Internat. J. Math. & Math. Sci.
Vol. 7 No. 4 (1984) 689-695

INTERNAL FUNCTIONALS AND BUNDLE DUALS

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(Received 8 February 1983, and in revised form August, 1984)

ABSTRACT. If $\pi: E \to X$ is a bundle of Banach spaces, X compact Hausdorff, a fibered space $\pi^*: E^* \to X$ can be constructed whose stalks are the duals of the stalks of the given bundle and whose sections can be identified with the "functionals" studied by Seda in [1] and [2] or elements of the "internal dual" $\mathrm{Mod}(\Gamma(\pi),C(X))$ studied by Gierz in [3]. If the given bundle is separable and norm continuous, then the fibered space $\pi^*: E^* \to X$ is actually a full bundle of locally convex topological vector spaces (Theorem 3). In the second portion of the paper two results are stated, both of them corollaries of theorems by Gierz, concerning functionals for bundles of Banach spaces which arise, in turn, from "fields of topological spaces."

KEY WORDS AND PHRASES. *Bundles of Banach spaces, fields of topological spaces, internal functionals, internal duals.*

1980 MATHEMATICS SUBJECT CLASSIFICATION. Primary 55R65, Secondary 28C05, 46H25.

1. INTRODUCTION.

If $\pi: E \to X$ is a bundle of Banach spaces, is there a bundle which can appropriately be called its dual? What are its sections and how do they relate to the sections of the given bundle? These questions are dealt with in the first portion of this paper. It turns out that the sections of our dual bundle (when it exists) can be identified either with Seda's "functionals" ([1]), which we shall refer to as "internal functionals," or with elements of the "internal dual" $\mathrm{Mod}(\Gamma(\pi),C(X))$ studied by Gierz in [3].

In the second portion of this paper some general theorems of Gierz are applied to a special class of Banach bundles which arise from "fields of topological spaces. A generalized Fubini type theorem (Theorem 5) results as well as a theorem which relates to Seda's integral representation of internal functionals.

2. DUAL BUNDLES.

Let $\pi: E \to X$ be a bundle of Banach spaces where X is compact Hausdorff. We let E^* be the disjoint union of the duals of the fibers, i.e.

$$E^* = \{(x,f): x \in X, f \in E_x^*\} .$$

We let $\pi^*: E^* \to X$ be the first coordinate projection and we let $F: E^* \to \Gamma(\pi)^*$ be the natural isometry defined by $F(x,f) = F_f$, where

$$F_f(\sigma) = f(\sigma(x))$$

for all $\sigma \in \Gamma(\pi)$. Following Seda [2] we give E^* the weak topology determined by the two maps π^* and F, where, in the case of the second map, $\Gamma(\pi)^*$ is given the weak-* topology. Thus, a net $\{(x_\alpha, f_\alpha)\}$ in E^* converges to a point (x,f) iff

$$\lim x_\alpha = x \quad \text{and} \quad \lim f_\alpha(\sigma(x_\alpha)) = f(\sigma(x))$$

for all σ in $\Gamma(\pi)$. A point (x_0, f_0) in E^* has a basis of neighborhoods consisting of sets of the form

$$V(U,\sigma_1,\ldots,\sigma_n,\varepsilon) = \bigcap_{k=1}^{n} \{(x,f): x \in U, |f(\sigma_k(x)) - f_0(\sigma_k(x_0))| < \varepsilon\}$$

where U is a neighborhood of x_0, $\sigma_1,\ldots,\sigma_n$ belong to $\Gamma(\pi)$ and $\varepsilon > 0$. We can further stipulate, of course, that the sections $\sigma_1,\ldots,\sigma_n$ be of norm one or less.

In certain cases the fibered space $\pi^*: E^* \to X$ will turn out to be a full bundle of locally convex topological vector spaces. When this happens $\pi^*: E^* \to X$ will be called the <u>dual bundle</u> of the given bundle. In all cases, the fibered space $\pi^*: E^* \to X$ has the following easily verified properties.

PROPOSITION 1. The map $\pi^*: E^* \to X$ is an open continuous surjection. The maps

$$\text{add}: E^* \vee E^* \to E^*$$

$$\text{scal}: C \times E^* \to E^*$$

of addition and multiplication by scalars are continuous. The space E^* is Hausdorff and for each $x \in X$, the topology which E_x^* inherits from E^* is the weak-* topology.

By an internal functional of the bundle $\pi: E \to X$ we shall mean a continuous map $T: E \to C$ which is linear on the stalks of E. Thus, for each $x \in X$ the restricted map

$$T_x := T/E_x$$

is an element of E_x^*. Since X is compact T is bounded, i.e.

$$||T|| := \sup\{||T_x|| : x \in X\} < +\infty$$

(by Proposition 2 in Seda [1]). Alternatively, T may be regarded as a bundle map from the given bundle $\pi: E \to X$ to the trivial bundle $pr_1: X \times C \to X$ (i.e. the corresponding bundle map carries each $\alpha \in E$ onto $(\pi(\alpha), T(\alpha))$). The bundle map, in turn, induces a $C(X)$-linear map from the space of sections $\Gamma(\pi)$ to the space

$C(X)$ of sections of the trivial bundle, namely, $\lambda_T\colon \Gamma(\pi) \to C(X)$ where

$$[\lambda_T(\sigma)](x) = T(\sigma(x))$$

for all $\sigma \in \Gamma(\pi)$ and $x \in X$. Thus, λ_T belongs to $\mathrm{Mod}_{C(X)}(\Gamma(\pi),C(X))$, the "internal dual" of $\Gamma(\pi)$. Moreover, $||\lambda_T|| = ||T||$.

Given an internal functional $T\colon E \to C$ we let $\tau_T\colon X \to E^*$ be defined by

$$\tau_T(x) = T_x = T_{/E_x}$$

for all $x \in X$. Note that τ_T is a selection of our fibered space $\pi^*\colon E^* \to X$.

PROPOSITION 2. For each internal function $T\colon E \to C$ the corresponding selection $\tau_T\colon X \to E^*$ is continuous. Moreover, every continuous selection of the fiber space $\pi^*\colon E^* \to X$ arises in this way.

PROOF. Suppose we have a convergent net $\{x_\alpha\}$ in X, say $\lim x_\alpha = x$. Then, for each $\sigma \in \Gamma(\pi)$, $\lim \sigma(x_\alpha) = \sigma(x)$ (by the continuity of σ), and consequently,

$$\lim[\tau_T(x_\alpha)](\sigma(x_\alpha)) = \lim T(\sigma(x_\alpha))$$

$$= T(\sigma(x)) = [\tau_T(x)](\sigma(x))$$

(by the continuity of T). Thus, $\lim \tau_T(x_\alpha) = \tau_T(x)$. Hence, τ_T is continuous.

Suppose, conversely, that $\xi\colon X \to E^*$ is a continuous selection of the fibered space $\pi^*\colon E^* \to X$. Define $T\colon E \to C$ by

$$T(\alpha) = \xi(\pi(\alpha))(\alpha) .$$

That is, $T = \xi \circ \pi$. Then T/E_x is the functional $\xi(x) \in E_x^*$ $(x \in X)$, and, since T is a composition of continuous maps, it is continuous. Clearly, $\xi = \tau_T$.

Thus, we have a bijective correspondence between internal functionals of the bundle $\pi\colon E \to X$ and continuous selections of the fiber space $\pi^*\colon E^* \to X$. Moreover, there is a one-to-one correspondence between the latter and elements of the internal dual $\mathrm{Mod}(\Gamma(\pi),C(X))$ (by Proposition 19.1 of Gierz [3]). Consequently, on those occasions when $\pi^*\colon E^* \to X$ is actually a bundle of locally convex topological vector spaces the sections of the bundle may be viewed in several different ways:

a) as internal functionals of the given bundle $\pi\colon E \to X$;

b) as bundle maps from $\pi\colon E \to X$ to the trivial line bundle over X;

c) as elements of the internal dual $\mathrm{Mod}(\Gamma(\pi),C(X))$ of $\Gamma(\pi)$.

When, then, is $\pi^*\colon E^* \to X$ an actual bundle?

THEOREM 3. Suppose that $\pi\colon E \to X$ is a norm continuous and separable bundle of Banach spaces, where X is compact Hausdorff. Then $\pi^*\colon E^* \to X$ is a full bundle of locally convex topological vector spaces.

PROOF. Under the stated hypotheses the topology of the fiber space E must be Hausdorff and through each point of E^* there passes a continuous selection of the fiber space $\pi^*\colon E^* \to X$ (by Corollary 19.16 of Gierz [3]). Thus $\pi^*\colon E^* \to X$ is a full bundle if, in fact, it is a bundle.

If $\sigma \in \Gamma(\pi)$ we define a seminorm ν_σ on E^* by setting

$$\nu_\sigma(x,f) = |f(\sigma(x))| .$$

We let S be the directed family of seminorms consisting of finite sups of the seminorms ν_σ where $\|\sigma\| \leq 1$. It suffices to show that each point (x_0,f_0) in E^* has a basis of neighborhoods consisting of tubes defined by continuous selections of $\pi^*: E^* \to X$ and seminorms from S.

Consider a neighborhood of (x_0,f_0). We may assume that it is

$$V(U,\sigma_1,\dots,\sigma_n,\varepsilon) = \bigcap_{k=1}^{n} \{(x,f): x \in U, |f(\sigma_k(x)) - f_0(\sigma_k(x_0))| < \varepsilon\}$$

where U is a neighborhood of x_0 and $\sigma_1,\dots,\sigma_n$ belong to the unit sphere of $\Gamma(\pi)$. Let $T: E \to C$ be an internal functional such that $T_{x_0} = f_0$. Because the map $x \to T_x$ is continuous, there exists a neighborhood $V \subset U$ of x_0 such that for all $x \in V$,

$$\tau_T(x) = T_x \in V(U,\sigma_1,\dots,\sigma_n,\varepsilon/2),$$

that is,

$$|\tau_T(x)(\sigma_k(x)) - \tau_T(x_0)(\sigma_k(x_0))| < \varepsilon/2, \quad k = 1,2,\dots,n .$$

Let $\nu = \sup\{\nu_{\sigma_1},\nu_{\sigma_2},\dots,\nu_{\sigma_n}\}$ and consider the tube

$$T(V,\tau_T,\nu,\varepsilon/2) = \{\xi \in E^*: \pi^*(\xi) \in V, \nu(\xi - \tau_T(\pi^*(\xi))) < \varepsilon/2\} .$$

If ξ belongs to this tube, then $x := \pi^*(\xi) \in V$ and

$$|\xi(\sigma_k(x)) - T_x(\sigma_k(x))| < \varepsilon/2, \quad k = 1,2,\dots,n ,$$

so

$$|\xi(\sigma_k(x)) - f_0(\sigma_k(x))| \leq |\xi(\sigma_k(x)) - T_x(\sigma_k(x))| + |T_x(\sigma_k(x)) - T_{x_0}(\sigma_k(x_0))|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad k = 1,2,\dots,n.$$

(Note that $T_{x_0} = f_0$.) Hence

$$T(V,\tau_T,\nu,\varepsilon/2) \subset V(U,\sigma_1,\dots,\sigma_n,\varepsilon).$$

Suppose, conversely, that we have a tube about (x_0,f_0), say $T(V,\tau_T,\nu,\varepsilon)$ where $\tau_T(x_0) = T_{x_0} = f_0$, and

$$\nu = \sup\{\nu_{\sigma_1},\dots,\nu_{\sigma_n}\} .$$

Since τ_T is continuous at x_0, there is a neighborhood $W \subset V$ of x_0 such that

$$|T_x(\sigma_k(x)) - T_{x_0}(\sigma_k(x_0))| < \varepsilon/2 ,$$

$k = 1,2,\dots,n$, whenever $x \in W$. Thus if

$$(x,f) \in V(W,\sigma_1,\dots,\sigma_n,\varepsilon/2),$$

it follows that

$$|f(\sigma_k(x)) - T_x(\sigma_k))| \leq |f(\sigma_k(x)) - f_0(\sigma_k(x_0))| + |T_{x_0}(\sigma_k(x_0)) - T_x(\sigma_k(x))|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad k = 1,2,\ldots,n,$$

and thus

$$\nu(f - \tau_T(f)) < \varepsilon \ ,$$

i.e.

$$(x,f) \in T(W,\tau_T,\nu,\varepsilon) \subset T(V,\tau_T,\nu,\varepsilon) \ .$$

Thus,

$$V(W,\sigma_1,\ldots,\sigma_n,\varepsilon/2) \subset T(V,\tau_T,\nu,\varepsilon) \ .$$

This proves that each point of E^* has neighborhood base consisting of tubes.

3. BUNDLES ARISING FROM FIELDS OF TOPOLOGICAL SPACES.

The closing portion of this paper deals with a special class of Banach bundles. These bundles, which arise from fields of topological spaces, have been studied by Seda in [1] and [2]. Theorem 4 is closely related to Seda's integral representation of internal functionals on such bundles. Theorem 5 is a generalized Fubini type theorem. Both are simple corollaries of theorems of Gierz.

Suppose we are given an open continuous surjection $p: Y \to X$, where X and Y are compact Hausdorff. (Such a triple (X,Y,p) is sometimes called a <u>field of topological spaces</u>.) For each $x \in X$, we denote by Y_x the fiber above x; thus $Y_x = p^{-1}(x)$. We know that $C(Y)$ is a locally $C(X)$-convex $C(X)$-module, where $C(X)$ acts on $C(Y)$ by pullback and pointwise multiplication. Specifically,

$$(fg)(y) = f(p(y))g(y)$$

for all $y \in Y$, $f \in C(X)$, and $g \in C(Y)$. If we denote by $\pi: E \to X$ the canonical bundle defined by the Banach module $(C(Y),C(X))$, then the Gelfand representation $\hat{\ }: C(Y) \to \Gamma(\pi)$ is an isometric isomorphism. Moreover, for each $x \in X$ there is a natural way in which the stalk E_x can be identified with the space $C(Y_x)$. Under this identification the Gelfand representation is described by:

$$\hat{f}(x) = f_{/Y_x} \ .$$

(See Kitchen and Robbins [4], §3, Example 2. Also see Seda [1] for an alternative description of the bundle $\pi: E \to X$.) Moreover, the bundle $\pi: E \to X$ is norm continuous (by Theorem 1 of Seda [1]).

If we assume that Y is, additionally, a metric space, then X is also a metric space and $C(Y)$ is separable. Consequently, $\pi: E \to X$ is a separable bundle. (Let $\{f_n\}$ be a countable dense subset of $C(Y)$. If g is an element of the fiber space E, say $g \in C(Y_x)$, then g can be extended to a continuous function $\bar{g}$ on Y (by the Tietze extension theorem). Given $\varepsilon > 0$, there exists an n such that $|f_n - \bar{g}| < \varepsilon$ uniformly on Y and hence uniformly on Y_x. Thus $\|\hat{f}_n(x) - g\| < \varepsilon$.)

THEOREM 4. Let X and Y be compact Hausdorff. Given $x_0 \in X$ and given a regular Borel measure μ on Y_{x_0}, there is an internal functional $T: E \to C$ such that

a) T_{x_0} = the functional on $C(Y_{x_0})$ defined by μ;

b) $\|T\| = \|\mu\|$.

Thus, there exists a family of regular Borel measures $\{\mu_x: x \in X\}$ such that

1) for each x, μ_x is a measure on Y_x;

2) $\mu_{x_0} = \mu$;

3) $\|\mu_x\| \leq \|\mu\|$ for all x;

4) for each $f \in C(Y)$ the function T_f defined by

$$T_f(x) = \int_{Y_x} f(y)d\mu_x(y)$$

is continuous on X.

The reader might wish to compare the preceding theorem with Theorem 2 in Seda [1].

Theorem 21.21 of Gierz [3] is also applicable in this situation with the following result:

THEOREM 5. Let X and Y be compact metric and let ϕ be any element of $C(Y)^*$. Then there exists a positive regular Borel measure μ on X and a family of measures $\{\mu_x: x \in X\}$ such that

1) μ_x is a measure on Y_x and $\|\mu_x\| \leq 1$;

2) for each $f \in C(Y)$ the map

$$x \to \int_{Y_x} f(y)d\mu_x(y)$$

is Borel measurable and bounded;

3) for all $f \in C(Y)$,

$$\phi(f) = \int_X \left(\int_{Y_x} f(y)d\mu_x(y)\right)d\mu(x) .$$

One might hope that the measures μ and $\{\mu_x: x \in X\}$ could always be chosen so that the map mentioned in 2) would be not only Borel measurable, but continuous. If so, the functional ϕ in $C(Y)^*$ would have a factorization $\phi = \alpha \circ \beta$, which belongs to the internal dual $\mathrm{Mod}_{C(X)}(C(Y),C(X))$ and α belongs to $C(X)^*$, the external dual of $C(X)$. Such a hope is naive as the following simple example shows.

EXAMPLE. Let $X = [0,1]$, let $Y = [0,1] \times [0,1]$, and let $p: Y \to X$ be the first coordinate projection. We let $\phi \in C(Y)^*$ be defined by

$$\phi(f) = \int_0^{1/2} f(x,0)dx + \int_{1/2}^1 f(x,1)dx .$$

Then the measure μ on X is Lebesgue measure and one possible choice of the measures $\{\mu_x\}$ is the following:

$$\mu_x = \begin{cases} \text{unit point mass at } (x,0), \text{ if } 0 \le x \le 1/2 \\ \text{unit point mass at } (x,1), \text{ if } 1/2 < x < 1 , \end{cases}$$

in which case the function mentioned in 2) is

$$g(x) = \begin{cases} f(x,0), \text{ if } 0 \le x \le 1/2 \\ f(x,1), \text{ if } 1/2 < x \le 1 . \end{cases}$$

If the function $f \in C(Y)$ has the value 0 on the set $\{(x,0): 0 \le x \le 1\}$ and the value 1 on the set $\{(x,1): 0 \le x \le 1\}$, then the function g is obviously discontinuous at $x = 1/2$. (Of course, the choice of measures $\{\mu_x: x \in X\}$ could be altered on a set of Lebesgue measure zero, but for no such choice could the resulting function g be continuous at $1/2$.)

ACKNOWLEDGMENTS. The authors wish to thank Professors Gierz and Seda for their courtesy in making unpublished work available to them.

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