Internat. J. Math. & Math. Sci.
Vol. 7 No. 2 (1984) 209-233

FIELDS OF SETS, SET FUNCTIONS, SET FUNCTION INTEGRALS, AND FINITE ADDITIVITY

WILLIAM D. L. APPLING

Department of Mathematics
North Texas State University
Denton, Texas 76203-5117

(Received April 1, 1983)

Dedicated to the memory of H. S. Wall

ABSTRACT: This expository paper discusses, for set functions defined on a field of sets, some of the basic properties of set functions and set function integrals.

KEY WORDS AND PHRASES: Field of sets, set function, set function integral, finite additivity.
AMS (MOS) SUBJECT CLASSIFICATION: Primary 28A25; secondary 46E99

1. INTRODUCTION.

This expository article, minus certain of its remarks and references to names that appear in the literature, is offered primarily as a Socratic method textbook on the theory of real number set-valued functions defined on fields of sets. It is largely self-contained for the reader who has had a rigorous introduction to the elementary algebraic, ordering and limiting properties of the real number system, together with basic facts about continuous functions from certain subsets of $\mathbb{R}^N$ into $\mathbb{R}$. We have arranged matters in a progressive fashion intended to permit the results to be worked through with reasonable effort. Certain of our theorems are stated in such a way as to indicate finitely additive technique; our methods neither require assumptions about closure properties of infinite sequences of elements of the underlying field of sets, nor assumptions about countable additivity.

We give a short list of references following the text of this paper. This list is by no means complete; it is intended, more than anything else, to permit one to observe the relation of certain results that we treat to classical theorems in the field. However, we most strongly urge the reader choosing the course of action described in the first paragraph above to refrain from consulting any references (animate or inanimate) at all during the time that he or she, in working through the results of this paper, is struggling to further develop his or her intuition and powers of invention and reasoning.

The main ideas of this paper are certain of the fundamental ways in which set functions, as described above, and set function integrals (see section 2) interact. We end this introduction by giving an outline of the sections of this paper that follow, each section title followed, when necessary, by an indication of the principal topics covered.

2. PRELIMINARIES: Fields of sets, set functions and integrals.

3. Σ-BOUNDEDNESS, SUM SUPREMUM AND SUM INFIMUM FUNCTIONALS, UPPER AND LOWER INTEGRALS AND DIFFERENTIAL EQUIVALENCE.

4. CONCERNING THE SPACE OF REAL-VALUED BOUNDED FINITELY ADDITIVE SET FUNCTIONS: Basic boundedness, closure, and integral existence properties.

5. CONCERNING STIELTJES-TYPE SET FUNCTION INTEGRALS: Preservation of integrability theorems.

6. ABSOLUTE CONTINUITY FOR ELEMENTS OF $(AB)(\mathbb{R})(F)$: Representation and decomposition theorems.

7. INTEGRABILITY AND ABSOLUTE CONTINUITY: Integral value and existence characterizations of absolute continuity.

8. SET FUNCTION MEASURABILITY AND A CHARACTERIZATION THEOREM: An extension, for set functions, of the classical notion of point function measurability.

9. SET FUNCTION SUMMABILITY AND THE SUMMABILITY OPERATOR: An extension, for set functions, of the classical notion of point function (Lebesgue) integrability.

10. A DOMINATED CONVERGENCE THEOREM FOR SUMMABLE SET FUNCTIONS: An analogue, for set functions, of the Lebesgue dominated convergence theorem.

11. SUMMABILITY AND PRODUCT FIELDS: A Fubini-type theorem.

12. THE SPACE H^2_μ AND EXPLICIT FORMS OF THE BOCHNER-RADON-NIKODYM THEOREM: An elementary proof of a fundamental approximation theorem.

13. CONCERNING CLOSEST APPROXIMATIONS: Theorems involving general sufficient conditions for certain subsets of $(AB)(\mathbb{R})(F)$ to yield "nearest points", basic properties of "nearest point" operators.

14. CONCERNING A CLASS OF TRANSFORMATIONS: Representation, commutativity and reversibility theorems for a special collection of linear transformations.

15. A MAPPING THEOREM FOR A PAIR OF CLASSES OF TRANSFORMATIONS: Theorems about a correspondence between two classes of linear transformations arising from a subset of the type discussed in section 13 that is linear.

16. CONCERNING AN INTEGRAL EQUATION: Characterization of the existence of a solution to a certain integral equation; uniqueness and absolute continuity.

17. MORE THEOREMS ABOUT INTEGRAL REPRESENTATIONS: Converse-type questions to the matters of section 16.

18. FINITE ADDITIVITY, SET FUNCTIONS AND UPPER AND LOWER DISTRIBUTION FUNCTIONS: The development of an analogue, for finite additivity and set functions, of the standard notion of distribution function; characterization and representation theorems for integrability, measurability and summability.

2. PRELIMINARIES.

DEFINITION 2.1. The statement that F is a field of subsets of U means that U is a set and F is a collection, each element of which is a subset of U such that:

i) If each of A and B is in F, then $A \cup B$ is in F, and

ii) if A is F and $A \neq U$, then U - A is in F.

Now, for the sake of brevity, we shall suppose given a field, F, of subsets of a set U, and it will be F, together with its attendant notions defined in the course of our discussion, that will be referred to in most of our theorems. The exception to this will be certain characterization theorems involving the family of all fields of sets. We trust that the reader will not be confused by our convention, that upon defining a notion for our given F, we shall consider it "correspondingly" defined for any field F' of subsets of a set U'.

We have the following easy theorem:

THEOREM 2.1. The following statements are true:

1) U is in F.

2) If each of A and B is in F and there is an element common to A and B, then $A \cap B$ is in F.

3) If n is a positive integer and $\{A_k\}_{k=1}^n$ is a sequence of sets of F, then $\bigcup_{k=1}^n A_k$ is in F and, if there is z such that z is in A_k, k = 1,...,n, then $\bigcap_{k=1}^n A_k$ is in F.

DEFINITION 2.2. The statement that D is a subdivision of V means that V is in F and D is a finite subcollection of F such that no two elements of D have an element in common and the union of the elements of D is V.

DEFINITION 2.3. The statement that E is a refinement of D means that there is a set V of F such that each of E and D is a subdivision of V and each element of E is subset of some element of D.

NOTATION: "$P \ll Q$" means that P is a refinement of Q. Note that $D \ll \{V\}$ iff D is a subdivision of V.

THEOREM 2.2. If $H_1 \ll \{V\}$ and $H_2 \ll \{V\}$, then there is K such that $K \ll H_1$ and $K \ll H_2$.

DEFINITION 2.4. $\exp(\mathbb{R})(F)$ denotes the set of all functions from F into $\exp(\mathbb{R})$. If γ is a function from F into $\mathbb{R}$, then we shall regard γ as equivalent to the element δ of $\exp(\mathbb{R})(F)$ given by $\delta(I) = \{\gamma(I)\}$.

DEFINITION 2.5. The statement that b is an α-function on H means that α is in $\exp(\mathbb{R})(F)$, $H \ll \{V\}$ for some V in F and b is a function with domain H such that for each I in H, b(I) is in $\alpha(I)$.

We now define the basic limiting concept that underlies the results of this paper; we trust that the summation notation will be self explanatory.

DEFINITION 2.6. The statement that K is an integral of α on V means that K is in $\mathbb{R}$, α is in $\exp(\mathbb{R})(F)$, V is in F, and if $0 < c$, then there is $D \ll \{V\}$ such that if $E << D$ and b is an α-function on E, then

$$|K - \Sigma_E b(I)| < c. \tag{2.1}$$

THEOREM 2.3. If each of K_1 and K_2 is an integral of α on V, then $K_1 = K_2$.

DEFINITION 2.7. If K is an integral of α on V, then, by Theorem 2.3, K is the only integral of α on V, and we denote K by $\int_V \alpha(I)$.

DEFINITION 2.8. The statement that α is integrable on V means that there is K such that K is an integral of α on V.

OBSERVATION 2.1. If α is integrable on V, γ is in $\exp(\mathbb{R})(F)$ and, for each V in F, $\gamma(I) \subseteq \alpha(I)$, then γ is integrable on V and

$$\int_V \gamma(I) = \int_V \alpha(I). \tag{2.2}$$

THEOREM 2.4. If α is integrable on each of V_1 and V_2 and V_1 and V_2 are mutually exclusive, then α is integrable on $V_1 \cup V_2$ and

$$\int_{V_1 \cup V_2} \alpha(I) = \int_{V_1} \alpha(I) + \int_{V_2} \alpha(I) \tag{2.3}$$

We digress to give a convention that we shall use when brevity is needed. We shall let the statement, "$\int_V \alpha(I) = K$" mean that α is integrable on V and $\int_V \alpha(I) = K$. We shall let the statement "$\int_V \alpha(I)$ exists" mean that α is integrable on V.

In the light of Theorem 2.4, it is natural to consider the question of whether, if α is integrable on V, W is in F and $W \subseteq V$, then α is integrable on W. There are a number of well known arguments for this fact. The particular kind that the work of this paper points to arises from the notions developed in the next section; indeed the above mentioned fact is really a corollary to these results. Another consequence of these results, Theorem 3.5 of section 3, is well known, is pivotal in a large number of our deductions, and can be roughly described as a set function analogue of the Fundamental Theorem of Calculus.

3. Σ-BOUNDEDNESS, SUM SUPREMUM AND SUM INFIMUM FUNCTIONALS, UPPER AND LOWER INTEGRALS, AND DIFFERENTIAL EQUIVALENCE.

DEFINITION 3.1. The statement that α is Σ-bounded on V with respect to D means that α is in $\exp(\mathbb{R})(F)$, V is in F, $D << \{U\}$ and $\{\Sigma_E b(I) : E << \{V\}, E \subseteq \text{some } H << D,$ b an α-function on E$\}$ is bounded.

In what follows, when in a given discussion or expression it is clear what subdivision or subdivision element or elements are being referred to, we shall feel free to dispence with distinguishing superscripts and subscripts.

THEOREM 3.1. If α is Σ-bounded on U with respect to D, then, for each V in F, α is Σ-bounded on V with respect to D.

DEFINITION 3.2. If α is Σ-bounded on U with respect to D, then $L(\alpha)$ and $G(\alpha)$ denote, respectively, the element of $\exp(\mathbb{R})(F)$ given, for each V in F by sup(S) and inf(S), where

$$S = \{\Sigma_E b(I) : E << \{V\}, E \subseteq \text{some } H << D, \text{ b an } \alpha\text{-function on } E\}. \tag{3.1}$$

THEOREM 3.2. If α is Σ-bounded on U with respect to D, W is in F and P << {W}, Q << {W}, H << P and H << Q, then

$$\Sigma_P G(\alpha)(I) \leq \Sigma_H G(\alpha)(I) \leq \Sigma_H L(\alpha)(I) \leq \Sigma_Q L(\alpha)(I), \quad (3.2)$$

so that each of $L(\alpha)$ and $G(\alpha)$ is integrable on W and

$$\int_W G(\alpha)(I) \leq \int_W L(\alpha)(I), \quad (3.3)$$

equality holding iff α is integrable on W, in which case

$$\int_W G(\alpha)(I) = \int_W \alpha(I) = \int_W L(\alpha)(I). \quad (3.4)$$

THEOREM 3.3. If α is integrable on W, then, for each V in F such that $V \subseteq W$, is integrable on V.

We must pause here to consider some conventions.

DEFINITION 3.3. Suppose that N is a positive integer, S is a set, $M \subseteq S^N$, f is a function with domain M, W is a set, and $\{\alpha_k\}_{k=1}^N$ is a sequence of functions from W into exp(S) such that if x is in W, then $\alpha_1(x) X \ldots X \alpha_N(x) \subseteq M$. Then $f(\alpha_1,\ldots,\alpha_N)$ denotes the function with domain W such that if x is in W, then $f(\alpha_1,\ldots,\alpha_N)(x) = f(\alpha_1(x),\ldots,\alpha_N(x))$, which, in turn is defined to be $\{f(z_1,\ldots,z_N) : (z_1,\ldots,z_N)$ in $\alpha_1(x) X \ldots X \alpha_N(x)\}$.

DEFINITION 3.4. If α is integrable on U, and therefore integrable on V for each V in F, then $\int\alpha$ denotes the function with domain F given, for each V in F, by $\int_V \alpha(I)$.

Finally, we trust that in various statements and expressions, minor modifications of the notations in the above definitions will not be confusing.

We now state an elementary linearity theorem.

THEOREM 3.4. If each of α and β is integrable on V and each of p and q is in $\mathbb{R}$, then $p\alpha + q\beta$ is integrable on V and

$$\int_V [p\alpha(I) + q\beta(I)] = p\int_V \alpha(I) + q\int_V \beta(I). \quad (3.5)$$

THEOREM 3.5. If α is Σ-bounded on U with respect to D, H << E << D and, for each I in E, $H(I) = \{J : J \text{ in } H, J \subseteq I\}$, and b is an α-function on E and b' is an α-function on H, then

$$\Sigma_E |b(I) - \Sigma_{H(I)} b'(J)| \leq \Sigma_E [L(\alpha)(I) - G(\alpha)(I)], \quad (3.6)$$

so that, if α is integrable on U, then

$$\int_U |\alpha(I) - \int_I \alpha(J)| = 0. \quad (3.7)$$

THEOREM 3.6. If α is in $\exp(\mathbb{R})(F)$, D << {U} and 0 < K, then the following two statements are equivalent:

1) α is Σ-bounded on U with respect to D and

$$\int_U [L(\alpha)(I) - G(\alpha)(I)] \leq K, \quad (3.8)$$

and

2) If $0 < c$, then there is β in $\exp(\mathbb{R})(F)$ such that β is Σ-bounded on U with respect to D,

$$\int_U [L(\beta)(I) - G(\beta)(I)] \leq K, \tag{3.9}$$

and there is $H \ll \{U\}$ such that if $Q \ll H$ and w is an α-function on Q, then there is v, a β-function on Q such that

$$\Sigma_Q |x(I) - v(I)| < c. \tag{3.10}$$

We end this section with two corollaries which, in spite of their rather specialized appearance, apply very frequently and fundamentally in the work that follows.

COROLLARY 3.5. If each of γ and δ is in $\exp(\mathbb{R})(F)$, γ has bounded range union and δ is integrable on U, then

$$\int_U |\gamma(I)||\delta(I) - \int_I \delta(J)| = 0, \tag{3.11}$$

so that if V is in F, then $\int_V \gamma(I)\delta(I)$ exists iff $\int_V \gamma(I) \int_I \delta(J)$ exists, in which case equality holds.

COROLLARY 3.6. If each of α and β is integrable on U, then

$$\int_U [|\min\{\alpha(I),\beta(I)\} - \min\{\int_I \alpha(J),\int_I \beta(J)\}| + \max\{\alpha(I),\beta(I)\} - \max\{\int_I \alpha(J),\int_I \beta(J)\}|] = 0, \tag{3.12}$$

so that, if V is in F and Q is "max" or "min", then $\int_V Q(\alpha(I),\beta(I)\}$ exists iff $\int_V Q\{\int_I \alpha(J),\int_I \beta(J)\}$ exists, in which case equality holds.

4. CONCERNING THE SPACE OF REAL-VALUED BOUNDED FINITELY ADDITIVE SET FUNCTIONS.

DEFINITION 4.1. $(AB)(\mathbb{R})(F)$ denotes the set to which ξ belongs iff ξ is a function from F into a bounded subset of $\mathbb{R}$ such that if V_1 and V_2 are mutually exclusive sets of F, then

$$\xi(V_1 \cup V_2) = \xi(V_1) + \xi(V_2). \tag{4.1}$$

THEOREM 4.1. If ξ is in $(AB)(\mathbb{R})(F)$ and $E \ll D \ll \{U\}$, then

$$\Sigma_D |\xi(J)| \leq \Sigma_E |\xi(V)| \leq 2 \sup\{|\xi(W)| : W \text{ in } F\} = 2X, \tag{4.2}$$

so that $\int_U |\xi(I)|$ exists; furthermore,

$$X \leq \int_U |\xi(I)| \leq 2X. \tag{4.3}$$

In accordance with our suppositions concerning the reader's knowledge, we state the well-known theorem below in explicit form.

THEOREM 4.2. The following statements are true:

1) If each of ξ and μ is in $(AB)(\mathbb{R})(F)$, and each of r and s is in $\mathbb{R}$, then $r\xi + s\mu$ is in $(AB)(\mathbb{R})(F)$, and

$$\int_U |r\xi(I) + s\mu(I)| \leq |r|\int_U |\xi(I)| + |s|\int_U |\mu(I)|. \tag{4.4}$$

2) Suppose that $\{\xi_k\}_{k=1}^{\infty}$ is a sequence of elements of (AB)($\mathbb{R}$)(F) such that $\int_U |\xi_n(I) - \xi_m(I)| \to 0$ as $\min\{m,n\} \to \infty$. Then there is μ in (AB)($\mathbb{R}$)(F) such that $\int_U |\xi_n(I) - \mu(I)| \to 0$ as $n \to \infty$.

We now make some observations concerning the existence of integrals of the type discussed in Corollary 3.6.

LEMMA 4.1. If each of a, b, c and d is in $\mathbb{R}$, then

$$\min\{a,b\} + \min\{c,d\} \leq \min\{a+c,b+d\} \leq \max\{a+c,b+d\} \leq \max\{a,b\} + \max\{c,d\}. \tag{4.5}$$

THEOREM 4.3. If each of ξ and μ is in (AB)($\mathbb{R}$)(F), $D \ll \{V\}$, $E_1 \ll D$, and $E_2 \ll D$, then

$$\begin{aligned} -[\textstyle\int_V |\xi(I)| + \int_V |\mu(I)|] &\leq \Sigma_{E_1} \min\{\xi(J),\mu(J)\} \leq \Sigma_D \min\{\xi(I),\mu(I)\} \\ &\leq \Sigma_D \max\{\xi(I),\mu(I)\} \leq \Sigma_{E_2} \max\{\xi(J),\mu(J)\} \leq \textstyle\int_V |\xi(I)| + \int_V |\mu(I)|, \end{aligned} \tag{4.6}$$

so that each of $\min\{\xi,\mu\}$ and $\max\{\xi,\mu\}$ is integrable on V; furthermore, each of $\int \min\{\xi,\mu\}$ and $\int \max\{\xi,\mu\}$ is in (AB)($\mathbb{R}$)(F).

DEFINITION 4.2. (AB)($\mathbb{R}$)(F)$^+$ denotes the set to which ξ belongs iff ξ is a function from F into $\mathbb{R}^+ = \{x : x \text{ in } \mathbb{R},\ 0 \leq x\}$ such that if V_1 and V_2 are mutually exclusive sets of F, then

$$\xi(V_1 \cup V_2) = \xi(V_1) + \xi(V_2). \tag{4.7}$$

THEOREM 4.4. The following statements are true:

1) (AB)($\mathbb{R}$) (F)$^+$ $\subseteq$ (AB)($\mathbb{R}$)(F).

2) If ξ is in (AB)($\mathbb{R}$)(F), then $\int |\xi|$ is in (AB)($\mathbb{R}$)(F)$^+$.

3) (AB)($\mathbb{R}$)(F) = (AB)($\mathbb{R}$)(F)$^+$ - (AB)($\mathbb{R}$)(F)$^+$, i.e., ξ is in (AB)($\mathbb{R}$)(F) iff for some μ_1 and μ_2, each in (AB)($\mathbb{R}$)(F)$^+$,

$$\xi = \mu_1 - \mu_2. \tag{4.8}$$

DEFINITION 4.3. If α is in exp ($\mathbb{R}$)(F), then sgn(α) denotes the function from F into exp($\{-1,1\}$) such that for each V in F, sgn(α)(V) contains -1 iff $x < 0$ for some x in α(V), and sgn(α)(V) contains 1 iff $0 \leq x$ for some x in α(V).

We end this section with a theorem that bears directly on our discussion and which is typical of the ways that we shall apply differential equivalence, in this case Corollary 3.5.

THEOREM 4.5. If ξ is in (AB)($\mathbb{R}$)(F) and V is in F, then

$$\int_V \operatorname{sgn}(\xi)(I)\xi(I) = \int_V |\xi(I)| \tag{4.9}$$

and

$$\int_V \operatorname{sgn}(\xi)(I) \int_I |\xi(J)| = \int_V \operatorname{sgn}(\xi)(I)|\xi(I)| = \xi(V). \tag{4.10}$$

5. CONCERNING STIELTJES-TYPE SET FUNCTION INTEGRALS.

DEFINITION 5.1. $\exp(\mathbb{R})(F)(B)$ denotes the set of elements of $\exp(\mathbb{R})(F)$ with bounded range union, $\exp(\mathbb{R})(F)^+$ denotes the set of elements of $\exp(\mathbb{R})(F)$ with range union a subset of the nonnegative numbers, and $\exp(\mathbb{R})(F)(B)^+$ denotes $\exp(\mathbb{R})(F)(B) \cap \exp(\mathbb{R})(F)^+$.

THEOREM 5.1. If each of α and β is in $\exp(\mathbb{R})(F)(B)$, μ is in $(AB)(\mathbb{R})(F)^+$ and each of $\int_U \alpha(I)\mu(I)$ and $\int_U \beta(I)\mu(I)$ exists, then each of $\int_U \max\{\alpha(I),\beta(I)\}\mu(I)$ and $\int_U \min\{\alpha(I),\beta(I)\}\mu(I)$ exists. (Hint: Each of $\int\alpha\mu$ and $\int\beta\mu$ is in $(AB)(\mathbb{R})(F)$; consider Theorem 4.3 and Corollary 3.6.)

We now state a well-known fact about approximations of functions.

THEOREM 5.2. Suppose that $a < b$ and f is a function with domain $[a;b]$ and range $\subseteq \mathbb{R}$. Suppose that $0 < c$ and W is a finite collection of nonoverlapping intervals with union $[a;b]$ such that if $[p;q]$ is in W and each of y and z is in $[p;q]$, then $|f(y) - f(z)| < c$. Then, if x is in $[a;b]$, then

$$|f(x) - [f(a) + \Sigma_W[f|_p^q/(q-p)]\max\{\min\{x-p,q-p\},0\}]| < c. \qquad (5.1)$$

A consequence of Theorems 5.1, 5.2, uniform continuity and Theorems 3.4 and 3.6 is the following integral existence theorem:

THEOREM 5.3. Suppose that $a < b$, f is a function with domain $[a;b]$ and range $\subseteq \mathbb{R}$, continuous on $[a;b]$. Suppose that α is in $\exp(\mathbb{R})(F)(B)$ and that the range union of $\alpha \subseteq [a;b]$. Suppose that μ is in $(AB)(\mathbb{R})(F)^+$ and $\int_U \alpha(I)\mu(I)$ exists. Then $\int_U f(\alpha(I))\mu(I)$ exists.

The question naturally arises as to whether the condition of continuity on f of Theorem 5.3 could be replaced by something weaker. Not only is the answer no, but, as we shall see at the end of this section, functions having the sort of integrability preservation property described in the above theorem must be continuous. The lemmas that follow lead to a mathematical induction argument for a characterization theorem of the type described above.

LEMMA 5.1. If f, α and μ satisfy the hypothesis of Theorem 5.3 and for each x in $[a,b]$, $f(x) = x^2$, then $\int_U f(\alpha(I))\mu(I)$ exists.

Note: For f as given in Lemma 5.1, $f(\alpha)$ is not to be confused with $\alpha.\alpha$.

The statement of the next lemma is given in a form suggesting a method of proof.

LEMMA 5.2. Suppose that f, α and μ satisfy the hypothesis of Lemma 5.1 and β is an element of $\exp(\mathbb{R})(F)(B)$ with range union $\subseteq [a;b]$ such that $\int_U \beta(I)\mu(I)$ exists. Then, if V is in F, then

$$\alpha(V)\beta(V) \subseteq (\tfrac{1}{2})[f(\alpha(I)+\beta(I)) - f(\alpha(I)) - f(\beta(I))]; \qquad (5.2)$$

this implies that $\int_U \alpha(I)\beta(I)\mu(I)$ exists.

LEMMA 5.3. If α is in $\exp(\mathbb{R})(F)(B)$ and ξ is in $(AB)(\mathbb{R})(F)$, then $\int_U \alpha(I)\xi(I)$ exists iff $\int_U \alpha(I)\int_I |\xi(J)|$ exists. (Hint: Let β be $\mathrm{sgn}(\xi)$.)

The reader, in handling a certain half of the proof of the theorem below, will have to show the existence of a particular field of subsets of a set as well as the existence of set functions having particular properties.

THEOREM 5.4. Suppose that N is a positive integer, $\{[a_i;b_i]\}_{i=1}^{N}$ is a sequence of number intervals and f is a function from $[a_1;b_1]X\ldots X[a_N;b_N]$ into $\mathbb{R}$. Then the following two statements areequivalent:

1) If F' is a field of subsets of U', ξ is in (AB)($\mathbb{R}$)(F') and $\{\alpha_i\}_{i=1}^{N}$ is a sequence of elements of exp($\mathbb{R}$)(F')(B) such that for each i=1,...,N, the range union of $\alpha_i \subseteq [a_i;b_i]$ and $\int_U \alpha_i(I)\xi(I)$ exists, then $\int_U f(\alpha_1(I),\ldots,\alpha_N(I))\xi(I)$ exists, and

2) f is continuous.

6. ABSOLUTE CONTINUITY FOR ELEMENTS OF (AB)($\mathbb{R}$)(F).

There are intimate connections between absolute continuity, as defined below, and the sort of integrability questions discussed in the preceding section. In this section we develop fundamental facts about absolute continuity and mutual singularity, as the terms apply to (AB)($\mathbb{R}$)(F).

DEFINITION 6.1. If μ is in (AB)($\mathbb{R}$)(F)$^+$, then A_μ^+ denotes the set to which ξ belongs iff ξ is an element of (AB)($\mathbb{R}$)(F)$^+$ such that if $0 < c$, then there is $d > 0$ such that if V is in F and $\mu(V) < d$, then $\xi(V) < c$.

DEFINITION 6.2. If η is in (AB)($\mathbb{R}$)(F), then A_η denotes the set to which belongs iff ξ is in (AB)($\mathbb{R}$)(F) and $\int|\xi|$ is in A_μ^+, where $\mu = \int|\eta|$.

THEOREM 6.1. If ν is in (AB)($\mathbb{R}$)(F), then the following statements are true:

1) If each of ξ and μ is in A_ν and each of r and s is in $\mathbb{R}$, then each of $\int\max\{\xi,\mu\}$, $\int\min\{\xi,\mu\}$ and $r\xi + s\mu$ is in A_ν.

2) Statement 2) of Theorem 4.2 is true if "(AB)($\mathbb{R}$)(F)" is replaced by "A_ν".

DEFINITION 6.3. If μ is in (AB)($\mathbb{R}$)(F), then LIP(μ) denotes the set to which ξ belongs iff ξ is in (AB)($\mathbb{R}$)(F) and for some $K \geq 0$ and all V in F,

$$|\xi(V)| \leq K\int_V|\mu(I)|. \tag{6.1}$$

THEOREM 6.2. Suppose that μ is in (AB)($\mathbb{R}$)(F). Then ξ is in Lip(ξ) iff ξ is in (AB)($\mathbb{R}$)(F) and $\int|\xi|$ is in Lip(μ). Also,

$$\text{Lip}(\mu) \subseteq A_\mu. \tag{6.2}$$

The following lemma, with statement 1) of Theorem 6.1, is useful in proving Theorem 6.3 below.

LEMMA 6.1 If μ is in (AB)($\mathbb{R}$)(F)$^+$, η is in A_μ^+ and $\int_U \min\{\eta(I),\mu(I)\} = 0$, then $\eta(U) = 0$.

THEOREM 6.3 Suppose that each of ξ and μ is in (AB)($\mathbb{R}$)(F)$^+$ and λ is the element of exp($\mathbb{R}$)(F) given by

$$\lambda(V) = \sup\{\int_V \min\{\xi(I),K\mu(I)\} : 0 < K\}. \tag{6.3}$$

The following statements are true:

1) λ is in A_μ^+ and $\xi - \lambda$ is in (AB)($\mathbb{R}$)(F)$^+$;

2) $$\int_U \min\{\xi(I) - \lambda(I),\mu(I)\} = 0; \tag{6.4}$$

3) if each of ν and $\xi - \nu$ is in $(AB)(\mathbf{R})(F)^+$, then ν is in A_μ^+ iff $\lambda - \nu$ is in $(AB)(\mathbf{R})(F)^+$;

4) if each of ν and $\xi - \nu$ is in $(AB)(\mathbf{R})(F)^+$, then

$$\int_U \min\{\xi(I) - \nu(I), \mu(I)\} = 0 \tag{6.5}$$

iff $\nu - \lambda$ is in $(AB)(\mathbf{R})(F)^+$;

5) $$\int_U \min\{\xi(I) - \lambda(I), \lambda(I)\} = 0. \tag{6.6}$$

Now, the equation of statement 5) of the above theorem, though it has arisen in a very specific way, is of significance in more than one setting. We shall meet it again in the next section, in which we treat integrability characterizations of absolute continuity, and in section 14, in which we develop a general closest approximation and decomposition theorem. We end this section with a characterization theorem for this equation.

THEOREM 6.3. If each of μ, ρ and $\mu - \rho$ is in $(AB)(\mathbf{R})(F)^+$, then the following two statements are equivalent:

1) $$\int_U \min\{\mu(I) - \rho(I), \rho(I)\} = 0, \tag{6.7}$$

and

2) There is a function β from F into $\{0,1\}$ such that if V is in F, then

$$\int_V \beta(I)\mu(I) = \rho(V). \tag{6.8}$$

(Hint: Showing that 2) implies 1) is a fairly routine application of Corollary 3.6; in showing that 1) implies 2), a desired β can be defined quite briefly.)

7. INTEGRABILITY AND ABSOLUTE CONTINUITY.

We trust that it is clear to the reader that on the basis of Definition 6.2 and Lemma 5.3, we are not losing generality in this section by considering elements of $(AB)(\mathbf{R})(F)^+$, rather than merely of $(AB)(\mathbf{R})(F)$.

We begin with an immediate consequence of Theorem 6.3.

COROLLARY 7.1. If ξ, μ and λ are as in Theorem 6.3, then ξ is in A_μ^+ iff $\xi = \lambda$.

The following lemma is an immediate consequence of Theorem 6.1, and along with the corollary above, should prove useful.

LEMMA 7.1. If each of ξ and μ is in $(AB)(\mathbf{R})(F)^+$, then ξ is in A_μ^+ iff $\int \max\{\xi, \mu\}$ is in A_μ^+.

We now state the first of the two characterizations theorems of this section.

THEOREM 7.1. If each of ξ and μ is in $(AB)(\mathbf{R})(F)^+$, then the following three statements are equivalent:

1) If α is in $\exp(\mathbf{R})(F)(B)^+$, $\int_U \alpha(I)\mu(I) = 0$ and $\int_U \alpha(I)\mu(I)$ exists, then $\int_U \alpha(I)\xi(I) = 0$.

2) If γ is a function from F into $\{0,1\}$, $\int_U \gamma(I)\mu(I) = 0$ and $\int_U \gamma(I)\xi(I)$ exists, then $\int_U \gamma(I)\xi(I) = 0$.

3) ξ is in A_μ^+.

The second of our characterization theorems is a consequence of the first. We first state a lemma.

LEMMA 7.2. If η is in $(AB)(\mathbb{R})(F)^+$ and $\eta(U) > 0$, then there is δ in $\exp(\mathbb{R})(F)(B)^+$ such that $\int_U \delta(I)\eta(I)$ does not exist.

THEOREM 7.2. If each of ξ and μ is in $(AB)(\mathbb{R})(F)^+$, then the following three statements are equivalent:

1) If γ is in $\exp(\mathbb{R})(F)(B)$ and $\int_U \gamma(I)\mu(I)$ exists, then $\int_U \gamma(I)\xi(I)$ exists.

2) If γ is in $\exp(\mathbb{R})(F)(B)^+$ and $\int_U \gamma(I)\mu(I)$ exists, then $\int_U \gamma(I)\xi(I)$ exists.

3) ξ is in A_μ^+.

8. SET FUNCTION MEASURABILITY AND A CHARACTERIZATION THEOREM.

The reader may, if he chooses, consult any of a number of standard treatises on real analysis for the notion of a real-valued measurable function. The definition given below is clearly an extension of this notion to set functions.

DEFINITION 8.1. If μ is in $(AB)(\mathbb{R})(F)^+$, then M_μ is the set to which α belongs iff α is an element of $\exp(\mathbb{R})(F)$ such that

i) if $K_1 \le 0 \le K_2$, then $\int_U \max\{\min\{\alpha(I),K_2\},K_1\}\mu(I)$ exists, and

ii) if $0 < c$, then there is $K > 0$ and $D \ll \{U\}$ such that if $E \ll D$, b is an α-function on E and $E' = \{I : I \text{ in } E, |b(I)| > K\}$, then $\sum_{E'} \mu(I) < c$.

LEMMA 8.1. If N is a positive integer, h is a bounded continuous function from $\mathbb{R}^N$ into $\mathbb{R}$, μ is in $(AB)(\mathbb{R})(F)^+$ and $\{\alpha_i\}_i^N$ is a sequence of elements of M_μ, then $\int_U h(\alpha_1(I),\ldots,\alpha_N(I))\mu(I)$ exists.

THEOREM 8.1. If N is a positive integer and g is a function from $\mathbb{R}^N$ into $\mathbb{R}$, then the following two statements are equivalent:

1) If F' is a field of subsets of U', μ is in $(AB)(\mathbb{R})(F')^+$, and $\{\alpha_i\}_{i=1}^N$ is a sequence of elements of M_μ, then $g(\alpha_1\ldots,\alpha_N)$ is in M_μ.

2) g is continuous.

Before we go on to the next section, where in one of the theorems we encounter set function measurability again, we remark that Theorem 8.1 implies that if μ is in $(AB)(\mathbb{R})(F)^+$, each of α and β is in M_μ, each of r and s is in $\mathbb{R}$, then each of $r\alpha + s\beta, \alpha\beta$, $\max\{\alpha,\beta\}$ and $\min\{\alpha,\beta\}$ is in M_μ.

9. SET FUNCTION SUMMABILITY AND THE SUMMABILITY OPERATOR.

We begin by stating a definition which is an extension, to set functions, of the standard notion of summability.

DEFINITION 9.1. If μ is in $(AB)(\mathbb{R})(F)^+$, then S_μ is the set to which α belongs iff α is in $\exp(\mathbb{R})(F)$ and there is a number interval $[p;q]$ such that if $K_1 \le 0 \le K_2$, then $\int_U \max\{\min\{\alpha(I),K_2\},K_1\}\mu(I)$ exists and is in $[p;q]$.

THEOREM 9.1. If μ is in $(AB)(\mathbb{R})(F)^+$, α is in $\exp(\mathbb{R})(F)$ and for each number interval $[K_1;K_2]$ containing 0, $\int_U \max\{\min\{\alpha(I),K_2\},K_1\}\mu(I)$ exists, then the following three statements are equivalent:

1) α is in S_μ.

2) There is μ_1 and μ_2, each in A_μ^+, such that if $K_1 \le 0 \le K_2$, then

$$\int_U |\mu_2(I) - \int_I \max\{\min\{\alpha(J),K_2\},0\}\mu(J)| +$$

$$\int_U |\mu_1(I) + \int_I \max\{\min\{\alpha(J),0\},K_1\}\mu(J)| \to 0 \text{ as } \min\{-K_1,K_2\} \to \infty. \tag{9.1}$$

3) There is ξ in A_μ such that if $K_1 \le 0 \le K_2$, then

$$\int_U |\xi(I) - \int_I \max\{\min\{\alpha(J),K_2\},K_1\}\mu(J)| \to 0 \text{ as } \min\{-K_1,K_2\} \to \infty. \tag{9.2}$$

THEOREM 9.2. If μ is in $(AB)(\mathbb{R})(F)^+$, then $S_\mu \subseteq M_\mu$ (so that, by Theorem 8.1, if N is a positive integer, g is a continuous function from $\mathbb{R}^N$ into $\mathbb{R}$ and $\{\alpha_i\}_{i=1}^N$ is a sequence of elements of S_μ, then $g(\alpha_1,\ldots,\alpha_N)$ is in M_μ).

THEOREM 9.3. If μ is in $(AB)(\mathbb{R})(F)^+$, then α is in S_μ iff for each number interval $[K_1;K_2]$ containing 0, $\int_U \max\{\min\{\alpha(I),K_2\},K_1\}\mu(I)$ exists, and $|\alpha|$ is in S_μ.

THEOREM 9.4. If N is a positive integer and f is a function from $\mathbb{R}^N$ into $\mathbb{R}$, then the following two statements are equivalent:

1) If F' is a field of subsets of U', μ is in $(AB)(\mathbb{R})(F')^+$, and $\{\alpha_i\}_{i=1}^N$ is a sequence of elements of S_μ, then $f(\alpha_1,\ldots,\alpha_N)$ is in S_μ.

2) f is continuous, and $\{|f(x_1,\ldots,x_N)|/(\Sigma_{k=1}^N |x_k|) : 1 \le \Sigma_{k=1}^N |x_k|\}$ is bounded.

Now, we have the following corollary to Theorem 9.1 and definition; the corollary is really just an amended restatement of statement 3) of that theorem.

COROLLARY 9.1 and DEFINITION 9.2. Under the hypothesis of Theorem 9.1, α is in S_μ iff there is ξ in A_μ such that if $K_1 \le 0 \le K_2$, then $\int_U |\xi(I) - \int_I \max\{\min\{\alpha(J),K_2\},K_1\}\mu(I)| \to 0$ as $\min\{-K_1,K_2\} \to \infty$; ξ is the only element η of $(AB)(\mathbb{R})(F)$ such that if V is in F, then $\int_V |\eta(I) - \int_I \max\{\min\{\alpha(J),K_2\},K_1\}\mu(J)| \to 0$ as $\min\{-K_1,K_2\} \to \infty$. We denote ξ by $\sigma_\mu(\alpha)$.

THEOREM 9.5. If μ is in $(AB)(\mathbb{R})(F)^+$ and each of β and γ is in S_μ, then the following statements are true:

1) $\beta + \gamma$ is in S_μ, and

$$\sigma_\mu(\beta + \gamma) = \sigma_\mu(\beta) + \sigma_\mu(\gamma). \tag{9.3}$$

2) If Q is "max" or "min", then $Q(\beta,\gamma)$ is in S_μ and

$$\sigma_\mu(Q(\beta,\gamma)) = \int Q(\sigma_\mu(\beta),\sigma_\mu(\gamma)). \tag{9.4}$$

3) If β is in $\exp(\mathbb{R})(F)(B)$, then $\beta\gamma$ is in S_μ, and

$$\sigma_\mu(\beta\gamma) = \int[\beta\sigma_\mu(\gamma)]. \tag{9.5}$$

4) If, for some $H > 0$ and all x in the range union of β, $|x| \ge H$, then $1/\beta$ is in $\exp(\mathbb{R})(F)(B)$ (clearly), $\int_U (1/\beta(I))\mu(I)$ exists and is $\int_U (1/\beta(I)^2)\sigma_\mu(\beta)(I)$.

We direct the reader's attention to certain fundamental sum and exponent inequalities. We also adopt the convention that if each of x and y is a number, then x/y = 0 if y = 0, and has the usual meaning otherwise.

We end this section with a final existence and equality theorem.

THEOREM 9.6. If μ is in $(AB)(\mathbb{R})(F)^+$ and each of β and γ is in $S_\mu \cap \exp(\mathbb{R})(F)^+$, then the following statements are true:

1) If $0 < p < 1$, then $\beta^p\gamma^{1-p}$ is in S_μ, and, for each V in F,

$$\sigma_\mu(\beta^p\gamma^{1-p})(V) = \int_V(\sigma_\mu(\beta)(I))^p(\sigma_\mu(\gamma)(I))^{1-p}. \tag{9.6}$$

2) If $1 < p$, then β^p is in S_μ iff $\int_U(\sigma_\mu(\beta)(I)/\mu(I))^p\mu(I)$ exists, in which case, if V is in F, then

$$\sigma_\mu(\beta^p)(V) = \int_V(\sigma_\mu(\beta)(I)/\mu(I))^p\mu(I). \tag{9.7}$$

10. A DOMINATED CONVERGENCE THEOREM FOR SUMMABLE SET FUNCTIONS.

In this section we prove a convergence theorem for summability operators. We begin with two well known definitions.

DEFINITION 10.1. If W is a set, then the statement that P is a partial ordering on W means that $P \subseteq W \times W$ such that:

i) if x is in W, then (x,x) is in P, and

ii) if each of (x,y) and (y,z) is in P, then (x,z) is in P.

DEFINITION 10.2. If W is a set, then the statement that P is a partial ordering with respect to which W is directed means that P is a partial ordering on W such that if each of x and y is in W, then there is z in W such that each of (x,z) and (y,z) is in P.

We now state a lemma.

LEMMA 10.1. Suppose that γ is in $\exp(\mathbb{R})(F)(B)$, μ is in $(AB)(\mathbb{R})(F)^+$, W is a set, P is a partial ordering with respect to which W is directed, α is a function from W into S_μ, and K is a number such that if x is in the range union of γ or $\alpha(t)$ for some t in W, then $|x| \leq K$. Suppose further that if $0 < \min\{c,d\}$, then there is z in W such that if (z,y) is in P, then there is $D_y \ll \{U\}$ such that if $E \ll D_y$, h is a γ-function on E and m is an $\alpha(y)$-function on E and $E^* = \{I : I \text{ in } E, |h(I) - m(I)| \geq c\}$, then $\Sigma_{E^*}\mu(I) < d$. Then $\int_U\gamma(I)\mu(I)$ exists, and $\int_U|\gamma(I) - \alpha(t)(I)|\mu(I) \to 0$ with respect to P, i.e., (and we trust that the reader can gather a general definition from this paraphrase) if $0 < c$, then there is v in W such that if (v,t) is in P, then

$$\int_U|\gamma(I) - \alpha(t)(I)|\mu(I) < c.$$

Now, the use of a set directed with respect to a partial ordering, instead of just the set of positive integers, is not generalization for its own sake. We will need such an ordering to develop a finitely additive analogue of Fubini's Theorem in section 11.

Here is our dominated convergence theorem.

THEOREM 10.1. Suppose that γ is in $\exp(\mathbb{R})(F)$, μ is in $(AB)(\mathbb{R})(F)^+$, W is a set, P is a partial ordering with respect to which W is directed, α is a function from W into S_μ, ρ is in A_μ^+ and for each t in W, $\rho - \int|\sigma_\mu(\alpha(t))|$ is in $(AB)(\mathbb{R})(F)^+$, Suppose further that if $0 < \min\{c,d\}$, then there is z in W such that if (z,y) is in P, then there is $D_y << \{U\}$ such that if $E << D_y$, h is a γ-function on E and m is an $\alpha(y)$-function on E and $E^* = \{I : I \text{ in } E, |h(I) - m(I)| \geq c\}$, then $\Sigma_{E^*}\mu(I) < d$. Then γ is in S_μ, and $\int_U|\sigma_\mu(\gamma)(I) - \sigma_\mu(\alpha(t))(I)| \to 0$ with respect to P.

11. SUMMABILITY AND PRODUCT FIELDS.

We begin this section with some special cases of well-known theorems about "product fields" of collections of fields of sets.

THEOREM 11.1. Suppose that F_1 is a field of subsets of U_1 and F_2 is a field of subsets of U_2. Let $R_p = \{X_1 \times X_2 : X_1 \text{ in } F_1 \text{ and } X_2 \text{ in } F_2\}$. Let F_3denote $\{\bigcup_H Y : H$ a finite subcollection of R_p, no two elements of which have an element in common$\}$. Then F_3 is a field of subsets of $U_1 \times U_2$ such that if F_0 is a field of subsets of $U_1 \times U_2$ such that $R_p \subseteq F_0$, then $F_3 \subseteq F_0$.

THEOREM 11.2. Suppose that F_1, U_1, F_2, U_2, R_p and F_3 are as in Theorem 11.1. Suppose that μ_1 is in $(AB)(\mathbb{R})(F_1)^+$ and μ_2 is in $(AB)(\mathbb{R})(F_2)^+$. Then there is exactly one element μ_3 in $(AB)(\mathbb{R})(F_3)^+$ such that if X_1 is in F_1 and X_2 is in F_2, then

$$\mu_3(X_1 \times X_2) = \mu_1(x_1)\mu_2(X_2). \tag{11.1}$$

Let us now suppose that F_1, U_1, F_2, U_2, F_3, μ_1, μ_2 and μ_3 are as in Theorem 11.2, α is in $\exp(\mathbb{R})(F_3)$, and β is a function with domain $\{(x,I,y,J) : I \text{ in } F_1, J \text{ in } F_2, (x,y) \text{ in } I \times J\}$ such that if I is in F_1, J in F_2, x is in I and y is in J, then $\beta(x,I,y,J) \subseteq \alpha(I \times J)$.

We now state two theorems, the second of which is a consequence and generalization of the first.

THEOREM 11.3. Suppose the α has bounded range union and $\int_{U_1 \times U_2}\alpha(V)\mu_3(V)$ exists. Then, if Q is either L or G (see section 3), then each of the integrals

$$\int_{U_1}[\int_{U_2}Q(\beta(x,I,\ldots)\mu_2)(W'')]\mu_1(I) \tag{11.2}$$

and

$$\int_{U_2}[\int_{U_1}Q(\beta(\ldots,y,J)\mu_1)(W')]\mu_2(J) \tag{11.3}$$

exists and is

$$\int_{U_1 \times U_2}\alpha(V)\mu_3(V). \tag{11.4}$$

THEOREM 11.4. Suppose that α is in S_{μ_3}, and that if $p \leq 0 \leq q$ and γ is a function whose range $\subseteq \exp(\mathbb{R})$, then $\gamma_{p,q} = \max\{\min\{\gamma,q\},p\}$. Then, if Q is L or G, then

$$\int_{U_1} |[\sigma_{\mu_3}(\alpha)(V\times U_2)/\mu_1(V)] - \int_{U_2} Q(\beta_{p,q}(x,V,\ldots)\mu_2)(W'')|\mu_1(V) =$$

$$\int_{U_1} |\sigma_{\mu_3}(\alpha)(V\times U_2) - \int_V[\int_{U_2} Q(\beta_{p,q}(x,I,\ldots)\mu_2)(W'')]\mu_1(I)| \leq \tag{11.5}$$

$$\int_{U_1\times U_2} |\sigma_{\mu_3}(\alpha)(Y) - \int_Y \alpha_{p,q}(Z)\mu_3(Z)| \to 0, \ \min\{-p,q\}\to\infty,$$

and

$$\int_{U_2} |[\sigma_{\mu_3}(\alpha)(U_1\times X)/\mu_2(X)] - \int_{U_1} Q(\beta_{p,q}(\ldots,y,X)\mu_1)(W')|\mu_2(X) =$$

$$\int_{U_2} |\sigma_{\mu_3}(\alpha)(U_1\times X) - \int_X[\int_{U_1} Q(\beta_{p,q}(\ldots,y,J)\mu_1)(W')]\mu_2(J)| \leq \tag{11.6}$$

$$\int_{U_1\times U_2} |\sigma_{\mu_3}(\alpha)(Y) - \int_Y \alpha_{p,q}(Z)\mu_3(Z)| \to 0, \ \min\{-p,q\}\to\infty.$$

12. THE SPACE H^2_μ AND EXPLICIT FORMS OF THE BOCHNER RADON NIKODYM THEOREM.

Suppose that μ is in $(AB)(\mathbb{R})(F)^+$. We let H^2_μ denote the set to which ξ belongs iff ξ is in $(AB)(\mathbb{R})(F)$, for each V in F, $\xi(V) = 0$ if $\mu(V) = 0$, and $\int_U[\xi(I)^2/\mu(I)]$ exists.

Certain of the results of this section are special cases of matters treated in statement 2) of Theorem 9.6. We begin, even so, with a lemma from which, by now, the reader should be able to deduce a useful refinement-sum inequality.

LEMMA 12.1. If each of p, q, r and s is a number such that $0 \leq \min\{r,s\}$ and $p = 0$ if $r = 0$ and $q = 0$ if $s = 0$, then $(p+q)^2/(r+s) \leq p^2/r + q^2/s$.

THEOREM 12.1. The following statements are true:

1) If ξ is in $(AB)(\mathbb{R})(F)$, then the following statements are equivalent:

i) ξ is in H^2_μ.

ii) For each V in F, $\xi(V) = 0$ if $\mu(V) = 0$, and $\{\Sigma_D[\xi(V)^2/\mu(V)] : D \ll \{U\}\}$ is bounded.

iii) For some ρ in $(AB)(\mathbb{R})(F)^+$, $\rho\mu-\xi^2$ is in $\exp(\mathbb{R})(F)^+$.

2) If each of η and ζ is in H^2_μ and each of r and s is in $\mathbb{R}$, then $r\eta + s\zeta$ is in H^2_μ.

3) If each of η and ζ is in H^2_μ, then $\int_U[\eta(I)\zeta(I)/\mu(I)]$ exists.

4) Suppose that $\{\eta_i\}_{i=1}^\infty$ is a sequence of elements of H^2_μ such that

$\int_U[(\eta_m(I) - \eta_n(I))^2/\mu(I)]\to 0$ as $\min\{m,n\}\to\infty$,

then there is ξ in H^2_μ such that $\int_U[(\xi(I) - \eta_n(I))^2/\mu(I)]\to 0$ as $n\to\infty$.

The reader, in working through the above theorem, has most likely seen that, in statement 2), a sum and exponent inequality of the type mentioned just before Theorem 9.6 comes into play. We suggest now that the reader formulate a set function integral extension of this inequality; it will be effective in establishing Theorem 12.2 below.

We now proceed with our development of a fundamental approximation theorem.

LEMMA 12.2. $\mathrm{Lip}(\mu) \subseteq H^2_\mu$.

THEOREM 12.2. If η is in $\mathrm{Lip}(\mu)$ and $D \ll \{U\}$, then

$$\Sigma_D \int_V |\eta(I) - [\eta(V)/\mu(V)]\mu(I)| \le [\int_U [\eta(I)^2/\mu(I)] - \Sigma_D[\eta(V)^2/\mu(V)]]^{\frac{1}{2}}[\mu(U)]^{\frac{1}{2}} \quad (12.1)$$

THEOREM 12.3. If ξ is in A_μ, then

$$\int_U |\xi(I) - \int_I \max\{\min\{\xi(J), K_2\mu(J)\}, K_1\mu(J)\}| \to 0 \text{ as } \min\{-K_1, K_2\} \to \infty. \quad (12.2)$$

THEOREM 12.4. If ξ is in A_μ, then

$$\int_U [\int_V |\xi(I) - [\xi(V)/\mu(V)]\mu(I)|] = 0, \quad (12.3)$$

i.e., if $0 < c$, then there is $D \ll \{U\}$ such that if $E \ll D$, then $\Sigma_E \int_V |\xi(I) - [\xi(V)/\mu(V)]\mu(I)| < c$.

THEOREM 12.5. If α is in $\exp(\mathbb{R})(F)$ and $\int_U \alpha(I)\mu(I)$ exists, then the following two statements are equivalent:

1) $\int \alpha\mu$ is in A_μ.

2) $$\int_U [\int_V | \int_I \alpha(J)\mu(J) - \alpha(V)\mu(I)|] = 0, \quad (12.4)$$

i.e., if $0 < c$, then there is $D \ll \{U\}$ such that if $E \ll D$ and b is an α-function on E then $\Sigma_E \int_V | \int_I \alpha(J)\mu(J) - b(V)\mu(I)| < c$.

The above theorem has considerable application; indeed we shall open the next section with a corollary of it which we shall use not only in that section, but in the next as well. Furthermore, we shall consider a special "nonintegrable" form of it on our development of an extension of the notion of distribution function to set functions.

13. CONCERNING CLOSEST APPROXIMATIONS.

We begin with the corollary mentioned at the end of section 12.

COROLLARY 13.1. Suppose that $W \subseteq (AB)(\mathbb{R})(F)$ and if λ is in W, then $A_\lambda \subseteq W$. Suppose that $K \ge 0$ and T is a transformation from W into $(AB)(\mathbb{R})(F)$ such that if each of ρ and μ is in W and V is in F, then

$$|T(\rho)(V) - T(\mu)(V)| \le K\int_V |\rho(I) - \mu(I)|. \quad (13.1)$$

Then, if λ is in W, α is in $\exp(\mathbb{R})(F)(B)$ and $\int_U \alpha(I)\lambda(I)$ exists, then

$$\int_U [\int_V |T(\int \alpha\lambda)(I) - T(\alpha(V)\lambda)(I)\ ] = 0, \quad (13.2)$$

i.e., if $0 < c$, then there is $D \ll \{U\}$ such that if $E \ll D$ and b is an α-function on E, then $\Sigma_E \int_V |T(\int \alpha\lambda)(I) - T(b(V)\lambda)(I)| < c$.

DEFINITION 13.1. The statement that M is a C-set means that $M \subseteq (AB)(\mathbb{R})(F)$ and M satisfies the following two conditions:

i) If ρ is in M, μ is in $(AB)(\mathbb{R})(F)$ and $\int|\rho| - \int|\mu|$ is in $(AB)(\mathbb{R})(F)^+$, then μ is in M, and

ii) If ξ is in $(AB)(\mathbb{R})(F)^+$ and α is the element of $\exp(\mathbb{R})(F)$ given by

$$\alpha(V) = \sup\{\gamma(V) : \gamma \text{ in } M \bigcap (AB)(\mathbb{R})(F)^+,\ \xi-\gamma \text{ in } (AB)(\mathbb{R})(F)^+\}, \tag{13.3}$$

then α is in $M \bigcap (AB)(\mathbb{R})(F)^+$.

Clearly, if μ is in $(AB)(\mathbb{R})(F)^+$, then A_μ is a C-set and is just one of many C-sets. We shall consider some further examples in this paper.

We now assume that M is a C-set.

The three lemmas that follow lead to the principal result of this section, namely a closest approximation theorem for C-sets.

LEMMA 13.1. If each of λ and μ is in $M \bigcap (AB)(\mathbb{R})(F)^+$, then so is $\int\max\{\lambda,\mu\}$.

DEFINITION 13.2. ℓ_M denotes the transformation with domain $(AB)(\mathbb{R})(F)$ and range $\subseteq \exp(\mathbb{R})(F)$ given by

$$\ell_M(\xi)(V) = \sup\{\gamma(V) : \gamma \text{ in } M \bigcap (AB)(\mathbb{R})(F)^+,\ \int|\xi| - \gamma \text{ in } (AB)(\mathbb{R})(F)^+\}. \tag{13.4}$$

Clearly the range of $\ell_M \subseteq M \bigcap (AB)(\mathbb{R})(F)^+$.

LEMMA 13.2. If ρ is in $(AB)(\mathbb{R})(F)^+$, μ is in $M \bigcap (AB)(\mathbb{R})(F)^+$ and $\mu \neq \ell_M(\rho)$, then

$$\int_U |\rho(I) - \ell_M(\rho)\ (I)| < \int_U |\rho(I) - \mu(I)|. \tag{13.5}$$

DEFINITION 13.3. a^*_M denotes the transformation with domain $(AB)(\mathbb{R})(F)$ and range $\subseteq \exp(\mathbb{R})(F)$ given by

$$a^*_M(\xi)(V) = \int_V \operatorname{sgn}(\xi)(I)\ell_M(\xi)(I) \tag{13.6}$$

(why does this integral exist?)

We shall for the remainder of this section, let ℓ denote ℓ_M and a^* denote a^*_M.

LEMMA 13.3. The range of $a^* \subseteq M$.

THEOREM 13.1. If ξ is in $(AB)(\mathbb{R})(F)$, μ is in M and $\mu \neq a^*(\xi)$, then

$$\int_U |\xi(I) - a^*(\xi)(I)| < \int_U |\xi(I) - \mu(I)|. \tag{13.7}$$

The question, stated in intuitive terms, that naturally arises is: "If ξ_1 is 'close to' ξ_2, is $a^*(\xi_1)$ 'close to' $a^*(\xi_2)$?" We begin by considering the following four lemmas:

LEMMA 13.4. If each of μ and ζ is in $(AB)(\mathbb{R})(F)^+$, then $\int\max\{\ell(\mu),\ell(\zeta)\} = \ell(\int\max\{\mu,\zeta\})$ and $\int\min\{\ell(\mu),\ell(\zeta)\} = \ell(\int\min\{\mu,\zeta\})$.

LEMMA 13.5. If each of ρ, μ, and $\rho - \mu$ is in $(AB)(\mathbb{R})(F)^+$, then so is $\rho - \mu - [\ell(\rho) - \ell(\mu)]$.

LEMMA 13.6. If each of ρ and μ is in $(AB)(\mathbb{R})(F)^+$, then so is $\int[|\rho-\mu|-|\ell(\rho)-\ell(\mu)|]$.

LEMMA 13.7. If each of ξ and ρ is in $(AB)(\mathbb{R})(F)$ and V is in F, then

$$\int_V L(|\mathrm{sgn}(\xi) - \mathrm{sgn}(\rho)|\int|\xi|)(I) \le \int_V[||\xi(I)|-|\rho(I)|| + |\xi(I)-\rho(I)|]. \qquad (13.8)$$

THEOREM 13.2. If each of ξ and ρ is in $(AB)(\mathbb{R})(F)$ and V is in F, then

$$\int_V |a^*(\xi)(I)-a^*(\rho)(I)| \le \int_V[2||\xi(I)|-|\rho(I)||+|\xi(I)-\rho(I)|]. \qquad (13.9)$$

We give, after stating two lemmas, a functional equation theorem that is an extension of Lemma 13.4.

LEMMA 13.8. If μ is in $(AB)(\mathbb{R})(F)^+$, β is a function from F into $\exp(\{0,1\})$ and $\int_U \beta(I)\mu(I)$ exists, then

$$\int\beta\ell(\mu) = \ell(\int\beta\mu). \qquad (13.10)$$

LEMMA 13.9. If μ is in $(AB)(\mathbb{R})(F)$, then

$$a^*(\int\max\{\mu,0\}) = \int\max\{a^*(\mu),0\}, \text{ and } a^*(\int\min\{\mu,0\}) = \int\min\{a^*(\mu),0\}. \qquad (13.11)$$

THEOREM 13.3. If each of ξ and μ is in $(AB)(\mathbb{R})(F)$, then

$$a^*(\int\max\{\xi,\mu\}) = \int\max\{a^*(\xi),a^*(\mu)\} \text{ and } a^*(\int\min\{\xi,\mu\}) = \int\min\{a^*(\xi),a^*(\mu)\}. \qquad (13.12)$$

We end this section with an "addition and scalar multiplication" closure characterization theorem and some observations.

THEOREM 13.4. The following four statements are equivalent:

1) If each of ρ and μ is in M and each of r and s is in $\mathbb{R}$, then $r\rho + s\mu$ is in M.

2) If ξ is in $(AB)(\mathbb{R})(F)^+$, then

$$\ell(\xi - \ell(\xi))(U) = 0. \qquad (13.13)$$

3) If ξ is in $(AB)(\mathbb{R})(F)^+$, then

$$\int_U \min\{\xi(I) - \ell(\xi)(I), \ell(\xi)(I)\} = 0 \qquad (13.14)$$

4) If μ is in $M \bigcap (AB)(\mathbb{R})(F)^+$, then so is 2μ.

THEOREM 13.5. Suppose that M satisfies one of the conditions of Theorem 13.4. Then the following statements are true:

1) If each of ρ and μ is in $(AB)(\mathbb{R})(F)^+$ and $0 \le r$, then each of $\ell(\rho) + \ell(\mu) - \ell(\rho + \mu)$ and $r\ell(\rho) - \ell(r\rho)$ is in $(AB)(\mathbb{R})(F)^+$.

2) If ξ is in $(AB)(\mathbb{R})(F)$, μ is in M and $\mu \neq a^*(\xi)$, then

$$0 = \int_U|a^*(\xi - a^*(\xi))(I)| < \int_U|a^*(\xi - \mu)(I)|. \qquad (13.15)$$

3) If each of ξ and μ is in $(AB)(\mathbb{R})(F)$ and each of r and s is in $\mathbb{R}$, then

$$a^*(r\xi + s\mu) = ra^*(\xi) + sa^*(\mu). \qquad (13.16)$$

We point out that the following subsets of $(AB)(\mathbb{R})(F)$ are C-sets, the last three also satisfy condition 1) of Theorem 13.4.

Ex. 13.1. For μ in $(AB)(\mathbb{R})(F)^+$ and $0 \leq K$, $\{\xi : \xi$ in $(AB)(\mathbb{R})(F)$, $|\xi(V)| \quad K\mu(V)$ for all V in $F\}$.

Ex. 13.2. Given a collection G of C-sets, $\bigcap_G X$.

Ex. 13.3. For μ in $(AB)(\mathbb{R})(F)^+$, A_μ.

Ex. 13.4. For α in $\exp(\mathbb{R})(F)(B)$, $\{\xi : \xi$ in $(AB)(\mathbb{R})(F)$, $\int_U \alpha(I)\xi(I)$ exists$\}$.

Ex. 13.5. $\{\xi : \xi$ in $(AB)(\mathbb{R})(F)$, $\int_U [\xi(I)]^2 = 0\}$.

14. CONCERNING A CLASS OF TRANSFORMATIONS.

We begin by considering a special case. Suppose that μ is in $(AB)(\mathbb{R})(F)^+$. As observed in section 13, A_μ satisfies each of the conditions of Theorem 13.4 and so, for $M = A_\mu$ and $a^*_\mu = a^*_M$, we see that a^*_μ has the properties given in Theorem 13.5.

Throughout this section, for each μ in $(AB)(\mathbb{R})(F)^+$, a^*_μ shall have the meaning given in the above papagraph.

We now define collection of transformations from $(AB)(\mathbb{R})(F)$ into $(AB)(\mathbb{R})(F)$ which, for each μ in $(AB)(\mathbb{R})(F)^+$, contains a^*_μ.

DEFINITION 14.1. C denotes the collection to which T belongs iff T is a transformation from $(AB)(\mathbb{R})(F)$ into $(AB)(\mathbb{R})(F)$ such that for some $K \geq 0$ and all ξ and μ in $(AB)(\mathbb{R})(F)$ and all r and s in $\mathbb{R}$,

i) $T(r\xi + s\mu) = rT(\xi) + sT(\mu)$, and

ii) $K\int|\xi| - \int|T(\xi)|$ is in $(AB)(\mathbb{R})(F)^+$.

OBSERVATION 14.1. If each of T_1 and T_2 is in C and each of r and s is in $\mathbb{R}$, THEN $rT_1 + sT_2$ is in C.

THEOREM 14.1. If μ is in $(AB)(\mathbb{R})(F)^+$, ξ is in A_μ and T is in C, then, for each V in F,

$$\int_V [T(\mu)(I)/\mu(I)]\xi(I) = T(\xi)(V). \tag{14.1}$$

THEOREM 14.2. If μ is in $(AB)(\mathbb{R})(F)^+$, T is in C, and η is in $(AB)(\mathbb{R})(F)$, then

$$T(a^*_\mu(\eta)) = a^*_\mu(T(\eta)). \tag{14.2}$$

THEOREM 14.3. If μ is in $(AB)(\mathbb{R})(F)^+$, each of S and T is in C, and for each ξ in $(AB)(\mathbb{R})(F)$, $a^*_\mu(S(\xi)) = S(\xi)$, then for each η in $(AB)(\mathbb{R})(F)$,

$$S(T(\eta)) = T(S(\eta)). \tag{14.3}$$

THEOREM 14.4. If each W and T is in C, then, for each ξ in $(AB)(\mathbb{R})(F)$,

$$W(T(\xi)) = T(W(\xi)). \tag{14.4}$$

THEOREM 14.5. If T is in C, then the following two statements are equivalent:

1) If each of ξ and μ is in $(AB)(\mathbb{R})(F)$ and $T(\xi) = T(\mu)$, then $\xi = \mu$.

2) If ρ is in $(AB)(\mathbb{R})(F)$, then ρ is in $A_{T(\rho)}$.

We end this section with an observation which is an easy consequence of various theorems of the preceding section and which will be the basis of the "mapping" theorem of the next section.

OBSERVATION 14.2. If M is a C-set satisfying condition 1) of Theorem 13.4, then a^*_M is in C.

15. A MAPPING THEOREM FOR A PAIR OF CLASSES OF TRANSFORMATIONS.

Again, for reasons given in the introduction, we "spell out" notions that have standard and well-known designations in the literature.

Suppose that M is a C-set satisfying condition 1) of Theorem 13.4.

We shall let M* denote the set to which f belongs iff f is a function from M into $\mathbb{R}$ satisfying the following conditions:

i) If each of μ and ρ is in M and r and s is in $\mathbb{R}$, then $f(r\mu + s\rho) = rf(\mu)+sf(\rho)$.

ii) $\{|f(\mu)| : \mu \text{ in } M, \int_U |\mu(I)| \leq 1\}$ is bounded.

Now, for each f in M*, we let $||f||$ denote $\sup\{|f(\mu)| : \mu \text{ in } M, \int_U |\mu(I)| \leq 1\}$.

OBSERVATION 15.1. If each of f_1 and f_2 is in M* and each of r and s is in $\mathbb{R}$, then $rf_1 + sf_2$ is in M*.

For each T in C, we let $||T||$ denote $\sup\{\int_U |T(\mu)(I)| : \int_U |\mu(I)| \leq 1\}$.

OBSERVATION 15.2. $||a^*_M|| \leq 1$.

For each μ in $(AB)(\mathbb{R})(F)$ and W in F, we let $\mu^{[W]}$ denote the element of $(AB)(\mathbb{R})(F)$ given by

$$\mu^{[W]}(V) = \mu(W \sqcap V). \tag{15.1}$$

We now consider the following subset of C: C_M is the set to which T belongs iff T is an element of C whose range $\subseteq$ M, i.e., for each μ in $(AB)(\mathbb{R})(F)$, $a^*_M(T(\mu)) = T(\mu)$.

Roughly speaking, the theorem that we are about to state says that M* and C_M are "indistinguishable", a notion made precise as follows:

THEOREM 15.1. Consider the mapping, Y, with domain M*, such that for each f in M*,

$$Y(f) = \{(\mu,\{f(a_M(\mu^{[V]})) : V \text{ in } F\}) : \mu \text{ in } (AB)(\mathbb{R})(F)\}. \tag{15.2}$$

Then Y is a mapping from M* onto C_M having the following properties:

i) If each of f_1 and f_2 is in M* and r and s is in $\mathbb{R}$, then,

$$Y(rf_1 + sf_2) = rY(f_1) + sY(f_2). \tag{15.3}$$

ii) If f is in M*, then

$$||Y(f)|| = ||f||. \tag{15.4}$$

iii) (as a consequence of i) and ii)) Y is one-one.

COROLLARY 15.1. If μ is in $(AB)(\mathbb{R})(F)^+$, and W is the mapping with domain A^*_μ such that if f is in A^*_μ, then

$$W(f) = \{(V,f(\mu^{[V]})) : V \text{ in } F\} \tag{15.5}$$

then W is a one-one mapping from A^*_μ onto Lip(μ) such that if each of g and h is in A^*_μ and each of r and s is in $\mathbb{R}$, then

$$W(rg + sh) = rW(g) + sW(h). \tag{15.6}$$

If f is in A^*_μ, then

$$||f|| = \sup\{|f(\mu^{[I]})/\mu(I)| : I \text{ in } F\}, \tag{15.7}$$

and, for each ξ in A_μ,

$$f(\xi) = \int_U [f(\mu^{[I]})/\mu(I)]\xi(I). \tag{15.8}$$

16. CONCERNING AN INTEGRAL EQUATION.

In this section we treat the following question: Given α in exp($\mathbb{R}$)(F)(B) and ξ in (AB)($\mathbb{R}$)(F), what is a necessary and sufficient condition that there be an element μ in (AB)($\mathbb{R}$)(F) such that for each V in F, $\int_V \alpha(I)\mu(I) = \xi(I)$? We begin with two lemmas.

LEMMA 16.1. If each of β and γ is in exp($\mathbb{R}$)(F) and V is in F, then $\int_V \beta(I)\gamma(I)$ exists iff $\int_V \mathrm{sgn}(\beta)(I)|\beta(I)|\gamma(I)$ exists, in which case equality holds.

LEMMA 16.2. If β is in exp($\mathbb{R}$)(F)(B), μ is in (AB)($\mathbb{R}$)(F), V is in F and $\int_V \beta(I)\mu(I)$ exists, then $\int_V \mathrm{sgn}(\beta)(I)\int_I \beta(J)\mu(J)$ exists.

We now state the main theorem of this section, which not only characterizes the existence of a solution, but makes a uniqueness assertion; once again, note the role that absolute continuity plays.

THEOREM 16.1. Suppose that α is in exp($\mathbb{R}$)(F)(B) and ξ is in (AB)($\mathbb{R}$)(F). The following statements are true:

1) The following two statements are equivalent:

i) There is η in (AB)($\mathbb{R}$)(F) such that for each V in F, $\int_V \alpha(I)\eta(I) = \xi(I)$.

ii) $\int_U \mathrm{sgn}(\alpha)(I)\xi(I)$ exists, and for some B in $\mathbb{R}$ and all $K > 0$, $\int_U [|\xi(I)|/\max\{|\alpha(I)|,K\}]$ exists and does not exceed B.

2) There is no more than one λ in A_ξ such that for each V in F, $\int_V \alpha(I)\lambda(I)$ exists and is $\xi(V)$.

3) If i), or equivalently, ii) of 1) holds, μ is the element of $(AB)(\mathbb{R})(F)^+$ given, for each V in F, by

$$\mu(V) = \sup\{\int_V [|\xi(I)|/\max\{|\alpha(I)|,K\}] : 0 < K\}, \tag{16.1}$$

and λ is the element of (AB)($\mathbb{R}$)(F) given, for each V in F, by

$$\lambda(V) = \int_V \mathrm{sgn}(\alpha)(I)\mathrm{sgn}(\xi)(I)\mu(I), \tag{16.2}$$

then λ is in A_ξ, and, for each V in F, $\int_V \alpha(I)\lambda(I)$ exists and is $\xi(V)$.

17. MORE THEOREMS ABOUT INTEGRAL REPRESENTATIONS.

In section 16 we considered the question, given α in $\exp(\mathbb{R})(F)(B)$ and ξ in $(AB)(\mathbb{R})(F)$, of when there is η in $(AB)(\mathbb{R})(F)$ such that for each V in F,

$$\int_V \alpha(I)\eta(I) = \xi(V). \tag{17.1}$$

In this section we consider converse-type remarks of this matter. We begin with a definition.

DEFINITION 17.1. If $S \subseteq \mathbb{R}$ and each of ξ and μ is in $(AB)(\mathbb{R})(F)$, then the statement that ξ is μ-dense in S means that if $0 < c$ and V is in F, then there is $D \ll \{V\}$ and a function b from D into S such that

$$\Sigma_D |\xi(I) - b(I)\mu(I)| < c. \tag{17.2}$$

Before we state the next theorem, we remind the reader about our remarks in the introduction concerning the elementary topological properties of $\mathbb{R}$.

THEOREM 17.1. If S is a closed and bounded subset of $\mathbb{R}$, each of ξ and μ is in $(AB)(\mathbb{R})(F)$ and ξ is μ-dense in F, then there is a function α from F into S such that if V is in F, then

$$\int_V \alpha(I)\mu(I) = \xi(V). \tag{17.3}$$

Now, notice that, trivially, under the hypothesis of Theorem 17.1, ξ is in $\mathrm{Lip}(\mu)$. The remaining theorems of this section concern the smallest (with respect to inclusion) closed subset of $\mathbb{R}$ "giving" a representation of the type described in the opening paragraph of this section, for ξ in $\mathrm{Lip}(\mu)$ and then for ξ in A_μ, in each case, for $\mu(U) > 0$.

We begin with two well-known theorems about closed and bounded subsets of $\mathbb{R}$.

THEOREM 17.C.1. If each of S_1 and S_2 is a closed and bounded subset of $\mathbb{R}$, then S_1 and S_2 have an element in common iff $0 = \inf\{|x - y| : x \text{ in } S_1,\ y \text{ in } S_2\}$.

THEOREM 17.C.2. Suppose that G is a collection of closed and bounded subsets of $\mathbb{R}$ such that if each of S_1 and S_2 is in G, then there is an element common to S_1 and S_2 and an element S_3 of G such that $S_3 \subseteq S_1 \cap S_2$. Then there is x such that x is in each set of G; furthermore, if $0 < c$, then there is some S^* in G such that if z is in S^*, then $\inf\{|y - z| : y \text{ in } \bigcap_G S\} < c$.

THEOREM 17.2. Suppose that μ is in $(AB)(\mathbb{R})(F)^+$, $\mu(U) > 0$, and ξ is in $\mathrm{Lip}(\mu)$. Let G denote the collection to which S belongs iff S is a closed and bounded subset of $\mathbb{R}$ for which there is a function α from F into S such that if V is in F, then

$$\int_V \alpha(I)\mu(I) = \xi(V). \tag{17.4}$$

Then G satisfies the hypothesis of Theorem 17.C.2 and $\bigcap_G S$ is in G.

THEOREM 17.3. Suppose that μ is in $(AB)(\mathbb{R})(F)^+$, $\mu(U) > 0$ and ξ is in A_μ. Then there is a closed subset P or $\mathbb{R}$ such that:

i) If Q is a closed subset of $\mathbb{R}$ for which there is β in S_μ with range union $\subseteq Q$ such that $\sigma_\mu(\beta) = \xi$, then $P \subseteq Q$, and

ii) there is a function γ from F into P such that if V is in F, then

$$\int_V \gamma(I)\mu(I) = \xi(V) \tag{17.5}$$

furthermore, γ is in S_μ and

$$\sigma_\mu(\gamma) = \xi. \tag{17.6}$$

18. FINITE ADDITIVITY, SET FUNCTIONS AND UPPER AND LOWER DISTRIBUTION FUNCTIONS.

The reader familiar with the notion of a distribution function, as it arises in a countably additive setting, will see the motivation behind the definition that we shall give after some preliminary results.

We first state a consequence of Theorem 13.2.

THEOREM 18.1. Suppose that μ is in $(AB)(\mathbb{R})(F)^+$, α is in $\exp(\mathbb{R})(F)(B)$,

$$g = \int G(\alpha\mu), \text{ and } h = L(\alpha\mu). \tag{18.1}$$

Then, if $D << \{U\}$, for each V in D, $E(V) << \{V\}$, $b(V)$ is in $\alpha(V)$ and for each I in $E(V)$, $c(I)$ is in $\alpha(I)$, then

$$\Sigma_D\Sigma_{E(V)}|b(V)\mu(I) - c(I)\mu(I)| \leq 2(\Sigma_D[L(\alpha\mu)(V) - G(\alpha\mu)(V)]) - [h(U) - g(U)] + \{\int_U[g(J)^2/\mu(J)] - \Sigma_D[g(V)^2/\mu(V)]\}^{\frac{1}{2}}\{\mu(U)\}^{\frac{1}{2}} + \{\int_U[h(J)^2/\mu(J)]-\Sigma_D[h(V)^2/\mu(V)]\}^{\frac{1}{2}} \{\mu(U)\}^{\frac{1}{2}}. \tag{18.2}$$

THEOREM 18.2. If μ is in $(AB)(\mathbb{R})(F)^+$ and α is in $\exp(\mathbb{R})(F)(B)$, then

$$\int_U[L(\alpha\mu)(J) - G(\alpha\mu)(J)] = \inf\{\sup\{\Sigma_E\Sigma_{E(V)}|b(V) - c(I)|\mu(I): E << D, \text{ for all V in E, } E(V) << \{V\}, b(V) \text{ is in } \alpha(V), \text{ for all I in } E(V), c(I) \text{ is in } \alpha(I)\}: D << \{U\}\}. \tag{18.3}$$

DEFINITION 18.1. For each α in $\exp(\mathbb{R})(F)$ and x in $\mathbb{R}$, $\beta(\alpha,x)$ denotes the element of $\exp(\mathbb{R})(F)$ such that if I is in F, then $\beta(\alpha,x)(I) \subseteq \{0,1\}$ and contains 1 iff there is y in $\alpha(I)$ such that $y < x$, and contains 0 iff there is y is $\alpha(I)$ such that $y \geq x$.

THEOREM 18.3. If α is in $\exp(\mathbb{R})(F)(B)$, μ is in $(AB)(\mathbb{R})(F)^+$ and $t < r$, then

$$\int_U[L(\beta(\alpha,t)\mu)(J)-G(\beta(\alpha,r)\mu)(J)] \leq (r - t)^{-1}\int_U[L(\alpha\mu)(J)-G(\alpha\mu)(J)]. \tag{18.4}$$

THEOREM 18.4. If α is in $\exp(\mathbb{R})(F)(B)$, μ is in $(AB)(\mathbb{R})(F)^+$ and h is a real-valued nondecreasing continuous function with domain $\mathbb{R}$, then

$$\int_{-\infty}^{\infty}h(x)d\{\int_U L(\beta(\alpha,x)\mu)(J)\} \leq \int_U G(h(\alpha)\mu)(J) \leq \int_U L(h(\alpha)\mu)(J) \leq \int_{-\infty}^{\infty}h(x)d \{\int_U G(\beta(\alpha,x)\mu)(J)\}. \tag{18.5}$$

Theorems 18.3 and 18.4 enable us to first show the integrability characterization theorem and representation theorem below.

THEOREM 18.5. If α is in $\exp(\mathbb{R})(F)(B)$ and μ is in $(AB)(\mathbb{R})(F)^+$, then the following three statements are equivalent:

1) $\int_U \alpha(I)\mu(I)$ exists.

2) If x is in $\mathbb{R}$, then

$$\int_U L(\beta(\alpha,x+)\mu)(J) = \int_U G(\beta(\alpha,x+)\mu)(J) \tag{18.6}$$

3) $$\int_{-\infty}^{\infty} x d\{\int_U L(\beta(\alpha,x)\mu)(J)\} = \int_{-\infty}^{\infty} x d\{\int_U G(\beta(\alpha,x)\mu)(J)\}. \tag{18.7}$$

THEOREM 18.6. If α and μ are as in the hypothesis of Theorem 18.5, k is a real-valued continuous function with domain $\mathbb{R}$ and $\int_U \alpha(I)\mu(I)$ exists, then we have the following existence and equivalence:

$$\int_U k(\alpha(I))\mu(I) = \int_{-\infty}^{\infty} k(x)d\{\int_U L(\beta(\alpha,x)\mu)(J)\} = \int_{-\infty}^{\infty} k(x)d\{\int_U G(\beta(\alpha,x)\mu)(J)\}. \tag{18.8}$$

We now prove two characterization theorems for set function measurability and summability, respectively.

THEOREM 18.7. If α is in $\exp(\mathbb{R})(F)$ and μ is in $(AB)(\mathbb{R})(F)^+$, then α is in M_μ iff

1) if x is in $\mathbb{R}$, then

$$\int_U L(\beta(\alpha,x+)\mu)(I) = \int_U G(\beta(\alpha,x+)\mu)(I), \tag{18.9}$$

and

2) $$\int_U G(\beta(\alpha,x)\mu)(I) \to \mu(U) \text{ as } x\to\infty \text{ and } \int_U L(\beta(\alpha,x)\mu)(I) \to 0 \text{ as } x\to-\infty. \tag{18.10}$$

THEOREM 18.8. If α is in $\exp(\mathbb{R})(F)$ and μ is in $(AB)(\mathbb{R})(F)^+$, then α is in S_μ iff α is in M_μ and $\int_{-\infty}^{\infty} x d\{\int_U L(\beta(\alpha,x)\mu)(J)\}$ (and hence $\int_{-\infty}^{\infty} x d\{\int_U G(\beta(\alpha,x)\mu)(J)\}$) exists.

THEOREM 18.9. Suppose that μ is in $(AB)(\mathbb{R})(F)^+$, α is in S_μ and h is a real-valued continuous function with domain $\mathbb{R}$ such that $\{|h(x)|/|x| : 1 \leq |x|\}$ is bounded. Then $\int_{-\infty}^{\infty} h(x)d\{\int_U L(\beta(\alpha,x)\mu)(J)\}$ (and hence $\int_{-\infty}^{\infty} h(x)d\{\int_U G(\beta(\alpha,x)\mu)(J)\}$) exists and is $\sigma_\mu(h(\alpha))(U)$.

REFERENCES

1. Bochner, S., and Phillips, R., Additive Set Functions and Vector Lattices, Ann. of Math., (2), 42, pp 316-324 (1941).

2. Caratheodory, C., Vorlesungen über reele Funcrionen, Second edition, Teubner, Leipzig, 1927.

3. Dunford, N., and Schwartz, J.T., Linear Operators, Part I, Interscience Publishers, New York, 1957.

4. Graves, L.M., The Theory of Functions of Real Variables, McGraw-Hill Company, New York, 1946.

5. Gunther, N., Sur les Intégrals de Stieltjes et Leurs Applications aux Problèms de la Physique Mathématique, Chelsea Publishing Company, New York, 1949.

6. Hellinger, E., Die Orthoginalinvarianten quadratischer Formen von unendlich vielen Variablen, Diss. Gottingen, 1907.

7. Henstock, R., Theory of Integration, Butterworths, London, 1963.

8. Hobson, E. W., The Theory of Functions of a Real Variable:, Volume I, Second edition, Cambridge University Press, 1921.

9. Kolmogoroff, A.N., Untersuchungen über den Integralbegriff, Math. Ann., 103, pp 654-696 (1930).

10. Leader, S., The Theory of L^p-Spaces for Finitely Additive Set Functions, Ann. of Math. (2) 58, pp 528-543 (1953).

11. Lebesgue, H., Lecons sur l'Integration et la Recherche des Fonctions Primitives, Gauthier-Villars, Paris, 1904, Second edition 1928.

12. McShane, E.J., Integration, Princeton University Press, Princeton, 1944.

13. Riesz, F., and Sz-Nagy, B., Lecons d'Analyse Fonctionnelle, Akademiai Kiado, Budapest, 1952.

14. Saks, S., Theory of the Integral, Second ed., Momografje Matematyczne, vol. 7, Warsaw, 1937. Reprinted Stechert-Hofner Pub. Co., New York.