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RINGS ALL OF WHOSE ADDITIVE ENDOMORPHISMS ARE LEFT MULTIPLICATIONS

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ABSTRACT. Motivated by Cauchy's functional equation $f(x+y) = f(x) + f(y)$, we study in §1 special rings, namely, rings for which every endomorphism f of their additive group is of the form $f(x) \equiv ax$. In §2 we generalize to R algebras (R a fixed commutative ring) and give a classification theorem when R is a complete discrete valuation ring. This result has an interesting consequence, Proposition 12, for the theory of special rings.

KEY WORDS AND PHRASES. *Ring, group, endomorphism, ideal, R-algebra, valuation ring.*
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1. INTRODUCTION.

Our starting point is Cauchy's functional equation

$$(*) \qquad f(x+y) = f(x) + f(y)$$

where f is a real function with domain $(-\infty,\infty)$ and the equality holds for every real x,y. If f is assumed to be continuous throughout $(-\infty,\infty)$, or even merely at one point, or merely bounded above or below on some set of positive measure, then necessarily $f(x) \equiv f(1)x$. On the other hand, using a Hamel basis, one can construct noncontinuous solutions of (*). For more details and references the reader is referred to [1], pp. 31-36.

By using (*) merely for integral (or rational) x and y, one easily concludes [1, pp. 31-32] that $f(x) = f(1)x$ for every integral (respectively, rational) x. This raises the following general problem: what are the rings A for which every endomorphism f of their additive group is of the form $f(x) \equiv ax$? We call such rings <u>special</u>.

In [2, Problem 45, p. 232] L. Fuchs raised the related question of characterizing the rings A which are isomorphic to the endomorphism ring $\mathrm{End}(A^+)$ of A^+, the additive group of A. This question was studied by P. Schultz [5] who found a number of interesting results.

In §1 we study special rings. Some of this material was considered in a different form by P. Schultz in [5]. In §2 we generalize our problem to R-algebras (R a fixed commutative ring) and give a classification theorem when R is a complete discrete valuation ring. This result has an interesting consequence, Proposition 12, for the theory of special rings. It would be nice to give a complete classification of special rings themselves, but this, so far, has eluded us.

2. SPECIAL RINGS.

PROPOSITION 1: If A is a special ring, then A is commutative and has an identity.

PROOF: The identity endomorphism is given by left multiplication by an element e. Similarly, the endomorphism $x \longrightarrow xe$ is given by left multiplication by an element e'. Thus $e = e^2 = e'e$. For every $x \in A$ we find $xe = e'x = e'ex = e^2x = ex = x$. This shows that e is an identity. We set $e = 1$ from now on.

If $a \in A$, consider the map $x \longrightarrow ax - xa$. This is in $\mathrm{End}(A^+)$ so $ax - xa = \lambda x$ for some λ and all x. Taking $x = 1$ shows $\lambda = 0$, and so $ax = xa$ for all x.

PROPOSITION 2: If A is special and $A = M \dotplus N$ as abelian groups, then M and N are ideals.

PROOF: Let π be the projection on M. Since $\pi \in \mathrm{End}(A^+)$, we have $\pi x = ex$ for some e and all x. It follows that $M = Ae$. ∎

PROPOSITION 3: Suppose A is special and $A = M \dotplus N \dotplus P$. Then $\mathrm{Hom}(M,N) = (0)$.

PROOF: Let $f \in \mathrm{Hom}(M,N)$. Define $f^*: A \to A$ by $f^*(m+n+p) = f(m)$ where $m \in M$, $n \in N$ and $p \in P$. Then $f^* \in \mathrm{End}(A^+)$ and so $f^*(x) = rx$ for some r and all x. Since M, N and P are ideals by Proposition 2, we see that $f(m) = rm$ for all $m \in M$ and $n \in N$. Take now $n = 0$. ∎

PROPOSITION 4: Suppose $A = M_1 \dotplus M_2 \dotplus \cdots \dotplus M_t$ as groups. If A is a special ring, then so is each M_i and, moreover, $\mathrm{Hom}(M_i, M_j) = (0)$ whenever $i \neq j$. Conversely, if each M_i is special and $\mathrm{Hom}(M_i, M_j) = (0)$ whenever $i \neq j$, then A is special.

PROOF: Straightforward, using the last two propositions. ∎

COROLLARY 1: If A is special and a Q-algebra, then $A \cong Q$. (Q is the field of rational numbers.)

COROLLARY 2: If A is special and A^+ is finitely generated, then either $A \cong Z$ or $A \cong Z/nZ$ for some $n \in Z$. (Z is the ring of ordinary integers.)

COROLLARY 3: If A is special and A^+ is torsion-free and completely decomposable, then $A \cong M_1 \dotplus M_2 \dotplus \ldots \dotplus M_t$, where the M_i are subrings of Q of incomparable types as abelian groups. The converse is also true.

PROOF: The decomposition of A into rank one groups has only finitely many

many summands because these summands must be ideals and A has an identity. The rest follows from Proposition 4 and Proposition 85.4 of [3].

PROPOSITION 5: Let A be a commutative ring with identity and $\mathcal{I} = \{I\}$ a collection of ideals, closed under finite intersection. Suppose: i) $f(I) \subseteq I$ for all $f \in End(A^+)$ and $I \in \mathcal{I}$, ii) A/I is special for all $I \in \mathcal{I}$, and iii) $A \cong \varprojlim A/I$. Then A is special.

PROOF: If $f \in End(A^+)$, then, by i), f induces $f_I \in End(A/I)$. Thus, by ii), there is an $r_I \in A/I$ such that $f_I(\bar{x}) = r_I\bar{x}$ for all $\bar{x} \in A/I$. The collection $\{r_I\}$ defines an element in the inverse limit and so, via iii), an element $r \in A$. It is easily seen that $f(x) = rx$ for all $x \in A$. ∎

COROLLARY 1: $\hat{Z} = \varprojlim Z/nZ$ and $Z_p = \varprojlim Z/p^mZ$ are both special rings. Here p is a prime and Z_p is the ring of p-adic integers.

COROLLARY 2: Suppose $\{A_j\}$ is a collection of special rings and, for each j_0, $Hom(\prod_{j \neq j_0} A_j, A_{j_0}) = (0)$. Then $\prod A_j$ is a special ring.

COROLLARY 3: For each prime p, let C_p be either cyclic of p-power order, or isomorphic to Z_p. Then $\prod C_p$ is special.

PROOF: Almost immediate, from Corollary 2. ∎

PROPOSITION 6: If A is special and $A = B \dotplus C$, with $B \cong Q$, then C is torsion.

PROOF: By Proposition 3, $Hom(C,B) \cong Hom(C,Q) = (0)$. It is standard, however, that $Hom(C,Q) = (0)$ if and only if C is torsion. ∎

PROPOSITION 7: A is special and A^+ is torsion if and only if A is cyclic.

PROOF: Let 1 be the identity of A. If A^+ is torsion, $n1 = 0$ for some positive integer n. But then $na = 0$ for all $a \in A$. Thus A^+ is a bounded torsion group and so, a direct sum of cyclic groups (Theorem 17.2 of [3]). Since A has an identity, and the summands are ideals, there are only finitely many of them. The result now follows rapidly from Proposition 3. ∎

PROPOSITION 8 (P. Schultz): If A is special, and A^+ is not reduced, then $A \cong Q \dotplus Z/nZ$ and conversely.

PROOF: Suppose A is special and let D be its maximal divisible subgroup. D is a direct summand of A and so, must be a special ring. If $D \neq (0)$, then D cannot be torsion, by the last proposition. Thus $D \neq (0)$ implies that D contains a copy of Q. Since Q is divisible, $A = B \dotplus C$ where $B \cong Q$ and C is torsion, by Proposition 6. By Proposition 7, C must be cyclic. The converse follows from Proposition 4. ∎

PROPOSITION 9: Suppose A is special and the torsion subgroup T of A^+ is a proper direct summand. Then $A \cong Z/nZ \dotplus C$ where C is a special ring which is torsion-free and divisible by n. The converse is also true.

PROOF: Immediate from Propositions 4 and 7. ∎

PROPOSITION 10: Suppose A is special and satisfies the descending chain condition on ideals. Then A is isomorphic to either Z/nZ, Q or $Z/nZ \dot{+} Q$. The converse is also true.

PROOF: Using Theorem 122.4 of [3], we see that A^+ is the direct sum of a torsion group and a Q-vector space. By previous results, the torsion group must be cyclic and the vector space must have dimension ≤ 1. The converse is clear. ∎

3. R-ALGEBRAS.

Let R be a fixed commutative ring. We consider the class of R-algebras A. The notion of a special ring can be generalized as follows. An R-algebra A is called special if every element of $End_R(A)$ is given by a left multiplication by an element of A. We may now ask for a classification of special R-algebras. Using a theorem in [4], we will give such a classification when R is a complete discrete valuation ring, for example, when R is some Z_p.

We note that, suitably modified, Proposition 3 remains valid in the context of R-modules and R-algebras.

PROPOSITION 11: Let R be a complete discrete valuation ring, and K its quotient field. Let C represent a cyclic torsion module. An R-algebra A is special if and only if A is isomorphic to either C, R, K or $K \dot{+} C$.

PROOF: If A is special as an R-algebra and not reduced as an R-module, then the proof of Proposition 8 is easily adapted to show that A is isomorphic to either K or $K \dot{+} C$. Thus we can assume A is reduced. By Corollary 1 to Theorem 23 in [4], every reduced R-module has a cyclic direct summand. If $A = B \dot{+} D$ where $B \cong R$, then $Hom_R(R,D) = (0)$ implies $D = (0)$, and so, $A \cong R$. If $A = C \dot{+} D$, then $Hom_R(C,D) = (0)$ implies D is torsion-free. If D were not trivial, it would have a direct summand isomorphic to R, but this cannot occur since $Hom_R(R,C) \neq (0)$. Thus, in this case, $A = C$.

Conversely, C, R, K and $K \dot{+} C$ are easily seen to be special. ∎

We can derive a consequence of this proposition for the theory of special rings.

Let p be a prime. Let us call a ring A p-complete if $A = \varprojlim A/p^m A$.

Such a ring is clearly a Z_p algebra. Moreover, every Z-endomorphism automatically commutes with the Z_p action. Since Z_p is a complete discrete valuation ring, we have

PROPOSITION 12: A ring A is special and p-complete if and only if A is isomorphic to either Z_p or $Z_p/p^m Z_p$ for some m.

PROOF: This follows from Proposition 11. Q_p, the quotient field of Z_p, cannot occur since it is infinite dimensional over Q. ∎

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