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SINGULARITIES IN A TWO-FLUID MEDIUM

RATHINDRA NATH CHAKRABARTI

Narikeldanga High School
50, Kavi Sukanta Sarani
Calcutta-700085, INDIA

and

BIRENDRANATH MANDAL

Department of Applied Mathematics, Calcutta University
92, Acharya Prafulla Chandra Road
Calcutta-700009, INDIA

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ABSTRACT. We compute the irrotational motion of two fluids with a horizontal plane surface of separation, under gravity. The fluids are nonviscous and incompressible, the upper one of finite depth with a free surface; they contain a line singularity or a point singularity. We obtain the velocity potentials for each singularity located in the upper or the lower fluid; if the upper depth tends to infinity, known results are recovered.

KEY WORDS AND PHRASES. Fluid dynamics, Laplace's equation, two-fluid problem, point singularity, line singularity, oscillating singularity, free surface.

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1. INTRODUCTION.

The reader of this paper should be familiar with Laplace's equation as applied to fluids. For the two dimensional case studied in Sections 3 and 4, elementary complex variable theory is needed as far as the theory of residues. For the three dimensional case treated in Sections 5 and 6, an elementary knowledge of spherical harmonies is assumed. In brief, you need to know typical singular solutions of Laplace's equation, since they dominate the potential in the neighborhood of the singularities. To motivate and better understand the integral representations of the potential assumed in

(3.1), (3.2), (4.1), (4.2), (5.1), (5.2), and (6.1), (6.2), see Thorne ([1], pp. 708, 710, 711) and Rhodes-Robinson ([2], p. 320). For a brief history of the subject, see Thorne, p. 715.

Some recent history is as follows. Many authors have investigated different types of singularities that can be used in the one-fluid problem. Thorne [1] and Rhodes-Robinson [2] gave surveys of the fundamental line and point singularities submerged in a fluid of finite or infinite depth. The two-fluid problem was discussed by Gorgui and Kassem [3], Mandal [4], and Chakrabarti [5]--the effect of surface tension being included by the last two authors as well as in [2]. References for other cases are mentioned in the Introduction of [3].

In this paper we shall discuss the basic line and point singularities when they are submerged in one of two fluids. The upper fluid is on finite constant depth 'h' with a free surface (FS); the lower fluid is of infinite depth. The time harmonic singularities are described by harmonic potential functions with period $\frac{\partial\pi}{\sigma}$ which satisfy the boundary conditions at the surface of separation (SS); in fact it is more convenient to use complex-valued potentials $\phi e^{-i\sigma t}$, the actual potential being the real part. The potential must also satisfy limiting conditions in the neighborhood of the singularity: it should behave like a typical singular harmonic function, as already mentioned; in the far field it should represent a spreading wave. Under these requirements, a unique solution will be found in all cases considered.

We note that our solution can be applied to cases when bodies are present in the fluids, whenever the two- or three-dimensional symmetry is such that the motion can be described by a series of singularities placed within the body in suitable positions. Whether the waves are generated by the body, or reflected by the body, does not matter.

2. STATEMENT AND FORMULATION OF THE PROBLEM.

Consider the irrotational motion of two non-viscous incompressible fluids under the action of gravity. Their SS is a horizontal plane, the lower fluid of infinite depth, the upper of finite height 'h'. Their motion is due to an oscillating singularity in one of the fluids; it is assumed to be simple harmonic with period $\frac{\partial\pi}{\sigma}$, so the velocity potentials ϕ_1 and ϕ_2 (of the lower and upper fluids respectively) will be too.

We take origin O in the mean SS, axis Oy pointing vertically downward into the lower fluid and chosen so it passes through the singularity which is then located at $(0,\eta)$ or $(0,-\eta)$ according as the singularity is in the lower or upper fluid respectively.

Then, for all x,

$$\nabla^2\phi_1 = 0, \quad 0 \le y \le h$$

$$\nabla^2\phi_2 = 0, \quad y < 0$$

except at the singular point. Also

$$\left.\begin{aligned} \frac{\partial\phi_2}{\partial y} + K\phi_2 &= 0 \quad \text{on } y = h \\ \nabla\phi_1 &= 0 \quad \text{on } y \to \infty \end{aligned}\right\} \tag{2.1}$$

$$\frac{\partial\phi_1}{\partial y} = \frac{\partial\phi_2}{\partial y} \quad \text{on } y = 0 \tag{2.2}$$

and

$$K\phi_1 + \frac{\partial\phi_1}{\partial y} = S\left(K\phi_2 + \frac{\partial\phi_2}{\partial y}\right) \quad \text{on } y = 0 \tag{2.3}$$

when $K = \frac{\sigma^2}{g}$, $s = \frac{\rho_2}{\rho_1}$, g is the acceleration due to gravity, and ρ_1 is the lower and ρ_2 the upper fluid density. Finally, ϕ_1 and ϕ_2 must satisfy the so-called radiation condition as $|x| \to \infty$. This condition is that the potential function represent diverging waves at a large distance from the singularity.

3. SUBMERGED LINE SINGULARITY, UPPER FLUID OF FINITE DEPTH.

We first consider a line singularity placed at the point $(0,-\eta)$ in the upper fluid or depth 'h'. Then $\phi_2 \to \log R$ as $R = \{x^2 + (y+\eta)^2\}^{\frac{1}{2}} \to 0$.

Now ϕ_1 and ϕ_2 can be represented as

$$\phi_1 = \sum_1^\infty f_j \log R_j + \sum_0^\infty g_j \log R'_j + \int_0^\infty A(k)e^{-ky}\cos kx\, dk \tag{3.1}$$

$$\phi_2 = \sum_{-\infty}^{\infty} c_j \log R_j + \sum_{-\infty}^{\infty} d_j \log R'_j + \int_0^\infty [B(k)\cosh k(h+y) + C(k)\sinh k(h+y)]\cos kx\, dk \tag{3.2}$$

where $d_0 = 1$, $R_j^2 = x^2 + (y + 2jh - \eta)^2$ and $R_j'^2 = x^2 + (y + 2jh + \eta)^2$, $j = 0, \pm1, \pm2, \ldots$ and A, B, C, f_h, g_i, c_j, d_j are to be found from the conditions (2.1), (2.2), and (2.3) and also the condition that the integrals are to be convergent. The radiation condition will be dealt with in the sequel.

The following integral representation used also in [3], page 34, will be needed in our calculations

$$\frac{\partial}{\partial y}\log R_j = \begin{cases} \int_0^\infty e^{-k(y+2jh-\eta)}\cos kx\,dk, & y > -2jh+\eta \\ -\int_0^\infty e^{k(y+2jh-\eta)}\cos kx\,dk, & y < -2jh+\eta \end{cases}$$

$$\frac{\partial}{\partial y}\log R'_j = \begin{cases} \int_0^\infty e^{-k(y+2jh+\eta)}\cos kx\,dk, & y > -(2jh+\eta) \\ -\int_0^\infty e^{k(y+2jh+\eta)}\cos kx\,dk, & y < -(2jh+\eta) \end{cases}$$

so that,

$$\frac{\partial}{\partial y}\log R_j\Big|_{y=-h} = \begin{cases} \int_0^\infty e^{-k(\overline{2j-1}\,h-\eta)}\cos kx\,dk, & j = 1,2,3,\ldots \\ -\int_0^\infty e^{k(\overline{2j-1}\,h-\eta)}\cos kx\,dk, & j = 0,-1,-2,-3,\ldots \end{cases}$$

$$\frac{\partial}{\partial y}\log R'_j\Big|_{y=-h} = \begin{cases} \int_0^\infty e^{-k(\overline{2j-1}\,h+\eta)}\cos kx\,dk, & j = 1,2,3,\ldots \\ -\int_0^\infty e^{k(\overline{2j-1}\,h+\eta)}\cos kx\,dk, & j = 0,-1,-2,-3,\ldots \end{cases}$$

$$\frac{\partial}{\partial y}\log R_j\Big|_{y=0} = \begin{cases} \int_0^\infty e^{-k(2jh-\eta)}\cos kx\,dk, & j = 1,2,3,\ldots \\ -\int_0^\infty e^{k(2jh-\eta)}\cos kx\,dk, & j = 0,-1,-2,-3,\ldots \end{cases}$$

$$\frac{\partial}{\partial y}\log R'_j\Big|_{y=0} = \begin{cases} \int_0^\infty e^{-k(2jh+\eta)}\cos kx\,dk, & j = 0,1,2,\ldots \\ -\int_0^\infty e^{k(2jh+\eta)}\cos kx\,dk, & j = -1,-2,-3,\ldots \end{cases}$$

Condition (2.1) gives

$$K\Big[\sum_0^\infty c_j \log\{x^2+(\overline{2j-1}\,h-\eta)^2\}^{\frac12} + \sum_1^\infty c_{-j}\log\{x^2+(\overline{2j+1}\,h+\eta)^2\}^{\frac12}$$

$$+\sum_0^\infty d_j\log\{x^2+(\overline{2j-1}\,h+\eta)^2\}^{\frac12} + \sum_1^\infty d_{-j}\log\{x^2+(\overline{2j+1}\,h-\eta)^2\}^{\frac12}$$

$$+\int_0^\infty B\cos kx\,dk\Big] + \sum_1^\infty c_j\int_0^\infty e^{-k(\overline{2j-1}\,h-\eta)}\cos kx\,dk - \sum_0^\infty c_{-j}\int_0^\infty e^{-k(\overline{2j+1}\,h+\eta)}\cos kx\,dk$$

$$+ \sum_1^\infty d_j \int_0^\infty e^{-k(\overline{2j-1}\,h+\eta)}\cos kx\,dk - \sum_0^\infty d_{-j}\int_0^\infty e^{-k(\overline{2j+1}\,h-\eta)}\cos kx\,dk$$

$$+ \int_0^\infty k\,c\cos kx\,dk = 0 \tag{3.3}$$

from which we obtain

$$\left.\begin{array}{ll} c_{j+1} + d_{-j} = 0, & j = 0,1,2,\ldots \\ d_{j+1} + c_{-j} = 0, & j = 0,1,2,\ldots \end{array}\right\} \tag{3.4}$$

Since $d_0 = 1$, we obtain
$$c_1 = -1 \tag{3.5}$$

Condition (2.3) gives

$$K\left[\sum_1^\infty f_j \log\{x^2 + (2jh-\eta)^2\}^{\frac12} + \sum_0^\infty g_j \log\{x^2 + (jh+\eta)^2\}^{\frac12} + \int_0^\infty A\cos kx\,dk\right]$$

$$+ \sum_1^\infty f_j \int_0^\infty e^{-k(2jh-\eta)}\cos kx\,dk + \sum_0^\infty g_j\int_0^\infty e^{-k(2jh+\eta)}\cos kx\,dk - \int_0^\infty kA\cos kx\,dk$$

$$= SK\left[\sum_1^\infty c_j \log\{x^2+(2jh-\eta)^2\}^{\frac12} - \sum_0^\infty c_{j+1}\log\{x^2+(2jh-\eta)^2\}^{\frac12} + \sum_1^\infty d_j \log\{x^2+(2jh+\eta)^2\}^{\frac12}\right.$$

$$\left. - \sum_0^\infty d_{j+1}\log\{x^2+(2jh+\eta)^2\}^{\frac12} + \int_0^\infty (B\cosh kh + c\sinh kh)\cos kx\,dk\right] + S\int_0^\infty\left[\sum_1^\infty c_j e^{-k(2jh-\eta)}\right.$$

$$+ \sum_2^\infty c_j e^{-k(\overline{2jh-1}\,h-\eta)} + \sum_1^\infty d_j e^{-k(2jh+\eta)} + \sum_2^\infty d_j e^{-k(\overline{2j-1}\,h+\eta)} + (1+d_1)e^{-k\eta}$$

$$\left. + k(B\sinh kh + c\cosh kh)\right]\cos kx\,dk. \tag{3.6}$$

from which we obtain by considering the coefficients of the different logarithmic terms

$$\left.\begin{array}{ll} S(c_j - c_{j+1}) = f_j, & j = 1,2,3,\ldots \\ S(d_j - d_{j+1}) = g_j, & j = 1,2,3,\ldots \\ S(d_0 - d_1) = g_0, & \end{array}\right\} \tag{3.7}$$

and then (3.6) reduces to

$$(k-K)A + S(K\cosh kh + k\sinh kh)B + S(K\sinh kh + k\cosh kh)C$$

$$= -2S\left[d_1 e^{-k\eta} + \sum_1^\infty e^{-2kjh}(c_{j+1}e^{-k\eta} + d_{j+1}e^{-k\eta})\right]. \tag{3.8}$$

Condition (2.2) now gives

$$\sum_1^\infty (1-S)(c_j e^{k\eta} + d_j e^{-k\eta})e^{-2kjh} + \sum_1^\infty (1+S)(c_{j+1}e^{k\eta} + d_{j+1}e^{-k\eta})e^{-2kjh}$$

$$+ d_1(1+S)e^{-k\eta} - c_1(1-S)e^{-k\eta} = -k(A + B\sinh kh + C\cosh kh). \tag{3.9}$$

Now for convergence of the integrals in the expressions for ϕ_1 and ϕ_2, G(k) must be zero for $k = 0$, where $G(k)$ is the expression in the left side of (3.9), so that

$$\sum_1^\infty (1 - S)(c_j + d_j) + \sum_1^\infty (1 - S)(c_{j+1} + d_{j+1}) - c_1(1 - S) + d_1(1 + S) = 0 .$$

This is satisfied by choosing

$$(1 - S)c_j + (1 + S)c_{j+1} = 0, \qquad j = 1,2,\ldots,\infty,$$

$$(1 - S)d_j + (1 + S)d_{j+1} = 0, \qquad j = 1,2,\ldots,\infty,$$

$$\text{and} \quad (1 - S)c_1 = (1 + S)d_1 .$$

From (3.4), $c_0 = -d_1 = \mu$ where $\mu = \frac{1 - S}{1 + S}$ so that we obtain

$$\left.\begin{aligned} c_j &= (-1)^j \mu^{j-1}, \qquad j = 1,2,\ldots \\ d_j &= (-1)^j \mu^j \;\;, \qquad j = 1,2,\ldots \end{aligned}\right\} \tag{3.10}$$

(3.8) can be written as

$$(k - K)A - s(K \cosh kh + k \sinh kh)B + s(K \sinh kh + k \cosh kh)C$$
$$= 2S\mu \left[e^{-k\eta} - \frac{e^{-2kh}(e^{k\eta} + \mu e^{-k\eta})}{1 + \mu e^{-2kh}} \right] \tag{3.11}$$

From (3.3), we obtain

$$KB + kc - \frac{2e^{-kh}(e^{k\eta} + \mu e^{-k\eta})}{1 + \mu e^{-2kh}} = 0 \tag{3.12}$$

From (3.9), we obtain

$$A + B \sinh kh + c \cosh kh = 0 \tag{3.13}$$

Solving for A,B,C from (3.11), (3.12) and (3.13), we obtain

$$A = \frac{2e^{-kh}(e^{k\eta} + \mu e^{-k\eta})\sinh kh}{K(1 + \mu e^{-2kh})} + \left(\frac{k}{K} \sinh kh - \cosh kh \right) \left[\frac{1}{k - K} + \frac{(S - 1)\sinh kh}{\Delta(k)} \right] F_1 \tag{3.14}$$

$$B = \frac{1}{K} \left[\frac{2e^{-kh}(e^{k\eta} + \mu e^{-k\eta})}{1 + \mu e^{-2kh}} - k \left\{ \frac{1}{k - K} + \frac{(S - 1) \sinh kh}{\Delta(k)} \right\} F_1 \right] \tag{3.15}$$

$$C = \left[\frac{1}{k - K} + \frac{(S - 1) \sinh kh}{\Delta(k)} \right] F_1 \tag{3.16}$$

where

$$F_1 = \frac{2S\mu e^{-k\eta} - \dfrac{2e^{-kh}(e^{k\eta} + \mu e^{-k\eta})}{1 + \mu e^{-2kh}} \left[S\mu e^{-kh} + \frac{1}{K} \{(K-k+Sk)\sinh kh + SK \cosh kh\} \right]}{(1 - 2S) \sinh kh - \cosh kh} \tag{3.17}$$

and
$$\Delta = \Delta(k) = \{k(1 - S) - SK\}\sinh kh - K \cosh kh \tag{3.18}$$

Now, $\Delta(k)$ has one simple pole at $k = k_0 > 0$, say on the real axis of k (there are also poles $k = k_1$ and complex poles $k_n = \alpha_n + i\beta_n$ where when $S \to 0$, $\beta_n \to k_n$ where $k_n \to (n-1)\pi$ as $n \to \infty$ (cf. [2])). The zeroes of the denominator of F_1 are purely imaginary. Hence A,B,C have simple poles $k = k_0$ and $k = K$ on the positive real axis of k. In the line integrals from 0 to ∞ we make indentation below these poles which account for the behaviors of the potential functions at infinity particularly as $|x| \to \infty$. This will be evident later.

Substituting the above results, we have

$$\phi_1 = S \sum_1^\infty [(-1)^j \mu^{j-1} + (-1)^j \mu^j] \log R_j + S \sum_0^\infty (-1)^j [\mu^j + \mu^{j+1}] \log R'_j$$
$$- \int_0^\infty \frac{2}{K} \cdot \frac{e^{-kh}(e^{k\eta} + \mu e^{-k\eta})}{1 + \mu e^{-2kh}} \sinh kh\, e^{-ky} \cos kx\, dk$$
$$+ \oint_0^\infty \frac{1}{k - K} \left(\frac{k}{K} \sinh kh - \cosh kh\right) F_1\, e^{-ky} \cos kx\, dk$$
$$+ \oint_0^\infty \frac{1}{\Delta(k)} (S - 1) \sinh kh \left(\frac{k}{K} \sinh kh - \cosh kh\right) F_1\, e^{-ky} \cos kx\, dk \tag{3.19}$$

$$\phi_2 = \sum_1^\infty (-1)^j \mu^{j-1} \log R_j + \sum_0^\infty (-1)^j \mu^{j+1} \log R_{-j} + \sum_1^\infty (-1)^j \mu^j \log R'_j$$
$$+ \sum_0^\infty (-1)^j \mu^j \log R'_{-j} + \int_0^\infty \frac{2}{K} \cdot \frac{e^{-kh}(e^{k\eta} + \mu e^{-k\eta})}{1 + \mu e^{-2kh}} \cosh k(h + y) \cos kx\, dk$$
$$+ \oint_0^\infty \frac{1}{k - K} \left[\sinh k(h + y) - \frac{k}{K} \cosh k(h + y)\right] F_1 \cos kx\, dk$$
$$+ \oint_0^\infty \frac{1}{\Delta(k)} (S - 1) \sinh k(h + y) - \frac{k}{K} \cosh k(h + y)] F_1 \cos kx\, dk \tag{3.20}$$

Now, as $h \to \infty$ it is possible to obtain

$$\phi_1 = \frac{2S}{1 + S} \log R'_0 + \frac{2S}{1 + S} \oint_0^\infty \frac{e^{-k(y+\eta)}}{k - M} \cos hx\, dk$$

$$\phi_2 = \log R'_0 + \frac{1 - S}{1 + S} \log R_0 - \frac{2S}{1 + S} \oint_0^\infty \frac{e^{k(y-\eta)}}{k - M} \cos kx\, dk$$

where $M = K \frac{1 + S}{1 - S}$ which are the results derived by Gorgui and Kassem [3]. Now to investigate the behavior of the integral for large $|x|$, we put

$$2 \cos kx = 3^{ik|x|} + e^{-ik|x|}$$

Then

$$\int_0^\infty \frac{1}{k - K} \left(\frac{k}{K} \sinh kh - \cosh kh \right) F_1 \, e^{-ky} \cos kx \, dk$$

$$= \int_0^\infty I_1 \, e^{ik|x|} \, dk + \int_0^\infty I_1 \, e^{-ik|x|} \, dk, \text{ say} \tag{3.21}$$

Where,

$$I_1 = \frac{1}{2}\left(\frac{k}{K} \sinh kh - \cosh kh \right) \frac{F_1}{k - K} e^{-ky} \tag{3.22}$$

For the first integral of (3.21), we consider in the complex k-plane a contour in the first quadrant bounded by the real axis of large length X_1 with an indentation below the pole $k = K$, an arc Γ of radius X_1 with center at the origin and the line joining the origin, with the point $X_1 e^{i\alpha}$ where $0 < \alpha < \frac{\pi}{2}$. Then for considering the behavior as $|x| \to \infty$, we only need to consider the behavior of the term arising from the residue at $k = K$, because the integral along the arc becomes exponentially small as $X_1 \to \infty$ and the integral along the line 0 to $X_1 e^{i\alpha}$ $(0 < \alpha < \frac{\pi}{2})$ will have a factor $e^{-X_1 \sin \alpha |x|}$ which becomes exponentially small for large $|x|$. Hence making $X_1 \to \infty$ we find that as $|x| \to \infty$

$$\int_0^\infty I_1 \, e^{ik|x|} \, dk \to 2\pi i \text{ Residue of } I_1 \, e^{ik|x|} \text{ at } k = K.$$

For the second integral of (3.21), we consider in the complex k-plane a contour in the fourth quadrant bounded by the real axis from 0 to X_1 with an indentation below the pole $k = K$, an arc Γ' or radius X_1 with center at the origin and the line joining the origin with the point $X_1 e^{-i\alpha}$ where $0 < \alpha < \frac{\pi}{2}$. Since now the singularities on the real axis are taken to be outside the contour and following a similar argument as above, we obtain that as $|x| \to \infty$

$$\int_0^\infty I_1 \, e^{-ik|x|} \, dk \to 0$$

Again,

$$\int_0^\infty \frac{1}{\Delta(k)} (S - 1) \sinh kh \left(\frac{k}{K} \sinh kh - \cosh kh \right) F_1 \, e^{-ky} \cosh kx \, dk$$

$$= \int_0^\infty I_2 \, e^{ik|x|} \, dk + \int_0^\infty I_2 \, e^{-ik|x|} \, dk \tag{3.23}$$

where

$$I_2 = \frac{1}{2}(S - 1) \sinh kh \left(\frac{k}{K} \sinh kh - \cosh kh \right) \frac{F_1}{\Delta} e^{-ky} \tag{3.24}$$

For the first integral of (3.24), we choose a similar contour as was chosen for the intetral with I_1 excepting that the indentation is now below $k = k_0$ instead of K. The contribution from the poles of $\Delta(k)$ which lie inside the contour has a factor $e^{-\beta_n|x|}$ where $\alpha_n + i\beta_n$ is a zero of $\Delta(k)$ in the first quadrant so that for large $|x|$ we may neglect it. The line may cross some singularities of $\Delta(k)$. To avoid this, if it crosses a zero of $\Delta(k)$, we indent the line about it so that it lies outside the region bounded by these contours and the contribution for this indentation will also contain a factor $e^{-\beta_m|x|}$ which becomes exponentially small for large $|x|$, $\alpha_m + i\beta_m$ being a singularity of this type. Hence, we find that as $|x| \to \infty$

$$\oint_0^\infty I_2\, e^{ik|x|}\, dk \to 2\pi i \text{ Residue of } I_2\, e^{ik|x|} \text{ at } k = k_0$$

By a somewhat similar argument as was used in the second integral in (3.21), we obtain

$$\oint_0^\infty I_2\, e^{-ik|x|} \to 0 \quad \text{as} \quad |x| \to \infty$$

Hence, we find that as $|x| \to \infty$, ϕ_1 tends to

$$2\pi i\,\frac{s(\sinh Kh - \cosh kh)}{(1-2s)\sinh Kh - \cosh Kh}\left[\mu e^{-K\eta} - \frac{e^{-Kh}(e^{K\eta}+\mu e^{-K\eta})}{1+\mu e^{-2Kh}}\left\{\mu e^{-Kh}+\sinh Kh+\cosh Kh\right\}\right]e^{-Ky}e^{ik|x|}$$

$$+\left\{2\pi i(s-1)\sinh k_0 h\left(\frac{k_0}{K}\sinh k_0 h - \cosh k_0 h\right)\left[s\mu e^{-k_0\eta} - \frac{e^{-k_0 h}(e^{k_0\eta}+e^{-k_0\eta})}{1+\mu e^{-2k_0 h}}\right.\right.$$
$$\left.\left.\left\{s\mu e^{-k_0 h} + \frac{1}{K}\left((K-k_0+sk_0)\sinh k_0 h + sK\cosh k_0 h\right)\right\}\right]e^{ik_0|x|}\right\}\Big/D,$$

where

$$D = [(1-2s)\sinh k_0 h - \cosh k_0 h][h\{k_0(1-s) - sK\}\cosh k_0 h + (1-s-hK)\sinh k_0 h]$$

Similarly as $|x| \to \infty$, ϕ_2 tends to

$$2\pi i s\,\frac{\sinh K(h+y)-\cosh K(h+y)}{(1-2s)\sinh Kh - \cosh Kh}\left[\mu e^{-K\eta} - \frac{e^{-Kh}(e^{K\eta}+\mu e^{-K\eta})}{1+\mu e^{-2Kh}}\left\{\mu e^{-Kh}+\sinh Kh+\cosh Kh\right\}\right]e^{iK|x|}$$

$$+\left\{2\pi i(s-1)\sinh k_0 h\left[\sinh k_0(h+y) - \frac{k_0}{K}\cosh k_0(h+y)\right]\left[s\mu e^{-k_0\eta} - \frac{e^{-k_0 h}(e^{k_0\eta}+\mu e^{-k_0\eta})}{1+\mu e^{-2k_0 h}}\right.\right.$$
$$\left.\left.\left\{s\mu e^{-k_0 h} + \frac{1}{K}\left((k-k_0+sk_0)\sinh k_0 h + sK\cosh k_0 h\right)\right\}\right]e^{ik_0|x|}\right\}\Big/D,$$

where

$$D = [(1-2s)\sinh k_0 h - \cosh k_0 h][h\{k_0(1-s) - sK\}\cosh k_0 h + (1-s-hK)\sinh k_0 h].$$

Thus ϕ_1 and ϕ_2 satisfy the radiation condition at infinity. Now as h tends to infinity (3.25) and (3.26) take respectively the following forms

$$2\pi i \frac{s}{1+s} e^{-k_o(y+\eta)} e^{ik_o|x|}$$

and

$$-2\pi i \frac{s}{1+s} e^{k_o(y-\eta)} e^{ik_o|x|}$$

where now $k_o = \frac{1+s}{1-s} K = M$, and these agree with results of Gorgui and Kassem [3].

4. WAVE SOURCE SUBMERGED IN LOWER FLUID.

In this case, there is a logarithmic type singularity at the point $(0,\eta)$.

Now ϕ_1 and ϕ_2 can be represented as

$$\phi_1 = \sum_0^\infty c_j \log R_j + \sum_0^\infty d_j \log R_j' + \int_0^\infty A(k)e^{-ky} \cos kx\, dk \tag{4.1}$$

$$\phi_2 = \sum_{-\infty} f_j \log R_j + \sum_1^\infty f_j \log R_j' + \sum_{-1}^{-\infty} g_j \log R_j' + \int_0^\infty$$

$$\cdot [B(k)\cosh k(h + y) + C(k)\sinh k(h + y)]\cos kx\, dk \tag{4.2}$$

where $C_o = 1$. Condition (2.1) gives

$$\left.\begin{aligned} f_o + g_1 &= 0 \\ f_1 &= 0 \\ f_{j+1} + g_{-j} &= 0, \qquad j = 1,2,\ldots\ . \\ g_{j+1} + f_{-j} &= 0, \qquad j = 1,2,\ldots\ . \end{aligned}\right\} \tag{4.3}$$

and

$$KB + \sum_1^\infty f_j\, e^{-k(\overline{2j-1}\,h - \eta)} - \sum_0^\infty f_{-j}\, e^{-k(\overline{2j+1} + \eta)} + \sum_1^\infty g_j\, e^{-k(\overline{2j-1}\,h + \eta)}$$

$$- \sum_1^\infty g_{-j}\, e^{-k(\overline{2j+1}\,h - \eta)} + kC = 0 \tag{4.4}$$

Condition (2.3) gives

$$\left.\begin{aligned} d_o &= Sf_o - 1 \\ c_j &= S(f_j - f_{j+1}), \qquad j = 1,2,\ldots\ . \\ d_j &= S(g_j - g_{j+1}), \qquad j = 1,2,\ldots\ . \end{aligned}\right\} \tag{4.5}$$

and

$$(k - K)A + S(K \cosh kh + k \sinh kh)B + S(K \sinh kh + k \cosh kh)C$$

$$= 2Sf_o\, e^{-k\eta} - 2e^{-k\eta} - 2S \sum_1^\infty f_{j+1}\, e^{-k(2jh - \eta)} - 2S \sum_1^\infty g_{j+1}\, e^{-k(2jh + \eta)} \tag{4.6}$$

Condition (2.2) gives

$$k(A + B \sinh kh + C \cosh kh) = -e^{-kh} + S \sum_1^\infty (f_j - f_{j+1})e^{-k(2jh - \eta)} + (Sf_o - 1)e^{-k\eta}$$
$$+ S \sum_1^\infty (g_j - g_{j+1})e^{-k(2jh + \eta)} - \sum_1^\infty f_j \, e^{-k(2jh - \eta)} + f_o \, e^{-k\eta}$$
$$- \sum_1^\infty g_{j+1} \, e^{-k(2jh + \eta)} - \sum_1^\infty g_j \, e^{-k(2jh + \eta)} - \sum_1^\infty f_{j+1} \, e^{-k(2jh - \eta)} \tag{4.7}$$

Now for convergence at $k = 0$, we obtain

$$\sum_1^\infty[(S - 1)f_j - (S + 1)f_{j+1}] + \sum_1^\infty[(S - 1)g_j - (S + 1)g_{j+1}] + (S + 1)f_o - 2 = 0 \tag{4.8}$$

This is satisfied by choosing

$$\left.\begin{aligned} (S - 1)f_j - (S + 1)f_{j+1} &= 0, \qquad j = 1,2,\ldots . \\ (S - 1)g_j - (S + 1)g_{j+1} &= 0, \qquad j = 1,2,\ldots . \\ f_o &= \frac{2}{1 + S} \end{aligned}\right\} \tag{4.9}$$

From (4.5), $d_o = -\mu$ where $\mu = \frac{1 - S}{1 + S}$.

From (4.3), (4.5) and (4.9), we obtain

$$\left.\begin{aligned} g_j &= \frac{2}{1 + S}(-1)^j\mu^j &&, \quad j = 1,2,\ldots . \\ g_{-j} &= 0 &&, \quad j = 1,2,\ldots . \\ f_j &= 0 &&, \quad j = 1,2,\ldots . \\ f_{-j} &= \frac{2}{1 - S}(-1)^j\mu^{j+1} &&, \quad j = 1,2,\ldots . \\ C_j &= 0 &&, \quad j = 1,2,\ldots . \\ d_j &= \frac{4S}{1 - S^2}(-1)^j\mu^j &&, \quad j = 1,2,\ldots . \end{aligned}\right\} \tag{4.10}$$

(4.4) can be written as

$$KB + kC - \frac{2}{1 + S} e^{-k(h + \eta)} + \frac{2\mu e^{-2kh}}{1 + \mu e^{-2kh}} \left[\frac{e^{-k(h + \eta)}}{1 + S} - \frac{e^{k(h - \eta)}}{1 - S} \right] = 0 \tag{4.11}$$

(4.6) can be written as

$$(k - K)A + S(K \cosh kh + k \sinh kh)B + S(K \sinh kh + k \cosh kh)C$$
$$= -2\mu e^{-k\eta} - \frac{4S\mu^2 e^{-k(2h + \eta)}}{(1 - S)(1 + \mu e^{-2kh})} \tag{4.12}$$

(4.7) gives

$$A + B \sinh kh + C \cosh kh = 0. \tag{4.13}$$

Solving for A,B,C from (4.11), (4.12) and (4.13), we obtain

$$A = \left(\frac{k}{K}\sinh kh - \cosh kh\right)\left[\frac{1}{k-K} - \frac{(S-1)\sinh kh}{\Delta(k)}\right] G_1 - \frac{\sinh kh}{K}$$

$$\cdot \left[\frac{2e^{-k(h+\eta)}}{1+S} + \frac{2\mu e^{-2kh}}{1+\mu e^{-2kh}}\left\{\frac{e^{k(h-\eta)}}{1-S} - \frac{e^{-k(h+\eta)}}{1+S}\right\}\right] \tag{4.14}$$

$$B = -\frac{k}{K}\left[\frac{1}{k-K} - \frac{(S-1)\sinh kh}{\Delta(k)}\right] G_1 + \frac{2e^{-k(h+\eta)}}{K(1+S)} + \frac{2\mu e^{-2kh}}{K(1+\mu e^{-2kh})}\left[\frac{e^{k(h-\eta)}}{1-S} - \frac{e^{-k(h+\eta)}}{1+S}\right] \tag{4.15}$$

$$C = \left[\frac{1}{k-K} + \frac{(S-1)\sinh kh}{\Delta(k)}\right] G_1 \tag{4.16}$$

where,

$$G_1 = -2\mu e^{-k\eta}\left(1 + \frac{2S}{1+S}\cdot\frac{e^{-2kh}}{1+\mu e^{-2kh}}\right) - \frac{1}{K}\left[(K - k + Sk)\sinh kh + SK\cosh kh\right]$$

$$\cdot \left[\frac{2}{1+S}e^{-k(h+\eta)} + \frac{2e^{-2kh}}{1+\mu e^{-2kh}}\left\{\frac{e^{k(h-\eta)}}{1-S} - \frac{e^{-k(h+\eta)}}{1+S}\right\}\right] / (1-2S)\sinh kh - \cosh kh \tag{4.17}$$

$\Delta(k)$ being given previously.

Thus using the above results, we obtain

$$\phi_1 = \log R_o + \frac{S-1}{S+1}\log R_o' + \frac{4S}{1-S^2}\sum_1^\infty (-1)^j \mu^j \log R_j'$$

$$+ \int_0^\infty \frac{1}{k-K}\left(\frac{k}{K}\sinh kh - \cosh kh\right) G_1 e^{-ky}\cos kx\, dk$$

$$+ \oint_0^\infty \frac{1}{\Delta(k)}(S-1)\sinh kh\left(\frac{k}{K}\sinh kh - \cosh kh\right) G_1 e^{-ky}\cos kx\, dk$$

$$- \int_0^\infty \frac{\sinh kh}{K}\ \frac{2e^{-k(h+\eta)}}{1+S} + \frac{2\mu e^{-2kh}}{1+\mu e^{-2kh}}\ \frac{e^{k(h-\eta)}}{1-S} - \frac{e^{-k(h+\eta)}}{1+S}\ e^{-ky}\cos kx\, dk \tag{4.18}$$

$$\phi_2 = \frac{2}{1+S}\log R_o + \frac{2}{1+S}\sum_1^\infty (-1)^j \mu^{j+1}\log R_j + \frac{2}{1-S}\sum_1^\infty (-1)^j\mu^j \log R_j'$$

$$+ \oint_0^\infty \left[\sinh k(h+y) - \frac{k}{K}\cosh k(h+y)\right]\frac{G_1}{k-K}\cos kx\, dk + \int_0^\infty \frac{1}{\Delta(k)}(S-1)\sinh kh$$

$$\cdot \left[\sinh k(h+y) - \frac{k}{K}\cosh k(h+y)\right] G_1 \cos kx\, dk + \int_0^\infty \left[\left\{\frac{2e^{-k(h+y)}}{K(1+S)} + \frac{2\mu e^{-2kh}}{K(1+\mu e^{-2kh})}\right.\right.$$

$$\left.\left.\cdot\left(\frac{e^{k(h-\eta)}}{1-S} - \frac{e^{-k(h+\eta)}}{1+S}\right)\right\}\cosh k(h+y)\right]\cos kx\, dk \tag{4.19}$$

Now as $h \to \infty$, we obtain

$$\phi_1 = \log R_o - \frac{1-S}{1+S}\log R_o' - \frac{2}{1+S}\oint_0^\infty \frac{e^{-k(y+\eta)}}{k-M}\cos kx\, dk$$

$$\phi_2 = \frac{2}{1+S}\log R_o + \frac{2}{1+S}\oint_0^\infty \frac{e^{k(y-\eta)}}{k-M}\cos kx\,dk$$

which are the results derived by Gorgui and Kassem [3].

Proceeding similarly as in the previous case we see that the above potentials have the following forms as $|x| \to \infty$

$$2\pi i(\sinh Kh - \cosh Kh)\Bigg[\mu e^{-K\eta}\Big(1 + \frac{2S}{1+S}\,\frac{e^{-2Kh}}{1+\mu e^{-2Kh}}\Big) + S(\sinh Kh + \cosh Kh)$$
$$\times\Bigg\{\frac{e^{-K(h+\eta)}}{1+S} + \frac{\mu e^{-2Kh}}{1+\mu e^{-2Kh}}\Bigg(\frac{e^{K(h-\eta)}}{1-S} - \frac{e^{-K(h+\eta)}}{1+S}\Bigg)\Bigg\}\Bigg]e^{-Ky}e^{iK|x|}/(2S-1)\sinh Kh + \cosh Kh$$
$$+\Bigg\{2\pi i(1-S)\sinh k_oh\Big(\frac{k_o}{K}\sinh k_oh - \cosh k_oh\Big)\Bigg[\mu e^{-k_o\eta}\Big(1 + \frac{2S}{1+S}\,\frac{\mu e^{-2k_oh}}{1+\mu e^{-2k_oh}}\Big) + \frac{1}{K}$$
$$\times\Big\{(K - k_o + Sk_o)\sinh k_oh + SK\cosh k_oh\Big\}\Bigg\{\frac{e^{-k_o(h+\eta)}}{1+S} + \frac{\mu e^{-2k_oh}}{1+\mu e^{-2k_oh}}$$
$$\times\Bigg(\frac{e^{k_o(h-\eta)}}{1-S} - \frac{e^{-k_oh(h+\eta)}}{1+S}\Bigg)\Bigg\}\Bigg]e^{-k_oy}\,e^{ik_o|x|}\Bigg\}/D \tag{4.20}$$

where $D = [(1-2S)\sinh k_oh - \cosh k_oh][h\,k_o(1-S) - SK\cosh k_oh + (1-S-hK)\sinh k_oh]$

and

$$\frac{\pi i\{\sinh K(h+y) - \cosh K(h+y)\}}{(1-2S)\sinh Kh - \cosh Kh}\Bigg[-2\mu e^{-K\eta}\Bigg(1 + \frac{2S}{1+S}\,\frac{e^{-2Kh}}{1+\mu e^{-2Kh}}\Bigg) - S(\sinh Kh + \cosh Kh)$$
$$\times\Bigg\{\frac{2}{1+S}e^{-K(h+\eta)} + \frac{2\mu e^{-2Kh}}{1+\mu e^{-2Kh}}\Bigg(\frac{e^{K(h-\eta)}}{1-S} - \frac{e^{-K(h+\eta)}}{1+S}\Bigg)\Bigg\}\Bigg]e^{iK|x|}$$
$$+\Bigg\{\pi i(S-1)\sinh k_oh\{\sinh k_o(h+y) - \frac{k_o}{K}\cosh k_o(h+y)\}\Bigg[-2\mu e^{-k_o\eta}\Big(1 + \frac{2S}{1+S}\,\frac{e^{-2k_oh}}{1+\mu e^{-2k_oh}}\Big)$$
$$-\frac{1}{K}\{(K - k_o + Sk_o)\sinh k_oh + SK\cosh k_oh\}\Bigg\{\frac{2}{1+S}e^{-k_o(h+\eta)}\,.\,\frac{2\mu e^{-2k_oh}}{1+\mu e^{-2k_oh}}$$
$$\times\Bigg(\frac{e^{k_o(h-\eta)}}{1-S} - \frac{e^{k_o(h+\eta)}}{1+S}\Bigg)\Bigg\}\Bigg]e^{ik_o|x|}\Bigg\}/D \tag{4.21}$$

where $D = [(1-2S)\sinh k_oh - \cosh k_oh][h\,k_o(1-S) - SK\cosh k_oh + (1-S-hK)\sinh k_oh]$.

Now, as h tends to infinity, (4.20) and (4.21) take respectively the following forms:

$$-\frac{2\pi i}{1+S}e^{-M(y+\eta)}\,e^{iM|x|}$$

and

$$2\pi i\,\frac{S}{1+S}\,e^{M(y-\eta)}e^{iM|x|}$$

5. SUBMERGED POINT SINGULARITIES, UPPER FLUID OF FINITE DEPTH.

Here, $\phi_2 \sim \dfrac{P_n(\cos\theta)}{R_o'^{\,n+1}}$ as $R_o' = \{r^2 + (y+\eta)^2\}^{\frac{1}{2}} \to 0$, $n = 0,1,2,\ldots,n$ where r is the distance from the y axis, and $\theta = \tan^{-1}\left(\frac{r}{y-\eta}\right)$. We assume

$$\phi_1 = \int_0^\infty A(k)\, e^{-ky}\, J_o(kr)\, dk \tag{5.1}$$

$$\phi_2 = \frac{P_n(\cos\theta)}{R_o'^{\,n+1}} + \int_0^\infty \{B(k)\cosh k(h+y) + C(k)\sinh k(h+y)\} J_o(kr)dk. \tag{5.2}$$

We use the following integral representations

$$\left.\begin{aligned} \frac{P_n(\cos\theta)}{R_o'^{\,n+1}} &= \frac{1}{n!}\int_0^\infty k^n\, e^{-k(y+\eta)}\, J_o(kr)dk, & y > -\eta \\ &= \frac{(-1)^n}{n!}\int_0^\infty k^n\, e^{k(y+\eta)} J_o(kr)dk, & y < -\eta \end{aligned}\right\} \tag{5.3}$$

From conditions (2.1), (2.2) and (2.3), we obtain

$$KB + kc = \frac{(-1)^{n+1}}{n!} k^n (k+K)\acute{e}^{-k(h-\eta)} \tag{5.4}$$

$$A + B\sinh kh + C\cosh kh = \frac{k^n}{n!} e^{-k\eta} \tag{5.5}$$

and

$$(k-K)A + s(K\cosh kh + k\sinh kh)B + s(K\sinh kh + k\cosh kh)C = \frac{S}{n!} k^n\, e^{-k\eta}(k-K) \tag{5.6}$$

Solving for A, B, C we obtain

$$\left.\begin{aligned} A &= \frac{k^n}{n!}\left[e^{-k\eta} + \frac{(-1)^n}{K}(k+K)e^{-k(h-\ )}\sinh kh\right] + \frac{W_1}{k-K} + \left(\frac{k}{K}\sinh kh - \cosh kh\right) \\ &\qquad + \frac{W_1}{\Delta}(S-1)\sinh kh\left(\frac{k}{K}\sinh kh - \cosh kh\right) \\ B &= \frac{(-1)^n k^n (k+K)}{n!\,K} e^{-k(h-\eta)} - \frac{k}{K}\left[\frac{1}{k-K} + \frac{(S-1)\sinh kh}{\Delta}\right] W_1 \\ C &= \left[\frac{1}{k-K} + \frac{(S-1)\sinh kh}{\Delta}\right] W_1 \end{aligned}\right\} \tag{5.7}$$

where

$$W_1 = \frac{\frac{k^n}{n!}\left[(S-1)(k-K)e^{-k\eta} + \frac{(-1)^n}{K}\{(K-k+Sk)\sinh kh + SK\cosh kh\}(k+K)e^{-k(h-\eta)}\right]}{(1-2S)\sinh kh - \cosh kh} \tag{5.8}$$

and Δ is the same expression used in §3, so that we obtain

$$\begin{aligned} \phi_1 = &\int_0^\infty \frac{k^n}{n!}\left[e^{-k\eta} + \frac{(-1)^n}{K}(k+K)\sinh kh\, e^{-k(h-\eta)}\right] e^{-ky} J_o(kr)\, dk \\ &+ \oint_0^\infty \frac{W_1}{k-K}\left(\frac{k}{K}\sinh kh - \cosh kh\right) e^{-ky}\, J_o(kr)\, dk \\ &+ \oint_0^\infty \frac{W_1}{\Delta}(S-1)\sinh kh\left(\frac{k}{K}\sinh kh - \cosh kh\right) e^{-ky}\, J_o(kr)\, dk \end{aligned} \tag{5.9}$$

and,

$$\phi_2 = \frac{P_n(\cos\theta)}{R_o'^{\,n+1}} + \int_0^\infty \frac{(-1)^{n+1}}{n!} \cdot \frac{k^n(k+K)}{K} e^{-k(h-\eta)} \cosh k(h+y) J_o(kr)dk$$

$$+ \int_0^\infty \frac{W_1}{k-K}\left[\sinh k(h+y) - \frac{k}{K}\cosh k(h+y)\right] J_o(kr)dk$$

$$+ \oint_0^\infty \frac{W_1}{\Delta}(S-1)\sinh kh\left[\sinh k(h+y) - \frac{k}{K}\cosh k(h+y)\right] J_o(kr)dk \qquad (5.10)$$

Now, as h tends to infinity, ϕ_1 and ϕ_2 take respectively the following forms:

$$-\frac{2SM}{n!(1+S)} \oint_0^\infty \frac{k^n}{k-M} e^{-k(y+\eta)} J_o(kr)dk \qquad (5.11)$$

and

$$\frac{P_n(\cos\theta)}{R_o'^{\,n+1}} + \oint_0^\infty \frac{k^n}{n!}\left[1 + \frac{2SM}{(1+S)(k-M)}\right] e^{k(y-\eta)} J_o(kr)dk \qquad (5.12)$$

where $M = K\frac{1+S}{1-S}$, which are the results derived by Gorgui and Kassem [3]. Now putting $2J_o(kr) = H_o^{(1)}(kr) + H_o^{(2)}(kr)$ and rotating the contour in the integral involving $H_o^{(1)}(kr)$ in the first quadrant and in the integrals involving $H_o^{(2)}(kr)$ in the fourth quadrant, we can reduce the integrals into suitable forms from which it is seen that as $r \to \infty$, ϕ_1 and ϕ_2 respectively take the following forms:

$$2\pi i S\frac{(-1)^n}{n!}K^{n+1} \cdot \frac{e^{K(\eta-h-y)} H_o^{(1)}(kr)}{(2S-1)\sinh Kh + \cosh Kh}$$

$$+\{\pi i \frac{k_o^{\,n}}{n!}(S-1)\sinh k_oh\left(\frac{k_o}{K}\sinh k_oh - \cosh k_oh\right)\left[(S-1)(k_o-K)e^{-k_o\eta}\right.$$

$$\left.+\frac{(-1)^n}{K}\{(K-k_o+Sk_o)\sinh k_oh + SK\cosh k_oh\}(k_o+K)e^{-k_o(h-\eta)}\right] e^{-k_oy} H_o^{(1)}(k_or)\,/D \qquad (5.13)$$

where $D = [(1-2S)\sinh k_oh - \cosh k_oh][h\{k_o(1-S) - SK\}\cosh k_oh + (1-S-hK)\sinh k_oh]$

and

$$2\pi i\frac{(-1)^n}{n!}K^{n+1} \cdot \frac{[\sinh Kh + \cosh Kh][\sinh K(h+y) - \cosh K(h+y)]e^{-K(h-\eta)}H_o^{(-1)}(kr)}{(1-2S)\sinh Kh - \cosh Kh}$$

$$+\{\pi i\frac{k_o^{\,n}}{n!}(S-1)\sinh k_oh[\sinh k_o(h+y) - \frac{k_o}{K}\cosh k_o(h+y)]\cdot[(S-1)(k_o-K)e^{-k_o\eta}$$

$$+\frac{(-1)^n}{K}\{(K-k_o+Sk_o)\sinh k_oh + SK\cosh k_oh\}](k_o+K)e^{-k_o(h-\eta)}H_o^{(1)}(k_or)\}/D \qquad (5.14)$$

where $D = [(1-2S)\sinh k_oh - \cosh k_oh][h\{k_o(1-S) - SK\}\cosh k_oh + (1-S-hK)\sinh k_oh]$.

Now as h tends to infinity, (5.13) and (5.14) take respectively the following forms:

$$\frac{\pi i}{2SK}(S-1)\frac{M^n}{n!}(M-K)^2 e^{-M(y+\eta)} H_o^{(1)}(Mr)$$

$$\frac{\pi i}{2SK}\frac{M^n}{n!}(1 - S)(M - K)^2 e^{M(y - \eta)} H_o^{(1)}(Mr).$$

6. MULTIPOLE SUBMERGED IN LOWER FLUID.

Here, $\phi_1 \sim \dfrac{P_n(\cos\theta)}{R_o^{\,n+1}}$ as $R_o = [r^2 + (y - \eta)^2]^{\frac{1}{2}} \to 0$, $n = 1,2,\ldots$, where

$\theta = \tan^{-1}\left(\dfrac{r}{y - \eta}\right)$.

We assume

$$\phi_1 = \frac{P_n(\cos\theta)}{R_o^{\,n+1}} + \int_0^\infty A(k)e^{-ky}\, J_o(kr)dk \tag{6.1}$$

$$\phi_2 = \int_0^\infty [B(k)\cosh k(h + y) + C(k)\sinh k(h + y)]J_o(kr)dk. \tag{6.2}$$

We use

$$\frac{P_n(\cos\theta)}{R_o^{\,n+1}} = \frac{1}{n!}\int_0^\infty k^n\, e^{-k(y - \eta)}\, J_o(kr)dk, \qquad y > \eta$$

$$= \frac{(-1)^n}{n!}\int_0^\infty k^n\, e^{k(y - \ )} J_o(kr)dk, \qquad y < \eta$$

Proceeding much as in the previous case, we obtain

$$\begin{aligned}\phi_1 = {} & \frac{P_n(\cos\theta)}{R_o^{\,n+1}} + \int_0^\infty \frac{(-1)^n}{n!} k^n\, e^{-k\eta}\, J_o(kr)dk \\ & + \oint_0^\infty \frac{V}{k - K}\left(\frac{k}{K}\sinh kh - \cosh kh\right)e^{-ky}\, J_o(kr)dk \\ & + \oint_0^\infty \frac{V}{\Delta}(S - 1)\sinh kh\left(\frac{k}{K}\sinh kh - \cosh kh\right)e^{-ky}J_o(kr)dk\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\phi_2 = {} & \oint_0^\infty \frac{V}{k - K}\left[\sinh k(h + y) - \frac{k}{K}\cosh k(h + y)\right]J_o(kr)dk \\ & + \oint_0 \frac{V}{\Delta}(S - 1)\sinh kh\left[\sinh k(h - y) - \frac{k}{K}\cosh k(h + y)\right]J_o(kr)dk\end{aligned} \tag{6.5}$$

where,

$$V = \frac{2(-1)^n K\, k^n e^{-k\eta}}{n![(1 - 2S)\sinh kh - \cosh kh]} \tag{6.6}$$

and $\Delta(k)$ is the same expression used previously.

Now, as h tends to infinity, ϕ_1 and ϕ_2 take respectively the following forms:

$$\frac{P_n(\cos\theta)}{R_o^{\,n+1}} + \frac{(-1)^n}{n!}\oint_0^\infty \left\{1 + \frac{2M}{(1 + S)(k - M)}\right\}k^n e^{-k(y+\eta)}\, J_o(kr)dk \tag{6.7}$$

and,

$$\frac{2(-1)^n M}{n!(1+S)} \oint_0^\infty \frac{k^n}{k-M} e^{k(y-\eta)} J_o(kr)dk \tag{6.8}$$

where $M = K\frac{1+S}{1-S}$ which are the results derived by Gorgui and Kassem [3].

Proceeding much as in the previous case, we find that as $r \to \infty$, ϕ_1 and ϕ_2 respectively take the following forms:

$$2\pi i \frac{(-1)^n}{n!} \cdot \frac{K^{n+1} (\sinh Kh - \cosh Kh)e^{-K(y+\eta)} H_o^{(1)}(kr)}{(1-2S)\sinh Kh - \cosh Kh}$$

$$+ \frac{2 i \frac{(-1)^n}{n!} (S-1)Kk_o^n \sinh k_oh(\frac{k_o}{K}\sinh k_oh - \cosh k_oh)e^{k_o(y+\eta)}H_o^{(1)}(k_or)}{[(1-2S)\sinh k_oh - \cosh k_oh][h\{k_o(1-S) - SK\}\cosh k_oh + (1-S-hK)\sinh k_oh]} \tag{6.9}$$

and

$$2\pi i \frac{(-1)^n}{n!} K^{n+1} \cdot \frac{\sinh K(y+y) - \cosh K(h+y)}{(1-2S)\sinh Kh - \cosh Kh} H_o^{(1)}(Kr)$$

$$+\{2\pi i \frac{(-1)^n}{n!} (S-1)Kk_o^n \sinh k_oh[\sinh k_o(h+y) - \frac{k_o}{K}\cosh k_o(h+y)]e^{-k_o\eta}H_o^{(1)}(k_or)\}/D \tag{6.10}$$

where $D = [(1-2S)\sinh k_oh - \cosh k_oh][h\{k_o(1-S) - SK\}\cosh k_oh + (1-S-hK)\sinh k_oh]$

Now, as $h \to \infty$, (6.9) and (6.10) take respectively the following forms:

$$\pi i KM^n \frac{(-1)^n}{Sn!} (\frac{M}{K} - 1) e^{M(y+\eta)}H_o^{(1)}(Mr)$$

and

$$\pi i KM^n \frac{(-1)^n}{S(1-S)n!} (\frac{M}{K} - 1)e^{M(y-\eta)} H_o^{(1)}(Mr).$$

7. CONCLUSION.

Integral representation of the potential function in a two layered fluid medium where the upper layer is of finite depth with a free surface and lower layer is of infinite depth have been obtained. The different results reduce to the known results of Gorgui and Kassem [3] when the FS in the upper layer is taken to infinity (i.e., $h \to \infty$). We note that these authors thought there was no difficulty except longer equations and a bulkier result in adding the surface tension term in their problem, closely related to ours.

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