Internat. J. Math. & Math. Sci.
Vol. 6 No. 3(1983) 511-519

KNOTS WITH PROPERTY R+

BRADD EVANS CLARK

Department of Mathematics and Statistics
University of Southwestern Louisiana
Lafayette, Louisiana 70504 USA

(Received October 29, 1982)

ABSTRACT. If we consider the set of manifolds that can be obtained by surgery on a fixed knot K, then we have an associated set of numbers corresponding to the Heegaard genus of these manifolds. It is known that there is an upper bound to this set of numbers. A knot K is said to have Property R+ if longitudinal surgery yields a manifold of highest possible Heegaard genus among those obtainable by surgery on K. In this paper we show that torus knots, 2-bridge knots, and knots which are the connected sum of arbitrarily many (2,m)-torus knots have Property R+ . It is shown that if K is constructed from the tangles $(B_1,t_1),(B_2,t_2),\dots,(B_n,t_n)$, then $T(K) \leq 1 + \sum_{i=1}^{n} T(B_i,t_i)$ where T(K) is the tunnel of K and $T(B_i,t_i)$ is the tunnel number of the tangle (B_i,t_i). We show that there exist prime knots of arbitrarily high tunnel number that have Property R+ and that manifolds of arbitrarily high Heegaard genus can be obtained by surgery on prime knots.

KEY WORDS AND PHRASES. Knot, surgery, Heegaard genus, tangle.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 57M25; 57N10.

1. INTRODUCTION.

A traditional method of constructing 3-manifolds is to perform Dehn surgery on a knot or link in S^3. As a result of this relationship between knots and 3-manifolds, several negatively defined properties of knots, namely Property P and Property R, have been studied. The purpose of this paper is to introduce a positively stated property, Property R+ . This property is clearly related to Property R in that it is at least as strong and generally stronger than Property R. If the knot K should have Property R+ it would mean that trivial surgery and longitudinal surgery yield respectively the least complex and the most complex 3-manifolds obtainable by surgery on K in terms of Heegaard genus. We shall demonstrate that infinitely many knots have Property R+ .

If X is a point set, we shall use cl(X) for the closure of X, int(X) for the interior of X and ∂X for the boundary of X. If K is a cube-with-knotted hole, the longitudinal curve of K will be the simple closed curve on ∂K, unique up to isotopy, which bounds an orientable surface in K. The genus of a 3-manifold is defined to be the minimal genus of a Heegaard splitting of the manifold. If X is a polyhedron contained in the P. L. 3-manifold M, then $N(X) \subset M$ is called a regular neighborhood of X in M if $X \subset N(X)$ and N(X) is a 3-manifold which can be simplicially collapsed to X. This paper deals with P. L. topology. Therefore, all manifolds in this paper are assumed to be simplicial and all maps to be piecewise linear.

2. PROPERTY R+ .

Let $K \subset S^3$ be a knot. If we consider the set of manifolds that can be obtained by surgery on K, then of course we will have a set of numbers corresponding to the Heegaard genus of these manifolds. We know by [2] that there is an upper bound to this set of numbers.

DEFINITION. K is said to have Property R+ if and only if longitudinal surgery on K yields a manifold of maximal Heegaard genus among those that can be obtained by surgery on K.

PROPOSITION. All two bridge knots, all torus knots, and all knots which are the connected sum of arbitrarily many (2, m)-torus knots have Property R+ .

PROOF. We know by [10] that all two bridge knots have Property R. Hence by [2] these knots have Property R+. As for torus knots, we know by [8] that all torus knots have Property R. Thus by [1] we know that torus knots have Property R+ . Finally, we know by [3] that the connected sum of arbitrarily many (2, m)-torus knots will be a knot which also has Property R+ .

While it is true that a knot of arbitrarily high complexity can be fashioned from the connected sum of (2, m)-torus knots, it would be more interesting to find a collection of prime knots of arbitrarily high complexity that also have Property R+. In order to find such a collection of knots we must first consider the theory of tangles and develop a concept of tunnel number for tangles.

3. THE TUNNEL NUMBER OF TANGLES.

The original concept of a tangle was developed by Conway in [4]. We shall use the somewhat modified definition of a tangle as given by Kirby and Lickorish in [6].

DEFINITION. A tangle is a pair (B, t) where B is a 3-cell and t is a pair of disjoint arcs in B such that $t \cap \partial B = \partial t$. The tangles (B_1, t_1) and (B_2, t_2) are said to be equivalent if there is a homeomorphism of pairs between (B_1, t_1) and (B_2, t_2). A tangle is trivial if it is equivalent to $(D \times I, \{x, y\} \times I)$ where D is a disk with $\{x, y\} \subset \text{int } D$. An arc $A \subset \text{int } B$ with $A \cap t = \partial A$ will be called a tunnel.

If t_1 and t_2 are the two arcs making up $t \subset B$, we note that up to isotopy there are only two ways of adding disjoint arcs a_1 and a_2 with $a_1 \cup a_2 \subset \partial B$ and each arc connecting t_1 to t_2. Obviously, $t_1 \cup a_1 \cup t_2 \cup a_2$ will be a knot $K \subset S^3$. It is then possible to add a set of pairwise disjoint tunnels $\{A_1, A_2, \ldots, A_n\}$ so that:

(i) $H = N(K \cup A_1 \cup A_2 \cup \ldots \cup A_n)$ bounds $n + 1$ pairwise disjoint disks $\{D_1, D_2, \ldots, D_{n+1}\}$ with $\text{int } D_i \subset \text{int } B$;

(ii) $N(H \cup D_1 \cup D_2 \cup \ldots \cup D_{n+1})$ is a spanning 3-cell in B that contains a spanning unknotted arc.

To see that this is always possible, one merely needs to look at a regular projection of (B, t) and add a tunnel at each double point of the projection.

The tunnel number of K relative to (B, t), $T(K, (B, t))$, is the smallest number of tunnels that need to be added to K in order to satisfy both (i) and (ii) above. It should be noted that in general $T(K, (B, t))$ will be a larger number than the tunnel number $T(K)$ as defined in [2]. Let K_1 and K_2 be the two knots that can be formed by adding arcs a_1 and a_2 in ∂B to $t \subset B$. As a rule $T(K_1, (B, t))$ and $T(K_2, (B, t))$ will be different numbers. For example, consider the tangle (B, t) in Figure 1. One way to complete t to a knot yields the square knot K_1. Since the square knot has the same fundamental group as a granny knot, we know by [3] that K_1 has tunnel number 2. Hence $T(K_1, (B, t))$ is also 2. The other completion of t, K_2, is the twist knot 6_1. Since 6_1 is a 2-bridge knot, 6_1 has tunnel number 1 and hence $T(K_2, (B, t)) = 1$.

DEFINITION. $T((B, t)) = \max \{T(K_1, (B, t)), T(K_2, (B, t))\}$ where $T((B, t))$ is the tunnel number of the tangle (B, t).

If (B_1, t_1) and (B_2, t_2) are tangles, it is possible to create a new tangle called the partial sum of (B_1, t_1) and (B_2, t_2) by identifying a (disk, point pair) in the boundary of one tangle with a (disk, point pair) in the boundary of the other tangle. Any of the different ways that this can be accomplished will be denoted $(B_1, t_1) + (B_2, t_2)$. If (S^3, K) is the result of identifying $\partial(B_1, t_1)$ to $\partial(B_2, t_2)$ by a homeomorphism h, the result will be denoted as $(B_1, t_1) \cup_h (B_2, t_2)$. Therefore, if $(B_1, t_1), (B_2, t_2), \ldots, (B_n, t_n)$ are tangles, then one can write $(S^3, K) = (\sum_{i=1}^{n-1} (B_i, t_i)) \cup_h (B_n, t_n)$ where $K \subset S^3$ is any one of the infinitely many knots that can be created in this fashion from the tangles $(B_1, t_1), (B_2, t_2), \ldots, (B_n, t_n)$.

THEOREM. If $(S^3, K) = (\sum_{i=1}^{n-1} (B_i, t_i)) \cup_h (B_n, t_n)$ then $T(K) \leq 1 + \sum_{i=1}^{n} T((B_i, t_i))$.

PROOF. Within each 3-cell B_i we add the appropriate number of tunnels so that we can find unknotted spanning arcs α_{1i} and α_{2i} (possibly not disjoint) with

$\partial(a_{1i} \cup a_{2i}) = \partial t_i$ and $a_{1i} \cup a_{2i} \subset t_i \cup A_1 \cup A_2 \cup \ldots \cup A_n$. Thus $(B_i, a_{1i} \cup a_{2i})$ is equivalent to a trivial (possibly pinched) tangle. Hence both $(B_n, a_{1n} \cup a_{2n})$ and $(\sum_{i=1}^{n-1} (B_i, a_{1i} \cup a_{2i}))$ are trivial tangles, and $(S^3, \bigcup_{i=1}^{n} (a_{1i} \cup a_{2i}))$ may be at most a 2-bridge knot pair. Therefore, the addition of at most one more tunnel A, will yield a 1-complex C such that both N(C) and $cl(S^3 - N(C))$ are handlebodies.

COROLLARY. If K is obtained by adding the tangles $(B_1, t_1), (B_2, t_2), \ldots, (B_n, t_n)$ and M is obtained by Dehn surgery on K, then $H(M) \leq 2 + \sum_{i=1}^{n} T((B_i, t_i))$ where H(M) is the Heegaard genus of M.

We note that the formula given in the above theorem is the best possible such formula. If both (B_1, t_1) and (B_2, t_2) are trivial tangles, then $T(B_1, t_1) = T(B_2, t_2) = 0$. Yet it is possible to construct a knot K from $t_1 \cup t_2$ with $T(K) = 1$. On the other hand the square knot K can be obtained from the tangle (B, t) in Figure 1 by adding a trivial tangle. But $T(K) = T((B, t))$.

4. A COLLECTION OF PRIME KNOTS WITH PROPERTY R+ .

In [6], Kirby and Lickorish pointed out a special class of tangles which they called prime tangles. The tangle (B, t) is said to be prime if and only if every 2-sphere in B which meets t transversely in two points, bounds in B a 3-cell meeting t in an unknotted spanning arc and no properly embedded disk in B separates the arcs of t.

In [7] Lickorish proved that if $(S^3, K) = (\sum_{i=1}^{n-1} (B_i, t_i)) \cup_h (B_n, t_n)$ where $n \geq 2$ and (B_i, t_i) is a prime tangle for $1 \leq i \leq n$, then K is a prime knot. He also demonstrated that the tangle shown in Figure 1 is a prime tangle. We shall be using these results in the formation of our collection of prime knots.

Let K_n be the knot formed from n prime tangles as shown in Figure 2. As we've seen before, K_1 is the prime knot 6_1. K_n for $n \geq 2$ satisfies the hypothesis of Lickorish's theorem and hence is also a prime knot. As we saw in Section 3, each prime tangle used in the construction of K_n has tunnel number 2. Therefore,

$T(K_n) \leq 2n + 1$. The half twist at the top of K_n in Figure 2 yields a knot which needs only one tunnel in its top tangle and no additional tunnel from the addition of the trivial tangle to complete (S^3, K_n). Therefore $T(K_n) \leq 2n - 1$. The placement of these $2n - 1$ tunnels is shown in Figure 3. If A_i is the i-th tunnel added to K_n, then clearly $N(K_n \cup A_1 \cup A_2 \cup \ldots \cup A_{2n-1})$ is a handlebody with $cl(S^3 - N(K_n \cup A_1 \cup A_2 \cup \ldots \cup A_{2n-1}))$ also a handlebody.

A Wirtinger presentation of the fundamental group of $S^3 - K_n$ is as follows: $\{a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2, \ldots, a_n, b_n, c_n, d_n, e_n, f_n, a_{n+1}, b_{n+1} \mid R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}, R_{21}, R_{22}, R_{23}, R_{24}, R_{25}, R_{26}, \ldots, R_{n1}, R_{n2}, R_{n3}, R_{n4}, R_{n5}, R_{n6}, a_1 b_{n+1}^{-1}\}$, where $R_{i1} = a_i d_i a_i^{-1} f_i^{-1}$, $R_{i2} = d_i a_i d_i^{-1} c_i^{-1}$, $R_{i3} = c_i b_i c_i^{-1} d_i^{-1}$, $R_{i4} = f_i a_{i+1} e_i^{-1} a_{i+1}^{-1}$, $R_{i5} = c_i e_i a_{i+1}^{-1} e_i^{-1}$, and $R_{i6} = e_i c_i b_{i+1}^{-1} c_i^{-1}$ for $1 \leq i \leq n$. This presentation can be simplified by use of Tietze transformations to: $\{d_1, a_2, d_2, a_3, d_3, \ldots, a_n, d_n, a_{n+1} \mid \tilde{R}_{11}, \tilde{R}_{12}, \tilde{R}_{21}, \tilde{R}_{22}, \ldots, \tilde{R}_{n-11}, \tilde{R}_{n-12}, \tilde{R}_{n1}\}$ where

$$\tilde{R}_{i1} = d_i a_i d_i^{-1} a_{i+1}^{-1} a_i d_i a_i^{-1} a_{i+1}^{-1} a_i d_i^{-1} a_i^{-1} a_{i+1} \quad \text{for } 1 \leq i \leq n$$

$$\tilde{R}_{i2} = a_{i+1}^{-1} = d_n a_n^{-1} d_n^{-1} a_{n+1}^{-1} a_n d_n a_n^{-1} a_{n+1} d_n a_n d_n^{-1} .$$

Using the free calculus as developed in [5] we calculate the elementary ideals of K_n. For K_1 we find that $E_1 = (2t^2 - 5t + 2)$, and $E_m = (1)$ for $m \geq 2$. For K_n we find $E_{2n-1} = (t-2, 2t-1)$, and $E_m = (1)$ for $m \geq 2n$. Therefore $\pi_1(cl(S^3 - N(K_n))$ is a $2n$ generator group and hence $T(K_n) = 2n - 1$.

As a result of [2], it is clear that $T(K) < br(K)$. For example, all torus knots have tunnel number 1 although there exist torus knots of arbitrarily high bridge number. Thus the tunnel number of a knot is a more strict measure of the complexity of a knot than is the bridge number.

THEOREM. There exist prime knots of arbitrarily high tunnel number that have Property R+ .

PROOF. Let M_n denote the manifold obtained by longitudinal surgery on K_n.

The infinite cyclic covering of M_n has the same $Z[t, t^{-1}]$ module-structure as the infinite cyclic cover of the K_n knot complement. This follows directly from the method of construction of such covers as demonstrated in [9]. Hence all of the Alexander invariants of M_n and the K_n knot complement are identical. Hence the Heegaard genus of M_n is at least 2n. Since $T(K_n) = 2n - 1$, we know by [2] that any manifold obtained by surgery on K_n will have at most a Heegaard genus of 2n. Therefore K_n has property R+ .

COROLLARY. Manifolds of arbitrarily high Heegaard genus can be obtained by surgery on prime knots.

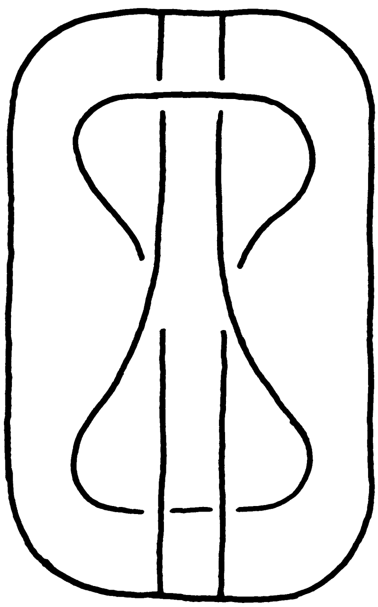

Figure 1

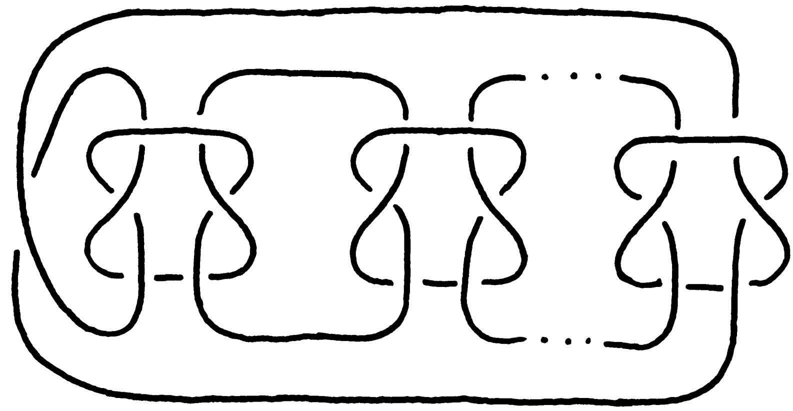

Figure 2

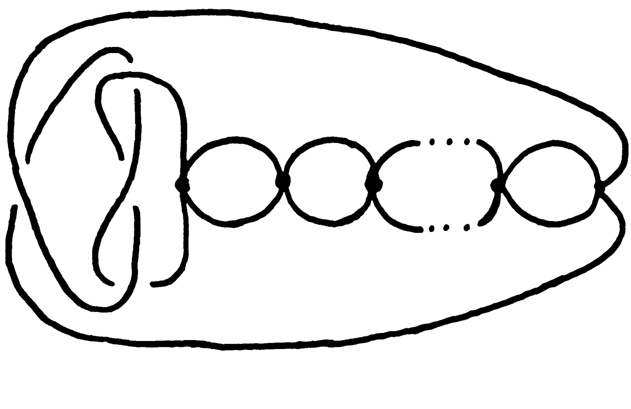

Figure 3

REFERENCES

1. CLARK, B. Surgery on Links Containing a Cable Sublink, Proc. Amer. Math. Soc. 72 (1978), 587-592.

2. CLARK, B. The Heegaard Genus of Manifolds Obtained by Surgery on Links and Knots, Internat. J. Math. and Math. Sci. 3 (1980), 583-589.

3. CLARK, B. Longitudinal Surgery on Composite Knots, Top. Proc. 6 (1981), 25-30.

4. CONWAY, J. An Enumeration of Knots and Links, and Some of Their Algebraic Properties, Computational Problems in Abstract Algebra, Pergamon Press, Oxford and New York, 1969, 329-358.

5. CROWELL, R. and FOX, R. Introduction to Knot Theory, Ginn and Company, Boston, 1963.

6. KIRBY, R. and LICKORISH, W. Prime Knots and Concordance, Math. Proc. Camb. Phil. Soc. 86 (1979), 437-441.

7. LICKORISH, W. Prime Knots and Tangles, Trans. Amer. Math. Soc. 267 (1981), 321-332.

8. MOSER, L. Elementary Surgery Along a Torus Knot, Pacific J. Math. 38 (1971), 737-745.

9. ROLFSEN, D. Knots and Links, Publish or Perish, Inc., Berkeley, Ca., 1976.

10. TAKAHASHI, M. Two Bridge Knots Have Property P, Mem. Am. Math. Soc. 239 (1981).