes R \otimes R \otimes \dots$. It follows from the definition of the infinite tensor product that $N \otimes R \otimes R \otimes \dots$ contains an increasing sequence $\{ N_k \}$ of subfactors such that

$$N_k \cong N \otimes \underbrace{R \otimes \dots \otimes R}_{k \text{ times}}$$

The relative commutant N_k^C of N_k in N is isomorphic to $R \otimes R \otimes \dots \cong R$.

Thus $N \otimes R \otimes R \otimes \dots = N_k \otimes N_k^C$. Also, $\{ N_k \}$ is an increasing sequence in $N \otimes R \otimes R \otimes \dots$ such that $\bigcup_k N_k$ is L^2 -dense, and $N_k \cong N \otimes \underbrace{R \otimes \dots \otimes R}_{k \text{ times}} \cong N$ for all k .

By the foregoing discussion, we may choose an increasing sequence $\{ M_k \}$ of subfactors of N with the following properties:

- (i) $M_k \cong N$, for all k .
- (ii) $\bigcup_k M_k$ is L^2 -dense in N .
- (iii) The relative commutant M_k^C of M_k in N is isomorphic to R , for all k .
- (iv) $N \cong M_k \otimes M_k^C$, for all k .

Now, let $x_1, \dots, x_n \in N$. By (ii), choose k and $x_1', \dots, x_n' \in M_k$ such that

$$\|x_j - x_j'\|_2 < \frac{\epsilon}{3}, \quad j = 1, \dots, n. \tag{10.1}$$

By (i) and (iii), there are isomorphisms

$$\theta_1: M_k \rightarrow N, \quad \theta_2: M_k^C \rightarrow R,$$

So we may extend $\theta_1 \otimes \theta_2$ to an isomorphism of $M_k \otimes M_k^C \rightarrow N \otimes R$ which preserves the trace (see the introduction), and hence the L^2 -norms. Thus, by identifying M_k with its image under the isomorphism $a \rightarrow \theta_1(a) \otimes 1_R$, $a \in M_k$, we may assume by (iv) that each x_j has the form $m_j \otimes 1_R$ in some factorization of $N = M \otimes R$, $M \cong N$. By Lemma 9.5, there exist $z_1, \dots, z_n \in R$ and a unitary $X \in M \otimes R$ such that

$$\|x_j - X(1_N \otimes z_j)X^*\|_2 < \frac{\varepsilon}{3}, \quad j = 1, \dots, n. \quad (10.2)$$

Since R is hyperfinite, there is a finite-dimensional subfactor Q of R and $q_1, \dots, q_n \in Q$ such that

$$\|z_j - q_j\|_2 < \frac{\varepsilon}{3}, \quad j = 1, \dots, n. \quad (10.3)$$

Then $X(\mathbb{C} \otimes Q)X^*$ is a finite-dimensional subfactor of $N = M \otimes R$, and by (10.1) - (10.3),

$$\|x_j - X(1_M \otimes q_j)X^*\|_2 < \varepsilon, \quad j = 1, \dots, n.$$

This verifies condition C for N .

QED

11. CLASSIFICATION OF INJECTIVE FACTORS OF TYPE II_∞ , III_λ , $\lambda \in [0,1)$.

With Theorem 8.1 and the fundamental structure Theorems 2.2, 2.3, and 2.4 now at our disposal, the precise structure of injective factors of type II_∞ and III_λ , $\lambda \in (0,1]$ can now be deduced in a fairly straightforward fashion.

Recall from Section 2 that $R_{0,1}$ is the hyperfinite II_∞ factor $R \otimes B(H)$, where H is a separable Hilbert space.

11.1. THEOREM ([10], Theorem 7.4). All separably acting injective factors of type II_∞ are isomorphic to $R_{0,1}$.

PROOF. Let M be an injective II_∞ factor acting on a separable Hilbert space H . By [4], Theorem IX, we can write $M \cong N \otimes B(H)$ for a separably acting II_1 factor N . M can thus be viewed as the algebra of $\infty \times \infty$ matrices

with entries in N . The map $F : (x_{ij}) \in M \rightarrow x_{11}$ defines a projection of norm 1 of M onto N , and composing this with a projection of $B(H \otimes H)$ onto M , we see that N is injective. By Theorem 8.1, N is isomorphic to R , whence M is isomorphic to $R_{0,1}$. QED

If $\{M_\alpha\}$ is an increasing family of injective von Neumann algebras, it follows by [14], Proposition 5.7 that the weak closure of $\bigcup_\alpha M_\alpha$ is injective. Since every finite-dimensional algebra is injective, we conclude that all hyperfinite von Neumann algebras are injective. Applying Theorem 11.1, we deduce

11.2. COROLLARY. All separably acting, hyperfinite, II_∞ factors are isomorphic.

11.3. THEOREM. ([10], Theorem 7.7). Let $\lambda \in (0,1)$. All separably acting injective factors of type III_λ are isomorphic to Powers factor R_λ .

PROOF. Let M be an injective III_λ factor acting on a separable Hilbert space with discrete decomposition $W^*(N,\theta)$ obtained via Theorem 2.3. We want to show that N is injective.

By Theorem 2.2 and [28], p. 303, we can find an abelian von Neumann algebra A and a one-parameter action α on $A \otimes N$ such that $M \cong W^*(A \otimes N, \alpha)$. By [28], Theorem 8.1, the fixed point algebra of $\hat{\alpha}$ on M is precisely $A \otimes N$. Hence, if μ is an invariant mean on $\ell^\infty(R)$, it is straightforward to verify that

$$(E(x)\xi, \eta) = \mu\{\hat{\alpha}_t(x)\xi, \eta\}, \quad x \in M, \quad \xi, \eta \in H.$$

defines a projection E of norm 1 of M onto $A \otimes N$. Since M is injective, we conclude that $A \otimes N$ is injective. Thus by [14], Proposition 5.6, N is injective.

We conclude by Theorem 11.1 that N is isomorphic to $R_{0,1}$. But by [9] Corollary 6 and [5], Théorème 4.4.1, R_λ is the only III_λ factor the II_∞ part of whose discrete decomposition is isomorphic to $R_{0,1}$. Hence M is isomorphic to R_λ . QED

11.4. THEOREM ([10], Theorem 7.5). If M is a separably acting injective

factor of type III_0 , then M is a Krieger factor.

PROOF. Let $W^*(N, \theta)$ be a discrete decomposition of M as in Theorem 2.4. As in the proof of Theorem 11.3, N is injective; however, it is not in general a factor. On the other hand, it is a direct integral of factors, and each factor appearing in its direct integral decomposition (except possibly for a set of measure 0) is injective by [10], Proposition 6.5 and of type II_∞ , and hence isomorphic to $R_{0,1}$. But these are precisely the hypotheses of [8], Theorem II.1, which asserts that under these conditions M is a Krieger factor QED

It follows from [30] that injective III_0 factors are classified up to isomorphism by ergodic non-transitive flows. For further details, the reader can consult [30], [10], and [37].

The only known injective factor of type III_1 is the factor R_∞ of Araki and Woods (see Section 2). It is not known whether this is the only possible one.

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