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NONOSCILLATION THEOREMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

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ABSTRACT. The authors give sufficient conditions for all oscillatory solutions of a sublinear forced higher order nonlinear functional differential equation to converge to zero. They then prove a nonoscillation theorem for such equations. A few intermediate results are also obtained.

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1. INTRODUCTION.

Consider the n-th order $(n \geq 2)$ nonlinear differential equation

$$(r(t)x^{(n-\nu)}(t))^{(\nu)} = f(t,x(t),x(g(t))) \tag{1.1}$$

where $1 \leq \nu \leq n-1$, and the functions $r, g : [t_0,\infty) \to R$ and $f : [t_0,\infty) \times R^2 \to R$ are continuous. We shall also require throughout that $r(t) > 0$ and $g(t) \to \infty$ as $t \to \infty$.

Our principal efforts here will be to obtain conditions which imply that certain classes of solutions of equation (1.1) are nonoscillatory and sufficient conditions to ensure that some classes of oscillatory solutions of (1.1) tend to zero as $t \to \infty$. The results obtained here for oscillatory solutions extend results previously given in [6, 8, 10, and 13], and the nonoscillation results extend those found in [1, 5-8, 10, and 13]. We note that the only nonoscillation theorems known for higher order ordinary nonlinear equations are due to Chen [1], Graef [4], and Staikos and Philos [13]. In the case of higher order functional equations the nonoscillation problem is considerably more difficult than it is for ordinary equations with the only results to date being those of Graef et al. [5-7] and Singh [10]. For a discussion of the particular difficults encountered in obtaining such results for functional differential equations the reader is referred to [8].

2. MAIN RESULTS

A solution $x(t)$ of (1.1) will be called oscillatory if its set of zeros is unbounded and it will be called nonoscillatory otherwise. Without further mention we note that the results here pertain only to the continuable solutions of (1.1), and that a number of them will be for classes of solutions of (1.1) which satisfy a growth condition of the form

$$|x(t)| = O(m(t)) \text{ as } t \to \infty \tag{2.1}$$

where $m : [t_0,\infty) \to R$ is positive and continuous. The asymptotic behavior of certain solutions of functional differential equations which satisfy conditions of type (2.1) has also been studied by other authors, for example Staikos and Sficas [14] and Graef et al. [6, 8].

We will assume in the remainder of this paper that the function f in (1.1) satisfies an estimate of the form

$$|f(t,x,y)| \le F(t,|x|,|y|) \tag{2.2}$$

where $F : [t_0,\infty) \times R_+^2 \to R_+$ is continuous and

$$F(t,u,v) \le F(t,u',v') \text{ for } 0 \le u \le u',\ 0 \le v \le v'.$$

It will also be convenient to use the notation

$$z(t) = r(t)x^{(n-\nu)}(t)$$

and

$$\omega_k(t) = \begin{cases} x^{(k)}(t), & \text{if } 0 \le k \le n-\nu-1 \\ z^{(k-n+\nu)}(t), & \text{if } n-\nu \le k \le n \end{cases}$$

throughout the paper. Before stating our first result we note that if $x(t)$ is an oscillatory solution of (1.1), then $\omega_k(t)$ also oscillates for $k = 1, 2, \ldots, n-1$. The following lemma provides some useful expressions for $\omega_k(t)$; they will be used in the remainder of this paper.

LEMMA 1. Let $T_1 \le T_2 \le \ldots \le T_n$ be such that

$$\omega_{n-j}(T_j) = 0,\ j = 1, 2, \ldots, n. \tag{2.3}$$

Then for any $t > T_n$ we have

$$\omega_{n-j}(t) = \int_{T_j}^{t} \int_{T_{j-1}}^{s_{j-1}} \cdots \int_{T_1}^{s_1} f(s,x(s),x(g(s)))ds ds_1 \ldots ds_{j-1} \tag{2.4}$$

for $1 \le j \le \nu$, and

$$\omega_{n-j}(t) = \int_{T_j}^{t} \cdots \int_{T_{\nu+1}}^{s_{\nu+1}} [1/r(s_\nu)] \int_{T_\nu}^{s_\nu} \cdots \int_{T_1}^{s_1} f(s,x(s),x(g(s)))ds ds_1 \ldots ds_{j-1} \tag{2.5}$$

for $\nu + 1 \le j \le n$.

PROOF. The conclusions of the lemma follow from (1.1) by integrating from T_j to t successively for $j = 1,2,\ldots,n$.

THEOREM 2. Suppose that (2.2) holds and

$$\int_{t_0}^{\infty} [s^{n-\nu-1}/r(s)] \int_{t_0}^{s} (s-u)^{\nu-1} F(u,cm(u),cm(g(u)))du ds < \infty \tag{2.6}$$

for any constant $c > 0$. If $x(t)$ is an oscillatory solution of (1.1) satisfying (2.1), then $\omega_k(t) \to 0$ as $t \to \infty$ for $k = 0,1,\ldots, n - \nu - 1$.

PROOF. Let $x(t)$ be an oscillatory solution of (1.1) satisfying (2.1). Then there exists $T > t_0$ and a constant $c > 0$ so that $|x(t)| \le cm(t)$ and $|x(g(t))| \le cm(g(t))$ for $t \ge T$. Choose T_j, $1 \le j \le n$, so that $T \le T_1 \le T_2 \le \ldots \le T_n$ and (2.3) holds. Integrating equation (1.1) j-times for $\nu + 1 \le j \le n$ we obtain (2.5). Note that if $t_1 < t_2 < t$, then

$$\left|\int_{t_2}^{t} \int_{t_1}^{s} P(u)du ds\right| \le \int_{t_2}^{t} \left|\int_{t_1}^{s} P(u)du\right| ds \le \int_{t_1}^{t} \left|\int_{t_1}^{s} P(u)du\right| ds \le \int_{t_1}^{t} \int_{t_1}^{s} |P(u)| du ds.$$

Hence from (2.1), (2.2), and (2.5) we have

$$|\omega_{n-j}(t)| \le \int_{T_1}^{t} \int_{T_1}^{s_{j-1}} \cdots \int_{T_1}^{s_{\nu+1}} [1/r(s_\nu)] \int_{T_1}^{s_\nu} \cdots$$

$$\int_{T_1}^{s_1} F(u,cm(u),cm(g(u)))du ds_1 \ldots ds_{j-1}. \tag{2.7}$$

Condition (2.6) implies that $\omega_k(t) = \omega_{n-j}(t) \to 0$ as $t \to \infty$ for $k = 0,1,\ldots, n - \nu - 1$.

REMARK. If $m(t) \equiv K$, where K is a positive constant, then from Theorem 2 we have that all bounded oscillatory solutions of (1.1) tend to zero as $t \to \infty$ together with $\omega_k(t)$ for $k = 1,2,\ldots, n - \nu - 1$.

In view of Theorem 2 it is reasonable to ask if it is possible to obtain a similar result but with $\omega_k(t) \to 0$ as $t \to \infty$ for all k independent of the value of i.e., $k = 0,1,\ldots, n-1$. If we choose $T_1 \le T_2 \le \ldots \le T_n$ such that $\omega_{n-j}(T_{n-j+1}) = 0$ for $j = 1,2,\ldots,n$ and then integrate equation (1.1) from $t < T_1$ to T_{n-j+1} for $j = 1,2,\ldots,n$ successively we would obtain expressions for $\omega_{n-j}(t)$ similar to (2.4) and (2.5) but having a factor of $(-1)^j$ and variable lower limits. If we would then replace (2.6) by

$$\int^{\infty}[s^{n-\nu-1}/r(s)]\int_s^{\infty}(u-s)^{\nu-1}F(u,cm(u),cm(g(u)))duds < \infty,$$

we would have that $\omega_k(t) \to 0$ as $t \to \infty$ for $k = 0,1,\ldots,n-1$. This approach however is not useful in obtaining the type of results we desire in the remainder of this paper. Due to an error in the way the T_j's were chosen and the integrations performed, several incorrect results have appeared in the literature (see the discussions in [6] and [12]). We will discuss this point further later in the paper.

In the next theorem we will ask that the function F satisfy the following sublinearity type condition. There exists a continuous function $H : [T_0,\infty) \to R$ such that

$$\limsup_{v \to \infty} F(t,v,v)/v \le H(t). \tag{2.8}$$

THEOREM 3. Let conditions (2.2), (2.6), and (2.8) hold with $m(t) \equiv K$ where K is any positive constant. If in addition

$$g(t) \le t \tag{2.9}$$

and

$$\int_{t_0}^{\infty}[s^{n-\nu-1}/r(s)]\int_{t_0}^{s}(s-u)^{\nu-1}H(u)duds < \infty, \tag{2.10}$$

then every oscillartory solution of (1.1) is bounded.

PROOF. Assume that the conclusion of the theorem does not hold. Then there is an unbounded oscillatory solution $x(t)$ of (1.1). Now (2.10) implies that there exists $t_1 > t_0$ so that

$$\int_{t_2}^{\infty}[s^{n-\nu-1}/r(s)]\int_{t_2}^{s}(s-u)^{\nu-1}H(u)duds < 1/4$$

for any $t_2 \ge t_1$, and (2.8) implies that there is a constant $v_1 > 0$ such that for $v \ge v_1$

$$F(t,v,v) \le 2vH(t)$$

on $[t_1,\infty)$. Let $t_1 < T < T_1 \le T_2 \le \ldots \le T_n$ be such that $g(t) \ge t_1$ for $t > T$ and (2.3) holds. Since $x(t)$ is oscillatory and unbounded, there exists an interval $[a,b]$ with $T_n < a < b$, $x(a) = x(b) = 0$, $|x(t)| > 0$ on (a,b), and

$$M = \max\{|x(t)| : a \le t \le b\} = \max\{|x(t)| : t_1 \le t \le b\} > v_1.$$

Notice that (2.9) ensures that $|x(g(t))| \le M$ for t in $[T,b]$. Moreover, from (2.2) and the properties of F we have

$$|f(t,x(t),x(g(t)))| \le F(t,|x(t)|,|x(g(t))|)$$
$$\le F(t,M,M)$$

for $T \le t \le b$. Next choose t_3 in (a,b) so that $M = |x(t_3)|$. Then from (2.7) it follows that

$$M = |x(t_3)| \le \int_{T_1}^{t_3} [s^{n-\nu-1}/r(s)] \int_{T_1}^{s} (s-u)^{\nu-1} F(u,M,M)\,du\,ds/(n-\nu-1)!(\nu-1)!$$
$$\le 2M \int_{T_1}^{\infty} [s^{n-\nu-1}/r(s)] \int_{T_1}^{s} (s-u)^{\nu-1} H(u)\,du\,ds < M/2$$

which is clearly impossible.

When equation (1.1) is sublinear we can combine Theroems 2 and 3 to obtain the following result.

THEOREM 4. If conditions (2.2), (2.6), and (2.8)-(2.10) hold with $m(t) \equiv K$ for any constant $K > 0$, then every oscillatory solution x(t) of (1) satisfies $\omega_k(t) \to 0$ as $t \to \infty$ for $k = 0,1,\ldots, n - \nu - 1$.

PROOF. Let x(t) be an oscillatory solution of (1.1), then by Theorem 3 x(t) is bounded. Thus the hypotheses of Theorem 2 are satisfied with $m(t) \equiv K$ for some constant $K > 0$, and hence the conclusion follows from Theorem 2.

REMARK. Theorems 2-4 above generalize Theorems 1 and 2 and Corollary 3 in [6], Theorems 1 and 3 and Corollary 2 in [8], Corollary 1 in [13], and special cases of Lemma 3.2 and Theorem 3.1 in [10] and Theorem 1 in [13]. For a discussion of known results of this type for second order equations, we refer the reader to references [6, 8, 10, and 13]. In these papers the authors unified and generalized much of what is known on this problem for second order equations.

Next we will give conditions which guarantee that certain classes of solutions of (1.1) are nonoscillatory. In so doing the connection between convergence to zero of oscillatory solutions and nonoscillation will become apparent. For this purpose we will need the existence of continuous functions $G : [T_0,\infty) \times R_+^2 \to R_+$ and $h : [t_0,\infty) \to R$ such that

$$G(t,u,v) \le G(t,u',v') \quad \text{for } 0 \le u \le u',\ 0 \le v \le v', \tag{2.11}$$

$$|f(t,x,y) - h(t)| \le G(t,|x|,|y|) \quad \text{for } x,\ y \in R, \tag{2.12}$$

$$\int_{t_0}^{\infty} [s^{n-\nu-1}/r(s)] \int_{t_0}^{s} (s-u)^{\nu-1} |h(u)|\,du\,ds < \infty, \tag{2.13}$$

and

$$\int_{t_0}^{\infty} [s^{n-\nu-1}/r(s)] \int_{t_0}^{s} (s-u)^{\nu-1} G(u,cm(u),cm(g(u)))\,du\,ds < \infty \tag{2.14}$$

for all $c > 0$.

THEOREM 5. Let (2.11) - (2.14) hold. Then all solutions of (1.1) that satisfy (2.1) are nonoscillatory provided there exists a constant $k > 0$ such that either

$$\liminf_{t \to \infty} \int_T^t [h(s) - G(s,k,k)]ds > 0 \tag{2.15}$$

or

$$\limsup_{t \to \infty} \int_T^t [h(s) + G(s,k,k)]ds < 0 \tag{2.16}$$

for all large T.

PROOF. Let $x(t)$ be an oscillatory solution of (1.1) satisfying (2.1) . Then from (2.11) and (2.12) we see that (2.2) holds with $F(t,u,v) = |h(t)| + G(t,u,v)$. This, in addition to (2.13) and (2.14), shows that all the hypotheses of Theorem 2 are satisfied and hence $x(t) \to 0$ as $t \to \infty$. Therefore there exists $t_1 \geq t_0$ so that $|x(t)| < k$ and $|x(g(t))| < k$ for $t \geq t_1$. Now choose $T > t_1$ so that $\omega_{n-1}(T) = (r(T)x^{(n-\nu)}(T))^{(\nu-1)} = 0$. From equation (1.1) and condition (2.12) we have

$$h(t) - G(t,k,k) \leq \omega_n(t) \leq h(t) + G(t,k,k)$$

for $t \geq T$. Integrating we obtain

$$\int_T^t [h(s) - G(s,k,k)]ds \leq \omega_{n-1}(t) \leq \int_T^t [h(s) + G(s,k,k)]ds.$$

If either (2.15) or (2.16) holds, then $\omega_{n-1}(t)$ eventually has fixed sign which contradicts the assumption that $x(t)$ is oscillatory.

The final theorem in this paper gives sufficient conditions for all solutions of (1.1) to be nonoscillatory.

THEOREM 6. Suppose that conditions (2.9) and (2.11) - (2.13) hold, G is sub-linear in the sense of condition (2.8), i.e. there exists $H_G : [t_0,\infty) \to R$ such that $\limsup_{v \to \infty} G(t,v,v)/v \leq H_G(t)$,

$$\int_{t_0}^{\infty} [s^{n-\nu-1}/r(s)] \int_{t_0}^{s} (s-u)^{\nu-1} H_G(u)duds < \infty, \tag{2.17}$$

and condition (2.14) holds with $m(t) \equiv K$ for any constant $K > 0$. If either (2.15) or (2.16) holds, then all solutions of equation (1.1) are nonoscillatory.

PROOF. Assume that (1.1) has an oscillatory solution $x(t)$. Clearly conditions (2.2) and (2.8) are satisfied with $F(t,u,v) = G(t,u,v) + |h(t)|$ and $H(t) = H_G(t) + |h(t)|$. Furthermore (2.13) and (2.17) imply that (2.10) holds; (2.13) and (2.14) holding with $m(t) \equiv K$ shows that (2.6) is satisfied. It then follows from Theorem 4 that $x(t) \to 0$ as $t \to \infty$. The remainder of the proof follows by proceeding as in the proof of Theorem 5 to again obtain a contradiction.

REMARK. Theorem 5 generalizes Theorem 5 in [6] and Theorem 4 in [8] while Theorem 6 generalizes Theorems 4 and 6 in [5], Theorem 4 in [6], Theorem 3 in [7], Theorem 5 in [8], Corollary 4 in [13], and special cases of Theorems 3,4,7, and 8 in [1] and Theorem 3.5 in [10].

As was mentioned earlier several incorrect results on the convergence to zero of oscillatory solutions of higher order ordinary and functional equations have appeared in the literature. This has in turn invalidated the nonoscillation criteria in these papers as well. In this regard we refer the reader to the discussions in [6] and [12] as well as the papers [2, 3, 9, and 11]. The only correct nonoscillation criteria for higher order equations to date appear in [1, 4-7, 10, and 13].

In order to illustrate Theorem 6 we will interpret this theorem for the forced higher order generalized Emden-Fowler type equation

$$[r(t)x^{(n-\nu)}]^{(\nu)} + q(t)b(x(g(t))) = a(t) \tag{2.18}$$

where $q, a : [t_0,\infty) \to R$ and $b : R \to R$ are continuous and r, g, and ν are as before. For equation (2.18) we ask that $g(t) \le t$, $b(x)$ is nondecreasing,

$$\limsup_{x \to \infty} |b(x)|/x < \infty,$$

$$\int_{t_0}^{\infty} [s^{n-\nu-1}/r(s)] \int_{t_0}^{s} (s-u)^{\nu-1} |a(u)| du ds < \infty,$$

$$\int_{t_0}^{\infty} [s^{n-\nu-1}/r(s)] \int_{t_0}^{s} (s-u)^{\nu-1} |q(u)| du ds < \infty,$$

and for some $k > 0$

$$\liminf_{t \to \infty} \int_T^t [a(s) - k|q(s)|] ds > 0 \tag{2.19}$$

for all large T. Then all solutions of (2.18) are nonoscillatory. Obviously (2.19) could be replaced by a condition analogous to (2.16).

As an example we consider the equation

$$[t^{\alpha}x^{(n-\nu)}]^{(\nu)} + (1/t^{\beta})x^{\gamma}(t^{\sigma}) = 1/t^{\varepsilon}, \quad t \ge 1 \tag{2.20}$$

where $0 < \sigma \le 1$ and γ is the ratio of odd positive integers with $0 < \gamma < 1$, $\alpha + \varepsilon > n$, $\alpha > n - 1$ and $\beta \ge \varepsilon$. Here (2.19) and all the other conditions listed above hold so by Theorem 6 all solutions of (2.20) are nonoscillatory. None of the nonoscillation criteria in [1, 4-7, 10, or 13] apply to this equation even when $\sigma = 1$.

In conclusion we note that although Theorem 5 does cover the case of superlinear equations (e.g. $b(x) = x^{\gamma}$, $\gamma > 1$) there are no known nonoscillation criteria for superlinear functional differential equations even in the case $n = 2$.

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