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SOME RADIUS OF CONVEXITY PROBLEMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper we consider some radius of convexity problems for certain classes of analytic functions. These classes, in general, are related with functions of bounded boundary rotation.

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1. INTRODUCTION.

Let V_k be the class of functions of bounded boundary rotation. Paatero [1] showed that a function f, analytic in $E = \{z:|z|<1\}, f(0) = 0, f'(0)=1$, $f'(z) \neq 0$; is in V_k if and only if, for $z = re^{i\theta}$,

$$\int_0^{2\pi} \left|\text{Re}\ \frac{(zf'(z))'}{f'(z)}\right| d\theta \leq k\pi$$

It is geometrically obvious that $k \geq 2$ and $V_2 \equiv C$, the class of univalent convex function.

A class T_k of analytic functions related with the class V_k has been introduced and discussed in [2]. Let f with $f(0) = 0$, $f'(0) = 1$ be analytic in E. Then $f \in T_k$, $k \geq 2$, if there exists a function $g \in V_k$ such that, for $z \in E$,

$$\text{Re}\ \frac{f'(z)}{g'(z)} > 0$$

It is clear that $T_2 \equiv K$, the class of close-to-convex functions introduced by Kaplan [3].

Let $P_{\alpha,n}$ denote the class of functions $p(z)$ in E given by $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots\ldots, n \geq 1$, which satisfy the inequality

$$|p(z) - \frac{1}{2\alpha}| < \frac{1}{2\alpha}\ , \quad 0 \leq \alpha < 1\ .$$

The class $P_{\alpha,n}$ has been introduced in [4]. If $\alpha=0$, the class $P_{\alpha,n}$ reduces to the classical class of functions with positive real part.

We shall need the following results in the next section.

Lemma 1.1[4]. Let $p \in P_{\alpha,n}$, then for $z \in E$, $|z| = r < 1$

(i) $\dfrac{1 - r^n}{1 + cr^n} \leq \mathrm{Re}\ p(z) \leq |p(z)| \leq \dfrac{1 + r^n}{1 - cr^n}$

(ii) $\left|\dfrac{p'(z)}{p(z)}\right| \leq \dfrac{(1+c)n\ r^{n-1}}{(1+cr^n)(1-r^n)}$,

where $c = 1-2\alpha$.

Lemma 1.2 [5]. If N and D are analytic in E and $N(0) = D(0) = 0$, and if D maps E onto many-sheeted region, which is starlike with respect to the origin, then $\mathrm{Re}\ \frac{N'}{D'} > 0 \Rightarrow \mathrm{Re}\ \frac{N}{D} > 0$, $z \in E$.

Lemma 1.3 [6]. Let $g \in V_k$. Then $G(z) = \frac{2}{z}\int_0^z g(t)dt$ is convex in the disc $|z| < \frac{1}{2}(k - \sqrt{k^2-4})$.

2. MAIN RESULTS

In all of the theorems, f and g will be analytic in E, $f'(0) = 1$, $f(0) = 0$. The univalence will not be assumed unless explicitly stated.

THEOREM 2.1. Let $g \in V_k$ and let $\frac{f'(z)}{g'(z)} \in P_{\alpha,1}$. Then f maps $|z| < r$ onto a convex domain, where r is the least positive root of

$$cx^3 - x^2(ck+c) - x(k+1) + 1 = 0\ . \tag{2.1}$$

PROOF: Let $\frac{f'(z)}{g'(z)} = p(z)$, $p(z) \in P_{\alpha,1}$.

Then

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zp'(z)}{p(z)}\ .$$

Hence

$$\mathrm{Re}\ \frac{(zf'(z))'}{f'(z)} \geq \mathrm{Re}\ \frac{(zg'(z))'}{g'(z)} - \left|\frac{zp'(z)}{p(z)}\right|\ . \tag{2.2}$$

Now, it is known [7] that if $g \in V_k$, then

$$\mathrm{Re}\ \frac{(zg'(z))'}{g'(z)} \geq \frac{r^2 - kr + 1}{1 - r^2}\ . \tag{2.3}$$

Using (2.3) and Lemma 1.1(ii) for $n=1$, (2.2) becomes

$$\text{Re}\,\frac{(zf'(z))'}{f'(z)} \geq \frac{cr^3 - (k+1)cr^2 - (k+1)r + 1}{(1-r^2)(1+cr)}$$

Thus f is convex if the right hand side of (2.1) is positive.

Corollary 2.1. Let $\alpha=0$ ($c=1$) which means $\text{Re}\,\frac{f'(z)}{g'(z)} > 0$. Then f maps $|z| < r = \frac{(k+2) - \sqrt{k^2 + 4k}}{2}$ onto a convex domain. This result was obtained in [2].

Corollary 2.2. For $\alpha = \frac{1}{2}$, we have $|\frac{f'(z)}{g'(z)} - 1| < 1$. Then f is convex for $|z| < r = \frac{1}{k+1}$. For $k=4$, V_4 consists of univalent functions and $r = \frac{1}{5}$. This result is known [4].

Corollary 2.3. If $\alpha=0$, and $k=2$, then f maps $|z| < r = 2-\sqrt{3}$ onto a convex domain. This result is well-known [8].

REMARKS 2.1. Let $\alpha=0$ and $k=4$. Then we obtain the known result $r=3-2\sqrt{2}$ of Ratti [9].

THEOREM 2.2. Let $g \in T_k$ and let $\frac{f'(z)}{g'(z)} \in P_{\alpha,1}$. Then f maps $|z| < r$ onto a convex domain where r is the least positive root of the equation

$$cx^3 - (k+3c)x^2 - (k+3)x + 1 = 0 .$$

PROOF: Let $\frac{f'(z)}{g'(z)} = p(z)$, where $p(z) \in P_{\alpha,1}$, $g \in T_k$.

Then

$$\text{Re}\,\frac{(zf'(z))'}{f'(z)} \geq \text{Re}\,\frac{(zg'(z))'}{g'(z)} - \left|\frac{zp'(z)}{p(z)}\right|$$

For $g \in T_k$, it is known [2] that

$$\text{Re}\,\frac{(zg'(z))'}{g'(z)} \geq \frac{r^2-(k+2)r+1}{1 - r^2} . \tag{2.4}$$

Using (2.4) and Lemma 1.1(ii), we obtain the result.

Corollary 2.4. Let $\alpha = \frac{1}{2}$ ($c=0$) and in this case f maps $|z| < r = \frac{1}{k+3}$ onto a convex domain. The special case for $k=2$ is known [4].

Corollary 2.5. For $\alpha=0$ and $k=2$, $T_2 \equiv K$ consists of close-to-convex univalent functions. Then the radius of convexity is $r=3-2\sqrt{2}$. This result is known [9].

THEOREM 2.3. Let $\text{Re}\,\frac{f'(z)}{g'(z)} > 0$ and $\text{Re}\,\frac{g'(z)}{S'(z)} > 0$, where S belongs to the class S^* of starlike functions. Then f maps $|z| < r = 4 - \sqrt{15}$ onto a convex domain.

PROOF: We have

$$f'(z) = S'(z)h_1(z)h_2(z), \quad \text{where} \quad \text{Re } h_1(z)>0, \ \text{Re } h_2(z)>0, \quad z\varepsilon E.$$

That is

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zS'(z))'}{S'(z)} + \frac{zh_1'(z)}{h_1(z)} + \frac{zh_2'(z)}{h_2(z)}$$

hence

$$\text{Re } \frac{(zf'(z))'}{f'(z)} \geq \text{Re } \frac{(zS'(z))'}{S'(z)} - \left|\frac{zh_1'(z)}{h_1(z)}\right| - \left|\frac{zh_2'(z)}{h_2(z)}\right| . \tag{2.5}$$

Now it is well known [8] that for $S\varepsilon S^*$,

$$\text{Re } \frac{(zS'(z))'}{S'(z)} \geq \frac{1-4r+r^2}{1-r^2} . \tag{2.6}$$

Also, if $\text{Re } h(z) > 0$, then it is known [10] that

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2r}{1-r^2} . \tag{2.7}$$

Using (2.6) and (2.7), (2.5) yields

$$\text{Re } \frac{(zf'(z))'}{f'(z)} \geq \frac{1-8r+r^2}{1-r^2} .$$

Hence f is convex for $|z| < r = 4 - \sqrt{15}$.

THEOREM 2.4. Let $\text{Re } \frac{f'(z)}{g'(z)} > 0$ and $\text{Re } \frac{g'(z)}{S'(z)} > 0$ where $S\varepsilon T_k$.
Then f maps $|z| < r = \frac{(k+6) - \sqrt{(k+6)^2-4}}{2}$ onto a convex domain.

The proof follows on the same lines of Theorem 2.3, by using (2.4).

Corollary 2.6. If $k=2$, then $S\varepsilon T_2 \equiv K$. In this case f maps $|z|<r=4-\sqrt{15}$ onto a convex domain.

THEOREM 2.5. Let $f\varepsilon V_k$ and $f_\alpha(z) = \int_0^z (f'(t))^\alpha dt$, $0\leq\alpha\leq1$. Then f_α maps $|z|<r$ onto a convex domain, where r is the least positive root of

$$(2\alpha-1)\, x^2 - \alpha kx + 1 = 0 . \tag{2.8}$$

PROOF: We have $f'_\alpha(z) = (f'(z))^\alpha$, $0\leq\alpha\leq1$.

Thus

$$\frac{(zf_\alpha'(z))'}{f_\alpha'(z)} = \alpha \frac{(zf'(z))'}{f'(z)} + (1-\alpha) .$$

Since $f \varepsilon V_k$, using (2.3), we have

$$\text{Re}\,\frac{(zf'_\alpha(z))'}{f'_\alpha(z)} \geq \alpha\,\frac{1-kr+r^2}{1-r^2} + (1-\alpha) = \frac{1-\alpha kr+(2\alpha-1)r^2}{1-r^2}$$

and this gives us the required result.

<u>THEOREM 2.6</u>. Let $f \varepsilon T_k$ and $f_\alpha(z) = \int_0^z (f'(t))^\alpha dt$. Then f_α maps $|z|<r$ onto a close-to-convex domain, where r is the least positive root of (2.8).

<u>PROOF</u>: Since $f \varepsilon T_k$, there exists $g \varepsilon V_k$ such that $\text{Re}\,\frac{f'(z)}{g'(z)} > 0$. Let $g_\alpha(z) = \int_0^z [g'(t)]^\alpha dt$. Then

$$(f'_\alpha(z)/g'_\alpha(z)) = (f'(z)/g'(z))^\alpha .$$

Using theorem 2.5, it follows that f_α is close-to-convex for $|z|<r$, where r is the least positive root of (2.8).

<u>Corollary 2.7</u>. Let $f \varepsilon T_4$, then f_α is close-to-convex for $|z|<r$, where r is the least positive root of

$$(2\alpha-1)x^2 - 4\alpha x + 1 = 0 .$$

In this case, if $\alpha = \frac{1}{2}$, then f_α is close-to-close for $|z| < \frac{1}{2}$.

<u>Corollary 2.8</u>. Let $f \varepsilon T_k$ and $\alpha = \frac{1}{2}$. Then f_α is close-to-convex for $|z|<r = \frac{2}{k}$. For k=2, we have a result proved in [11].

<u>Corollary 2.9</u>. For k=2, $f_\alpha \varepsilon K$, see [11].

<u>THEOREM 2.7</u>. Let $f \varepsilon T_k$ and $F(z) = \frac{2}{z}\int_0^z f(t)dt$. Then F maps $|z|<r = \frac{1}{2}(k - \sqrt{k^2-4})$ onto a close-to-convex domain.

<u>PROOF</u>: Since $f \varepsilon T_k$, there exists a $g \varepsilon V_k$ such that $\text{Re}\,\frac{f'(z)}{g'(z)} > 0$. Let $G(z) = \frac{2}{z}\int_0^z g(t)dt$. We know, from Lemma 1.3, that G is convex for $|z| < r = \frac{k-\sqrt{k^2-4}}{2}$. Now

$$\frac{F'(z)}{G'(z)} = \frac{\left(\frac{2}{z}\int_0^z f(t)dt\right)'}{\left(\frac{2}{z}\int_0^z g(t)dt\right)'} = \frac{N}{D}$$

and

$$\frac{N'}{D'} = \frac{f'(z)}{g'(z)} .$$

So $\operatorname{Re}\frac{N'}{D'} > 0$. Applying Lemma 1.2 for $|z| < r = \frac{k-\sqrt{k^2-4}}{2}$

we have $\operatorname{Re}\frac{N}{D} > 0$, which implies that F is close-to-convex for

$|z| < r = \frac{k-\sqrt{k^2-4}}{2}$.

Corollary 2.10. When $k=2$, $f \varepsilon T_2 \equiv K$ and hence $F \varepsilon K$ for $z \varepsilon E$. This result was obtained in [5] by Libera.

REFERENCES

1. PAATERO, V. Uber die Konforme Abbildung von Gebieten deren Rander von beschrankter Drehung Sind, Ann Acad. Sci. Fenn., Ser. A 33 (1933), 77 pp.

2. NOOR, K.I. On a Generalization of Close-to-Convexity, Int.J. Math. & Math. Sci. 6 (1983), 327-334.

3. KAPLAN, W. Close-to Convex Schlicht Functions, Michigan Math. J. 1(1952): 169-185.

4. SHAFFER, D.B. Radii of Starlikeness and Convexity for Special Classes of Analytic Functions, J. Math. Analysis and Applications, 45(1974): 73-80.

5. LIBERA, R.J. Some Classes of Regular Univalent Functions, Proc. Amer. Math. Soc. 16(1965) : 755-58.

6. KARUNAKARAN, V. and PADMA, K. Functions of Bounded Radius Rotation, Indian J. Pure Appl. Math., 12(5), 1981:621-627.

7. TAMMI, O. On certain Combinations for the Coefficients of Schlicht Functions, Ann. Acad. Sci. Ser. AI(1952).

8. HAYMAN, W.K. Multivalent Functions, Cambridge, 1967.

9. RATTI, J.S. The Radius of Convexity of Certain Analytic Functions, Indian J. Pure Appl. Math. 1(1970), 30-37.

10. MACGREGOR, T.H. The Radius of Univalence of Certain Analytic Functions, Proc. Amer. Math. Soc. 14(1963); 514-520.

11. ROYSTER, W.C. On the Univalence of a Certain Integral, Michigan Math.J. 12,(1965): 385-387.