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A NOTE ON AN EXTENSION OF LINDELÖF'S THEOREM TO MEROMORPHIC FUNCTIONS

MOHAMMAD SALMASSI

Department of Mathematics
University of Rhode Island
Kingston, Rhode Island 02881 U.S.A.

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ABSTRACT. S. M. Shah [3] has given an extension of Lindelöf's Theorem to meromorphic functions. He also obtained an expression for the characteristic function of a meromorphic function of integer order. In this note we give estimates for $\log |f(re^{i\theta})|$ of such functions.

KEY WORDS AND PHRASES. *meromorphic functions, proximate order, slowly changing functions.*
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1. INTRODUCTION.

In [3;theorem 1] S. M. Shah obtained an expression for the characteristic function $T(r,f)$ of a meromorphic function $f(z)$ of integer order ρ. Following the argument of Cartwright [2;theorem 45,46] we can obtain the following results for $\log |f(re)^{i\theta})|$. We write

$$n(r) = n(r,1/f) + n(r,f);\ N(r) = N(r,1/f)+N(r,f).$$

Since ρ is a positive integer, we can write $f(z)$ in the form (see [3])

$$f(z) = z^k \exp(cz^\rho + \dots) \prod_1^\infty E(z/a_n,\rho) \prod_1^\infty E(z/b_n,\rho). \tag{1.1}$$

Let $\rho(r)$ be a proximate order [3] for $N(r)$ and let $n_L = \limsup_{r\to\infty} n(r)/r^\rho L(r)$, where $L(r)$ is a slowly changing function.

2. MAIN RESULTS.

THEOREM. Let $f(z)$ be a meromorphic function of integer order $\rho > 0$ and let

$$S(r) = c + \frac{1}{\rho} \sum_{|a_n| \le r} a_n^{-\rho} - \frac{1}{\rho} \sum_{|b_n| \le r} b_n^{-\rho},$$

i. Suppose $n_L < \infty$. Then for every $\eta > 0$, there is a $K(\rho,\eta)$ such that for every $\varepsilon > 0$,

$$\left|\log |f(re^{i\Theta})| - Re(r^{\rho} e^{i\Theta} S(r))\right| < K(\rho,\eta)(n_L + \varepsilon) r^{\rho} L(r) \tag{2.1}$$

for $0 \le r \le R$, except perhaps in circles the sum of whose radii is less than ηR, provided that $R > R_o(\varepsilon,\eta)$.

ii. Suppose $N(r)$ is of order ρ. Then there is a $K(\rho,\eta)$ such that

$$\left|\log |f(re^{i\Theta})| - Re(r^{\rho} e^{i\Theta} S(r))\right| < K(\rho,\eta) r^{\rho(r)} \tag{2.2}$$

for $0 \le r \le R$, except perhpas in circles the sum of whose radii is less than ηR, provided that $R > R_o(\eta)$.

iii. Let $\lim_{r\to\infty} \sup \log N(r)/\log r = c_1 < \rho$ and let $c_1 < c_2 < \rho \le 1 + c_2$. Then for every $\eta > 0$, there is a $K(c_2,\eta)$ such that

$$\left|\log f(re^{i\Theta})| - Re(r^{\rho} e^{i\Theta} S(r))\right| < K(c_2,\eta) r^{c_2}$$

for $0 \le r \le R$, except perhaps in circles the sum of whose radii is less than ηR, provided that $R > R_o(c_2,\eta)$. The proof depends on the following lemma of Cartan (see [1;p.46], also [2;pp.73-77]):

LEMMA (H. Cartan). Let $p(z) = \prod_{k=1}^{n} (z-z_k)$; for any positive H, the inequality

$$|p(z)| > (H/e)^n$$

holds outside at most n circles the sum of whose radii is at most 2H.

We omit the details of the proof of the theorem.

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