TRANSLATION PLANES OF DIMENSION TWO AND CHARACTERISTIC TWO

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ABSTRACT. This article discusses translation planes of dimension two and characteristic two. Let G be a subgroup of the linear translation complement of such a plane π. The nature of G and its possible action on π are investigated. This continues previous work of the authors. It is shown that no new groups occur.

1. INTRODUCTION

Part I. A translation plane of dimension two over GF(q) = F may be represented by a vector space of dimension four over F. The lines through the zero vector form a spread—a class of two-dimensional subspaces which are pairwise independent. The collineations fixing the zero vector (the translation complement) constitute a subgroup of $\Gamma L(4,q)$; the intersection with GL(4,q) is the linear translation complement.

Finite translation planes of dimension two have been studied more than those of other dimensions. A complete classification may not be possible if this is understood to include an enumeration of all existing planes. Yet we have reached the stage where we can give a reasonable list of classes when q is even.

Hering [6] has determined the possible groups generated by the elations in the translation complement and one can at least make general statements as to the nature

of the planes which may admit these groups, especially when the group is non-solvable.

The authors [8] have investigated the case where the translation complement is not solvable and contains no elations.

In the present paper, G denotes a subgroup of the linear translation complement. We investigate the nature of G and of the planes of dimension two and characteristic two in the following situations:

Part II. $|G|$ is odd.

Part III.   All involutions in G are elations, (G may be solvable).

Part IV.   All involutions in G are Baer involutions, (G may be solvable).

Part V.   G contains elations and Baer involutions. Again no restriction on solvability.

In all of the above, especially in Part V) little attention is paid to the case where G is both solvable and reducible. Some remarks are made about this situation in Part VI.

These results give no recipe for constructing all planes of dimension two and characteristic two. However, there are no new groups and at least some aspects of the possible actions are determined. We suspect that this is as fine a classification as is worthwhile trying to give.

2.   REDUCIBILITY, $|G|$ ODD.

DEFINITION 2.1. A group G of linear transformation acting irreducibly on a vector space V is said to be imprimitive if V is the direct sum of subspaces, to be called subspaces of imprimitivity, such that the image of each subspace of imprimitivity under each element of G is always a subspace of imprimitivity. Otherwise G is said to be primitive.

DEFINITION 2.2. A group G of linear transformations is said to be fixed-point-free (f.p.f.) if no non-identity element fixes any non-zero vector. A normal subgroup G_1 is said to be minimal non f.p.f. with respect to G if G_1 is not f.p.f. and every normal subgroup of G included in G_1 is either equal to G_1 or is f.p.f.

REMARK. The above two definitions require no restrictions on dimension or characteristic. We wish to remind the reader, however, that throughout this paper G is assumed to be a group of linear translations acting on a vector space of dimension 4 over GF(q), where q is a power of 2. Indeed G is assumed to be a subgroup of the linear translation complement of a translation plane of dimesion two defined on this vector space.

LEMMA 2.3. Let G be primitive and solvable and let $\mathfrak{A}$ be a maximal abelian normal subgroup. Assume that $\mathfrak{A}$ contains Z, where Z is the group of scalars. Then $\mathfrak{A} \neq Z$.

PROOF. The proof is by contradiction, so suppose $\mathfrak{A}$ = Z is the unique maximal abelian normal subgroup. Since G is irreducible, G $\neq$ Z. If G is solvable G has a normal subgroup A such that A/A $\cap$ Z is abelian and characteristic in G/Z. (Let A be the full pre-image of A/A $\cap$ Z.)

Suppose that A is f.p.f. Then A must be odd, since no involution is f.p.f. at characteristic 2. An f.p.f. group is a Frobenius complement. See [9] Lemma 2.2. A Frobenius complement which is not cyclic has a characteristic abelian group not in its center. See [10] Theorem 3.7. This contradicts the condition $\mathfrak{A}$ = Z. Hence A is not f.p.f.

Then A contains a subgroup G_1 which is minimal non-f.p.f. with respect to G. Note that A/A $\cap$ Z is abelian implies that A is nilpotent and hence that the Sylow subgroups of A are characteristic. Hence the minimal property of G_1 implies that G_1 must be a u=group for some prime u. If H is the maximal normal subgroup of G included in G_1 but not equal to G_1, then G_1/H is elementary abelian. (See [10], Theorem 3.3.)

If H is trivial, G_1 is abelian and by our assumption that Z is the unique maximal abelian normal subgroup, we could have $G_1 \subseteq$ Z. This cannot happen since Z is f.p.f.

Thus H is an f.p.f. u-group. Two-groups are not f.p.f. at characteristic two and the Sylow subgroups of odd order in a Frobenius complement are cyclic. Hence H is abelian and by our present assumeption, H $\subseteq$ Z.

This implies u|(q - 1) and that the minimal G_1-space have dimension 1. This last comes from the facts that $q^2 + 1 \equiv 2$ mod u implies that a u-group fixes at least two

components of the spread and $q + 1 \equiv 2 \mod u$ implies that a fixed 2-space must

include at least two fixed 1-spreads. Cliffords Theorem (see [3] page 70) implies

that if G is primitive, all of the minimal G_1-spaces must be isomorphic as G_1-modules

and hence that G_1 must be faithful on its minimal G_1-spaces. Since this does not

happen, we have a contradiction to the assumption that $\mathfrak{U} = Z$.

THEOREM 2.4. Let G be primitive and solvable. Then G is isomorphic to a sub-
group of $\Gamma L(1,q^4)$.

PROOF. Let $\mathfrak{U}$ be a maximal abelian normal subgroup of G such that $\mathfrak{U}$ contains

Z. By the previous Lemma, $\mathfrak{U} \neq Z$.

If $\mathfrak{U}$ is not f.p.f., it leaves invariant the subspace pointwise fixed by one of

its elements and hence is not faithful on its minimal subspaces. Thus Cliffords

Theorem implies that $\mathfrak{U}$ is f.p.f.

Then (see Hering [3] Hilfsatz 5) G is isomorphic to a subgroup of $\Gamma L(1,q^4)$ or

$\Gamma L(2,q^2)$ since G is primitive.

If $G \cong \Gamma L(1,q^4)$, we are done, so suppose G is a subgroup of $\Gamma L(2,q^2)$. We may

also assume $G \cap GL(2,q^2)$ centralizes $\mathfrak{U}$.

$G > G \cap SL(2,q^2)$. The solvable subgroups of $SL(2,q^2) = PSL(2,q^2)$ (for q even)

are cyclic or dihedral except in cases where there is a characteristic 2-group. G

has no normal 2-group if it is primitive so $G \cap SL(2,q^2)$ is cyclic or dihedral and

hence contains a characteristic abelian group $\mathfrak{U}_1$. Since $\mathfrak{U}$ is maximal and $\mathfrak{U}$ is

centralized by $\mathfrak{U}_1$, $\mathfrak{U}\mathfrak{U}_1$ is abelian so $\mathfrak{U}_1 \subseteq \mathfrak{U}$. But $\mathfrak{U}$ is actually included in the

center of $GL(2,q^2)$, so $\mathfrak{U} \cap SL(2,q^2)$ is trivial. Hence $G \cap SL(2,q^2)$ is trivial. At

characteristic 2, $GL(2,q^2)$ is the direct product of $SL(2,q^2)$ and the center of

$GL(2,q^2)$. Hence, in our case, $G \cap GL(2,q^2) = \mathfrak{U}$. In that case $G = G \cap \Gamma L(2,q^2)$ can-

not act irreducibly on the vector space of dimension 4 over GF(q). Thus the only

possibility left is G is isomorphic to a subgroup of $\Gamma L(1,q^4)$.

THEOREM 2.5. If G is irreducible but imprimitive then G has a reducible sug-
group of index 2.

PROOF. The theorem is trivial if the vector space is the direct sum of two

subspaces of imprimitivity.

Hence assume that the basic vector space V is the direct sum $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ of subspaces of imprimitivity of dimension 1. Let G_0 be the normal subgroup of G which fixes each of V_1, V_2, V_3, V_4.

By Clifford's Theorem, V is a direct sum $W_1 \oplus \cdots \oplus W_k$ of invariant G_0 spaces; the minimal G_0-spaces in W_i for fixed i are all isomorphic on G_0-modules and each minimal G_0-space belongs to one of the W_i's; the W_i all have the same dimension. Note that k = 1, 2, or 4. If k = 2 we are done.

If k = 4 then there are no minimal G_0-spaces besides V_1, V_2, V_3, V_4. $|G_0|$ cannot be even, since an involution in G_0 would have to fix V_1, V_2, V_3, V_4 pointwise.

If G_0 is odd, then G_0 must fix some other 1-space on the component of the spread which contains V_1. We conclude that if k = 4, there must be two components, each containing two of the V_i. In this case G has a subgroup of index two fixing these two components.

We have left the case where k = 1, so that all of the minimal spaces are isomorphic as G_0-modules. We may assume that G and G_0 contain all of the scalars. (If they do not, adjoin the scalars.) If the minimal G_0-spaces are isomorphic as G_0-modules, then G_0 must be faithful on its minimal subspaces. With the minimal subspaces being 1-dimensional, it follows that if G_0 contains the scalars 2, then G_0 = Z.

But G/G_0 is a subgroup of the symmetric group S_4 and its order is divisible by 4. $|S_4|$ = 24 and any subgroup whose order is divisible by 4 is one of the following: (a) a two-group, (b) A_4, (c) S_4. In each case there is a normal 2-group (not necessarily a Sylow 2-group).

Let $\overline{G} = G/Z = G/G_0$. Let S be a 2-group in G, where $\overline{S}$ is normal in $\overline{G}$. If σ is an involution in S and $\lambda \in G$ then $\lambda^{-1}\sigma\lambda = \sigma_1 \nu$ for some σ_1 in S and some $\nu \in Z$. Hence $1 = (\sigma_1\nu)^2 = \sigma_1^2\nu^2 = \nu^2$ and $\nu = 1$. Then the subgroup of S generated by its involutions is normal in G. Hence the subspace which is pointwise fixed by this 2-group is invariant under G and G is reducible, contrary to hypothesis. Hence the case k = 1 does not happen and the Theorem is proved.

THEOREM 2.6. Suppose that $\mathcal{O}(G)$ is irreducible. Then $G \stackrel{\sim}{\subseteq} \Gamma L(1,q^4)$.

PROOF. If $\mathcal{O}(G)$ is irreducible, it may be primitive by the previous Theorem. Then $\mathcal{O}(G) \stackrel{\sim}{\subseteq} L(1,q^4)$ by (2.4). Since G must preserve 1-spaces over GF(1),

$\mathbb{Q}(G) \cap GL(1,q^4)$ is a subgroup of $\mathbb{Q}(G) = \mathbb{Q}(G) \cap \Gamma L(1,q^4)$ whose index divides 4.
But $|\mathbb{Q}(G)|$ is odd, so $\mathbb{Q}(G) \overset{\sim}{\subseteq} GL(1,q^4)$ and $\mathbb{Q}(G)$ is cyclic. The theorem then
follows from Proposition 19.8 in Passman [13].

THEOREM 2.7. Suppose that $G = \mathbb{Q}(G)$ contains the scalars and that G is redu-
cible. Then G is cyclic or G is not faithful on one of its minimal subspaces.

PROOF. If G is reducible, it has some invariant 2-space V_1. If G is not faith-
ful on V_1 we are done. Otherwise G is a subgroup of $GL(2,q)$. The subgroups of
$PSL(2,q)$, $q = 2^r$ of odd order are cyclic. See Dickson [1]. But $PSL(2,q) = SL(2,q)$.
Thus $G \cap SL(2,q)$ is cyclic, with order dividing $q - 1$ or $q + 1$. As a subgroup of
$GL(2,q)$ acting on a vector space of dimension two, G must either be cyclic or have
a pair of invariant 1-spaces and not be faithful on at least one of them.

3. G GENERATED BY ELATIONS

In this section we are concerned with the case where the translation complement
contains elations. Actually, we will limit ourselves to the case where G is the sub-
group of the translation complement generated by the elations.

Hering 6 has investigated this situation. In one of his cases, G has a normal
subgroup of index two and odd order. In unpublished work he has shown that, under
much broader circumstances than we have here, G must be dihedral. Since this work
is not published, we will have a proof that G is dihedral for a plane of dimension
two over GF(q), where q is a power of 2.

LEMMA 3.1. Suppose that (a) G has a normal subgroup of index and odd order;
(b) G is primitive; (c) The involutions in G are elations; (d) G contains more
than one elation. Then G is dihedral.

PROOF. Let σ be an involution and τ an element of odd order such that $<\tau>$ has
index 2 in $<\sigma,\tau>$. Then the involutions have the form $\sigma\tau^i$ for those i such that
$\sigma\tau^i\sigma = \tau^{-i}$. Hence the group generated by the involutions is dihedral.

By (2.4) G is isomorphic to a subgroup of $\Gamma L(1,q^4)$. Then $G \cap GL(1,q^4)$ is a
cyclic normal subgroup of odd order. The fact that G is a group of linear trans-
formations over GF(q) implies that the index of $G \cap GL(1,q^4)$ in G must divide 4.
Because of (a) this index is 2. Hence (d) and the above argument implies that G is
dihedral since it is generated by its elations.

LEMMA 3.2. G is dihedral if G is reducible and the conditions other than (b) of (3.1) are satisfied.

PROOF. If there are two elations with the same axis, 4 divides $|G|$. If there are two elations with different axes, no component of the spread is invariant. If G has an invariant 1-space, the component containing this 1-space is invariant. Hence if G is reducible, it has an invariant subspace which is a Baer subplane π_0.

If G is faithful on π_0, the hypotheses are sufficient to guarantee that G is dihedral.

Otherwise, let G_1 be a maximal dihedral subgroup of G acting faithfully on π_0. Specifically, let $G_1 = \langle\sigma,\tau\rangle$ where ρ is an involution and $\langle\tau\rangle$ is the cyclic stem. If G is not dihedral, there is some involution σ such that $\langle\sigma,G_1\rangle$ is not dihedral and some element fixes π_0 pointwise.

Suppose that $\sigma\rho\tau^i$ fixes π_0 pointwise. Then the restrictions to π_0 of the two involutions σ and $\rho\tau^i$ are identical. In particulat, they fix the same points of π_0 so that the axes of the elations σ and $\rho\tau^i$ must be the same. But this requires that either $\sigma = \rho\tau^i$ or that G has a subgroup of order 4.

Neither of these conditions is satisfied. Suppose that $\sigma\tau^i$ fixes π_0 pointwise. Then the restrictions of σ and τ^{-i} to π_0 are identical, so τ^{-2i} must fix π_0 pointwise. But $G_1 = \langle\rho,\tau\rangle$ is faithful on π_0 so i = 0. In this case, σ fixes π_0 pointwise, contrary to the assumptions that σ is an elation.

LEMMA 3.3. Suppose that G is irreducible but imprimitive and the conditions of (3.1) excluding (b) are satisfied. Then G is dihedral.

PROOF. By (2.5) G has a reducible subgroup H of index 2. We may assume that $V = V_1 \oplus V_2$, where V_1 and V_2 have dimension 2, are interchanged by each involution G and some invariant under H.

Lemmas (2.3), (2.4), (2.5), (2.6) for a translation plane of dimension 2 over its kernel and a vector space of dimension 4 over GF(q) apply as well as to a vector space of dimension two, i.e., to a Desarguesian affine plane over GF(q).

Hence if H is irreducible and faithful on V_1 and has odd order, then H is cyclic as in (2.6).

If H is faithful on V_1 and is reducible on V_1 then it has at least two invariant

1-spaces and is a subgroup of the direct product of two cyclic groups. Hence H is

abelian. If H is f.p.f. and abelian, then H is cyclic.

Since H has two fixes 1-spaces in V_2, V must be the direct sum of 4 H-invariant

1-spaces. If H has an additional 1-space in V_1 (or V_2) then H fixes each 1-space in

V_1 (V_2). If H is not cyclic, it is not f.p.f. on V_1. Thus if H fixes each 1-space

in V_1, then H is not faithful on V_1.

If H has precisely 2 invariant 1-spaces in V_1 and precisely 2 invariant 1-spaces

in V_2 then therefore 1-spaces must occur in two orbits of length 2 under G. In this

case, the 2-space containing an invariant pair is invariant so G is reeucible and

we can return to the previous case.

We have left the case where H is not faithful on V_1 or V_2.

Let G_1 = <ρ,τ> be a maximal (dihedral) subgroup where <τ> $\subseteq$ H acts faithfully on

V_1 and ρ is an involution. If $G \neq G_1$ and G is generated by its involutions, there

is an involution σ such that <σ,G_1> contains an element which fixes V_1 pointwise.

Note that V_1 is not invariant under any element of the form $\sigma\tau^i$ or $\rho\tau^i$. If

$\sigma\rho\tau^i$ fixes V_1 pointwise, we get a contradiction just as in the proof of the previous

Lemma. We conclude that G must be dihedral.

THEOREM 3.4. Let G be a subgroup of the linear translation complement of a

translation plane of dimension 2, characteristic 2. Suppose that G is generated by

its elations. Then one of the following holds:

(a) G is a Suzuki group.

(b) $G \cong SL(2,2^s)$ for some s.

(c) G is elementary abelian and is a group of elations all with the same axis.

(d) G is dihedral.

PROOF. This is just a restatement of a Theorem of Hering [6, 3] except for

case (d). In Hering's remaining case, G has a subgroup of odd order and index 2.

If |H| is odd and [G:H] = 2 then G is dihedral so we get case (d) by applying (3.1),

(3.2), and (3.3).

REMARK. By Hering [7], Theorem (5.1) in case (b) of (3.4), G has an invariant

subplane which is a Lüneburg plane.

4. ALL INVOLUTIONS ARE BAER INVOLUTIONS

Here we are concerned with a subgroup of the linear translation complement which has even order but contains no elations. As pointed out in [8], the Sylow 2-groups are elementary abelian. The case where G is non-solvable is covered in [8], so we shall be mostly concerned with the solvable case.

LEMMA 4.1. Let G be a subgroup of the linear translation complement of a finite translation plane of even square order. Suppose that each Sylow 2-group fixes a Baer subplane pointwise and that there is more than one Sylow 2-group. Then no Sylow 2-group contains a non-trivial element which normalizes another Sylow 2-group.

PROOF. Suppose that S_1 and S_2 are Sylow 2-groups and that S_1 contains an involution ρ which normalizes S_2. Then ρ centralizes some involution σ in S_2. Hence $\langle \rho, \sigma \rangle$ is a group of order 4 contained in some Sylow 2-group S_3.

Then the Baer subplanes pointwise fixed by σ, ρ, S_1, S_2, S_3 must all be identical. By Foulser [2], Theorem 3, a Sylow 2-group of the pointwise stablizer of a Baer subplane is normal and hence unique (in the pointwise stabilizer). Hence S_1 and S_2 are not distinct.

LEMMA 4.2. If G is solvable and each Sylow 2-group fixes a Baer subplane pointwise, then either the Sylow 2-group is normal or $|G|$ is not divisible by 4.

PROOF. See Hering [5]. Theorem 1. If S is an elementary abelian 2-group and is a Frobenius complement then $|S| = 2$.

REMARK. Lemma (4.1) does not require that the plane have dimension 2 over its kernel. Foulser's Theorem 3 [2] implies that the Sylow 2-group must be elementary abelian in this case. Hering's Theorem 1 then implies that if G is non-solvable and generated by its Sylow 2-group it must be isomorphic to $SL(2,2^s)$ for some s.

LEMMA 4.3. If G contains no elations, the Sylow 2-groups are elementary abelian. If, in addition, G is solvable and generated by its involutions, then $G/\mathfrak{G}(G)$ is a Sylow 2-group.

PROOF. A Sylow 2-group must have at least one fixed component. If it contains no elations, it acts faithfully on a 2-space and is a subgroup of a Sylow 2-group of $GL(2,q)$. Hence the Sylow 2-groups are elementary abelian.

The last part follows from Walter's characterization of groups with abelian Sylow 2-groups [16].

LEMMA 4.4. If G is irreducible and solvable, the Sylow 2-groups have order 2 or 4. In the latter case G interchanges a pair of two dimensional subspaces and the collineation group induced on them by their stablizer is dihedral.

PROOF. If G is primitive, then $G \cong \Gamma L(1,q^4)$ by (2.4). But G is acting as a group of linear transformations over $GF(q)$. When G is represented as a subgroup of $\Gamma L(1,q^4)$, the semi-linear transformations over $GF(q^4)$ must be those involving automorphisms of $GF(q^4)$ which fix the elements of $GF(q)$. The multiplicative group of $GF(q^4)$ is odd—i.e., $GL(1,q^4)$ has odd order. Thus the Sylow 2-groups are cyclic. By (4.3), they are elementary abelian. Hence the Lemma is proved for the case where G is primitive, since the Sylow 2-groups must have order 2.

By (2.5) if G is irreducible but imprimitive then G has a reducible subgroup G_0 of index w and G_0 has a pair of invariant 2-spaces V_1 and V_2. The subgroup fixing V_1 (or V_2) pointwise must have odd order since an involution fixing V_1 pointwise would have to act f.p.f. on V_2 and an involution cannot act f.p.f. on an invariant space at characteristic two.

Suppose that G_0 has a subgroup of order 4 fixing pointwise a 1-space in V_1. (It will also fix pointwise a subspace of V_2.) If this subspace is invariant under G_0, then G_0 will have a characteristic 2-group; the subspace fixed pointwise by this subgroup will be invariant under G so G is reducible, contrary to hypotheses.

Otherwise G_0 must induce a dihedral group on V_1, since G_0 is solvable. Hence the Sylow 2-groups in G_0 have order 2; those in G have order 4.

LEMMA 4.5. Suppose that G is reducible and solvable and that the Sylow 2-groups have order >2. Then G has an invariant subspace pointwise fixed by a 2-group. If G is not a 2-group, either it contains a group of affine homologies or its minimal invariant subspaces have dimension 2 and the induced group is dihedral.

PROOF. If G is reducible, it has an invariant 2-space since if its minimal space is 1-dimensional, G must fix the component which contains this 1-space.

Let V_1 be an invariant 2-dimensional subspace. If the subgroup fixing V_1 pointwise has even order, we are done.

If H is the subgroup fixing V_1 pointwise and $|H|$ is odd, consider the induced group G/V_1. If the 2-group in G/V_1 has order greater than 2, then G/V_1 has an invariant subspace in V_1, since G is assumed to be solvable. This subspace, as a vector space in the whole plane, will be invariant under G and pointwise fixed by a Sylow 2-group.

If a minimal G-space is 1-dimensional and G is not a 2-group then G cannot act faithfully on the component which contains this 1-space. Hence G contains a group of homologies. (In this section G contains no elations.)

If the minimal G-spaces have dimension 2 and G is not a 2-group but is solvable, the induced group must be dihedral.

THEOREM 4.6. Let G be a nontrivial subgroup of the linear translation comple-ment for a plane π of dimension 2 over GF(q), where q is even. Suppose that G is generated by its involutions and they are all Baer involutions. Then at least one of the following holds:

(a) $G = SL(2,2^S)$ for some power of s, π has an Ott subplane of order 2^S on
     which G acts in the normal manner. G acts irreducibly on π.

(b) $G = SL(2,2^S)$. G is reducible. Each Sylow 2-group fixes a Baer subplane
     pointwise. All of these Baer subplanes are included in a derivable net.
     In the derived plane all of the involutions are elations.

(c) $\mathfrak{G}(G)$ has index 2 in G and G is dihedral.

(d) The Sylow 2-groups in G have order 4. G interchanges a pair of 2-dimen-
     sional subspaces and the group induced on them be their stabilizer is
     dihedral.

(e) G is reducible and one of its invariant subspaces is pointwise fixed by a
     2-group. If G is not a 2-group either it contains affine homologies or
     the minimal G spaces have dimension 2 and G induces a dihedral group on
     its minimal space.

PROOF. If G is non-solvable, we have (a) or (b) by [8]. If G is solvable and each Sylow 2-group fixes a Baer subplane pointwise apply (4.2). In the remaining cases, apply (4.4) and (4.5).

5. GROUPS GENERATED BY INVOLUTIONS

In the last two sections, we have assumed that the involutions in the linear translation complement are all of the same kind—i.e., either they are all elations or all Baer involutions. As before G always denotes a subgroup of the linear translation complement.

LEMMA 5.1. If G contains both elations and Baer involutions then the group generated by the Baer involutions contains elations.

PROOF. If G contains involutions of both kinds, let G_1 be the normal subgroup generated by all of the elations in G and let G_2 be the normal subgroup generated by all of the Baer involutions in G.

Let S be a Sylow 2-group and let $S_1 = S \cap G_1$, $S_2 = S \cap G_2$. By Hering's results [6], G_1 contains no Baer involutions. If G_2 contains no elations, then $S_1 S_2 = S_1 \times S_2$. Furthermore S_1 and S_2 are both elementary abelian. See (4.3) and again refer to Hering [6]. It follows that $S_1 \times S_2$ is elementary abelian and contains involutions which do not belong to either S_1 or S_2 (unless one of them is trivial). But every involution belongs to G_1 or G_2 so every involution in S belongs to S_1 or S_2. Hence G_2 must contain elations.

DEFINITION 5.2. In the rest of this section G_1 is the group generated by the elations; $\mathcal{C}(G_1)$ is the centralizer of G_1 in G and H is the subgroup of $\mathcal{C}(G_1)$ generated by the involutions.

LEMMA 5.3. The center of G_1 is trivial unless G_1 is a 2-group. Thus $G_1 \mathcal{C}(G_1) = G_1 \times \mathcal{C}(G_1)$

PROOF. By (3.4) G_1 is a Suzuki group, $SL(2,2^s)$, dihedral, or a 2-group. The two non-solvable cases are simple. A dihedral group with cyclic stem of odd order has a trivial center.

LEMMA 5.4. Suppose that G_1 is not trivial, that $\mathcal{C}(G_1)$ contains involutions and neither G_1 nor $\mathcal{C}(G_1)$ has a non-trivial normal 2-group. Then the axes of the elations in G_1 can be embedded in a derivable net which contains the Baer subplanes pointwise fixed by involutions in $\mathcal{C}(G_1)$. If $\mathcal{C}(G_1)$ is non-solvable, this derivable net is embedded in the given translation plane.

PROOF. If $C(G_1)$ is non-solvable this follows from (2.7) in [8], since $C(G_1)$ must be reducible. Each axis of an elation in G_1 is invariant under $C(G_1)$. If G_1 has no normal 2-group, there are at least three such axes. Hence the non f.p.f. elements in $C(G_1)$, including the involutions, must fix Baer subplanes pointwise. The Lemma is now a direct consequence of the classical theory concerning a regular determined by three skew projective lines, etc.

In the rest of this section, the subgroup G_1 is assumed to be non-trivial.

LEMMA 5.5. If G_1 is not a 2-group, $C(G_1)$ is isomorphic to a subgroup of $GL(2,q)$.

PROOF. By (5.2), $G_1 \cap C(G_1)$ is trivial, so $C(G_1)$ contains no homologies. If $C(G_1)$ contains an affine homology whose axis ℓ is the axis of an elation in G_1 then ℓ must be invariant under G_1 and all of the elations in G_1 fix ℓ. Hence G_1 is a 2-group, contrary to hypotheses.

Thus, $C(G_1)$ must be faithful on each elation axis. This implies that $(G_1) \widetilde{\subseteq} GL(2,q)$.

LEMMA 5.6. If G_1 is not a two-group, $G/G_1 C(G_1)$ is abelian. If H is not a 2-group (or trivial) $G/H C(H)$ is abelian.

PROOF. If $\lambda \in G$, the mapping $G_1 \to \lambda^{-1} G_1 \lambda$ is an inner automorphism of G_1 if and only if belongs to $G_1 C(G_1)$. Hence $G/G_1 C(G_1)$ is isomorphic to a subgroup of the group of outer automorphisms of G_1. By (3.4), G_1 is dihedral, a Suzuki group, or $SL(2,2^s)$ for some s.

If G_1 is dihedral, the cyclic stem has odd order. Let $G_1 = \langle \rho, \tau \rangle$ where ρ is an involution and $\langle \tau \rangle$ is the cyclic stem. All of the involutions in G_1 are equivalent under the inner automorphisms so the outer automorphisms fix ρ and induce automorphisms of $\langle \tau \rangle$. The automorphism group of a cyclic group is abelian, so the automorphism group of G_1 is abelian in this case.

The outer automorphisms of $SL(2,2^s)$ are essentially the field automorphisms; this group is cyclic.

According to Suzuki [15], Theorem 11, page 139, the same remark holds for the

H is normal in G; if H is not a 2-group then H is dihedral or $SL(2,2^s)$ for some

s since H is a subgroup of $GL(2,q)$ generated by involutions. (Note that $H \cap G_1$ must

be trivial unless H and G_1 are both 2-groups.) Thus the arguments just used also

apply to H.

THEOREM 5.7. Except in the cases where H is trivial and G_1 is dihedral or where

G has a normal 2-group, $G_1 \, \mathcal{C}(G_1)$ contains the group generated by the involutions in

G and if G_1 is not a Suzuki group $G = G_1 \, \mathcal{C}(G_1)$.

PROOF. As in the proof of Lemma (5.6), $G/G_1 \, \mathcal{C}(G_1)$ is isomorphic to a group of

outer automorphisms of G_1.

Thus the first part of the Theorem is true unless G contains some involution

not in $G_1 \, \mathcal{C}(G_1)$ which induces an outer automorphism of G_1.

If $G_1 = Sz(2^s)$ then s must be odd. The Theorem of Suzuki referred to before

implies that the outer automorphism group has odd order.

If $G_1 = SL(2,2^s)$ then the net whose components are elation axes is Desarguesian.

This net is invariant under G.

By Ostrom [11], the induced permutation group on the affine elation centers is

isomorphic to a subgroup of $P\Gamma L(2,2^s)$ for some s.

If $|G_1|$ and $|\mathcal{C}(G_1)|$ are both even we can apply (5.4) to infer that the elation

axes are embedded in a regular (derivable) net invariant under G. (We do not need

the condition that all components of this derivable net are components of the spread

which defines the plane.) Again the induced group on the affine elation centers is

a subgroup of $P\Gamma L(2,2^s)$. An element of G fixes all of the elation centers if and

only if it is in $\mathcal{C}(G_1)$. Thus if G_1 is not a Suzuki group $G/\mathcal{C}(G_1)$ is a subgroup of

$P\Gamma L(2,2^s)$. Let $\widetilde{G} = G/\mathcal{C}(G_1)$.

Furthermore if $G_1 = SL(2,2^s)$, $GF(2^s)$ must be a subfield of $GF(q^2)$. If $2^s = q$

the plane must be Desarguesian. See Prohaska [14]. If the plane is not Desarguesian,

$GF(2^s)$ will be a proper subfield of $GF(q)$. This will also be the case where H is non-

trivial and G_1 is dihedral by (5.4). Since we are looking at linear transformations

over $GF(q)$, it follows that an element of G which fixes three elation centers must

fix all of them and thus be in $\mathcal{C}(G_1)$. Hence $\widetilde{G}$ is in $PGL(2,2^s) = PSL(2,2^s)$, not

merely in $P\Gamma L(2,2^s)$.

If $\widetilde{G}$ should have a normal 2-group, so would G. Recall that $G_1 \cap C(G_1)$ is trivial, so $\widetilde{G}$ has a normal subgroup isomorphic to G_1. From Dickson's [1] list of subgroups of $PSL(2,2^s)$, $\widetilde{G}$ is dihedral or $PSL(2,2^t)$ for some t. Hence if G_1 is $SL(2,2^s)$, $\widetilde{G}$ is $SL(2,2^s)$; if G_1 is dihedral, $\widetilde{G}$ is dihedral and the cyclic stem must have odd order since $\widetilde{G}$ can have no normal 2-group. Hence the involutions in $\widetilde{G}$ are all conjugate. Hence they must all be in G_1. Hence $\widetilde{G} = G_1$.

COROLLARY 5.8. If the group generated by the elations is A Suzuki group, G contains no Baer involutions.

PROOF. By (5.7) $G = G_1 C(G_1) = G_1 \times C(G_1)$ where G_1 is, in this case a Suzuki group. If G contains Baer involutions, then $C(G_1)$ will contain Baer involutions. This implies that G_1 induces a Suzuki group on a Desarguesian plane of order q; that is, on a Baer subplane pointwise fixed by some element of $C(G_1)$.

LEMMA 5.9. Suppose that π has a derivable net $\mathcal{H}_1$. Let $\mathcal{H}_2$ be the net whose components are the Baer subplanes in $\mathcal{H}_1$. ($\mathcal{H}_1$ and $\mathcal{H}_2$ correspond to a regulus and its opposite regulus.) Then G cannot contain 2-group Q_1 and Q_2 with the following properties:

   (a) Q_1 and Q_2 centralize each other.

   (b) $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant under $Q_1 Q_2$.

Q_1 consists of elations with a given axis ℓ in $\mathcal{H}_1$; Q_2 fixes pointwise some Baer subplane $V(Q_2)$ of $\mathcal{H}_1$ (line of $\mathcal{H}_2$).

   (c) $Q_1 = Q_2 = q_0 > 2$. Q_1 has an invariant subplane of order q_0 embedded in $V(Q_2)$. In its action on $\mathcal{H}_2$, Q_2 has an invariant subplane of order q_0 embedded in ℓ, where ℓ now is a Baer subplane of $\mathcal{H}_2$.

We can choose a representation so that the components of $\mathcal{H}_1$ are $x = 0$ and $y = xW$, where W runs over the various scalar matrices over GF(q). Furthermore, we can choose a basis so that Q_1 is represented by the various matrices of the form

$$\begin{pmatrix} 1 & 0 & \beta & 0 \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\beta \in GF(q_0)$ and $V(Q_2)$ is the set of points (x_1, x_2, y_1, y_2) such that $x_2 = y_2 = 0$. This representation for Q_1 is unchanged under conjugation with respect to any matrix

$$\begin{pmatrix} A & \mathbb{0} \\ \mathbb{0} & A \end{pmatrix} \quad \text{where A is 2 by 2 over GF(q).}$$

Hence Q_2 can be represented by matrices of the form

$$\begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where   varies over $GF(q_0)$.

Let $\mathcal{M} = \{M \mid y = xM$ is a component of $\pi\}$. Note that if $M \in \mathcal{M}$ then $M + I\beta$ and

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \qquad \text{both belong to } \mathcal{M}. \text{ Let } M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \quad \text{and suppose that}$$

$m_3 \in GF(q_0)$. Then

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m_1 + \alpha m_3 \\ m_3 \end{pmatrix} \begin{pmatrix} m_1 + \alpha^2 m_3 + m_2 + \alpha m_4 \\ m_3 \alpha + m_4 \end{pmatrix}$$

and $M + \alpha m_3 I = \begin{pmatrix} m_1 + \alpha m_3 \\ m_3 \end{pmatrix} \begin{pmatrix} m_2 \\ m_3 \alpha + m_4 \end{pmatrix}$.

But the difference of distinct matrices in $\mathcal{M}$ must be non-singular. Thus if matrices in $\mathcal{M}$ have identical first columns (second rows) they must be identical. Hence $m_2 = m_1 + {}^2 m_3 + m_2 + m_4$ for all α in $GF(q_0)$. Thus $\alpha[(m_1 + m_4) + \alpha m_3] = 0$. This can only happen if $m_3 = 0$, $m_1 = 4$. But if the q^2 distinct matrices in $\mathcal{M}$ have distinct second rows every ordered pair of elements of $GF(q)$ must appear as the second row for some matrix in $\mathcal{M}$. In particular, there must be matrices in which m_3 belongs to $GF(q_0)$ but is not equal to zero. We have a contradiction which establishes the Lemma.

REMARK. A similar proof works for odd characteristic.

THEOREM 5.10. If G_1 and H (see (5.2)) are both non-solvable, then $G_1 = SL(2,q)$, $G_2 = SL(2,q_2)$ where $GF(q_1)$ and $GF(q_2)$ are subfields of $GF(q)$ such that $GF(q_1) \cap GF(q_2) = GF(2)$.

PROOF. By (3.4), (4.6) and (5.8) we have only to prove that $GF(q_1) \cap GF(q_2)$ cannot contain a subfield of order greater than 2. If $GF(q_1) \cap GF(q_2) = GF(q_0)$, where $q_0 > 2$ we can apply Lemma (5.9) to get a contradiction.

6. CONCLUSION

We have tended to ignore subgroups of the translation complement which are both solvable and reducible or even solvable and irreducible but imprimitive. We have

generally assumed that there is no normal 2-group. Here the subspace pointwise fixed by the normal 2-group is invariant.

In all cases the induced group (the factor group modulo the subgroup fixing a minimal invariant subspace pointwise) is cyclic or dihedral in the solvable case. Thus the group theoretic structure is relatively uncomplicated. Yet "most" of the known planes of even order and dimension two are in this class. This includes most semi-field planes, generalized Hall planes and generalized André planes. We would conjecture that there are many more planes with an invariant component or Baer subplane, and we will not attempt to enumerate all of the possibilities.

Our investigations have led to two interesting possibilities for which we do not know whether or not examples exist. The first is described in (4.4); the second in (5.10).

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ADDED IN PROOF. Several people have called our attention to a paper by B. Mwene "On the subgroups of the group $PSL(4,2^m)$" <u>J. Alg.</u> <u>41</u> (1976), 79. The arguments in [8] could have been simplified if we had used Mwene's results. They probably also could be used in the present paper.