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RESEARCH NOTES
ON THE CONVERGENCE OF FOURIER SERIES

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ABSTRACT. We define the space $B^p = \{f:(-\pi,\pi] \to R,\ f(t) = \sum_{n=0}^{\infty} c_n b_n(t),\ \sum_{n=0}^{\infty} |c_n| < \infty\}$. Each b_n is a special p-atom, that is, a real valued function, defined on $(-\pi,\pi]$, which is either $b(t) = 1/2\pi$ or $b(t) = \frac{-1}{|I|^{1/p}} \chi_R(t) + \frac{1}{|I|^{1/p}} \chi_L(t)$, where I is an interval in $(-\pi,\pi]$, L is the left half of I and R is the right half. $|I|$ denotes the length of I and χ_E the characteristic function of E. B^p is endowed with the norm $\|f\|_{B^p} = \text{Int} \sum_{n=0}^{\infty} |c_n|$, where the infimum is taken over all possible representations of f. B^p is a Banach space for $1/2 < p < \infty$. B^p is continuously contained in L^p for $1 \leq p < \infty$, but different. We have

THEOREM. Let $1 < p < \infty$. If $f \in B^p$ then the maximal operator $Tf(x) = \sup_n |S_n(f,x)|$ maps B^p into the Lorentz space $L(p,1)$ boundedly, where $S_n(f,x)$ is the n^{th}- sum of the Fourier Series of f.

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1. INTRODUCTION.

We define the space $B^p = \{f:(-\pi,\pi] \to R,\ f(t) = \sum_{n=0}^{\infty} c_n b_n(t),\ \sum_{n=0}^{\infty} |c_n| < \infty\}$. Each b_n is a special p-atom that is, a real valued function b, defined on $(-\pi,\pi]$, which is either $b(t) = 1/2\pi$ or $b(t) = \frac{-1}{|I|^{1/p}} \chi_R(t) + \frac{1}{|I|^{1/p}} \chi_L(t)$, where I is an interval in $(-\pi,\pi]$, L is the left half of I and R is the right half. $|I|$ denotes the length of I and χ_E the characteristic function of E. B^p is endowed with the norm $\|f\|_{B^p} = \text{Inf} \sum_{n=0}^{\infty} |c_n|$, where the infimum is taken over all possible representations of f. B^p is a Banach space for $1/2 < p < \infty$.

These spaces were originally introduced by the author in [2]. Also see [3], [4], and [5].

In this note we are interested in p belonging to the interval $(1,\infty)$, that is, $1 < p < \infty$.

The Carleson-Hunt theorem on the almost everywhere convergence of a function f in L^p for $1 < p < \infty$, asserts that if $f \in L^p$ then the Fourier Series of f, denoted by $S(f,x)$, converges to f almost everywhere. As is well known, this is a very powerful theorem in the theory of Fourier Analysis. However, many analysts agree that the proof of this theorem is too complicated to be accessible to the general audience, so that many researchers in the field have tried to give a suitable proof, but as far as I know nobody has done so. We refer the interested reader to [1] and [6].

In this note we present a proper subset of L^p for $1 < p < \infty$, namely the B^p functions, for which the proof of the convergence of Fourier Series is relatively simple.

The proof follows basically the idea of Carleson-Hunt, namely, we will prove that if $f \in B^p$ then the operator defined by $Tf(x) = \sup_n |S_n(f,x)|$ where $S_n(f,x)$ is the n^{th}-sum of the Fourier series of f, is a bounded operator into the Lorentz space $L(p,1)$.

Recall that a measurable function f belongs to the Lorentz space $L(p,q)$ if $\|f\|_{pq} = (\frac{1}{q}\int_0^\infty (f^*(t)t^{1/p})^q \frac{dt}{t})^{1/q} < \infty$, for $0 < p \leq \infty$, $0 < q \leq \infty$, where f^* is the decreasing rearrangement of f, defined by $f^*(t) = \mathrm{Inf}\{y, |\{x: |f(x)| > y\}| \leq t\}$, with $|\cdot|$ being the Lebesgue measure. Observe that $L(p,p)$ is the usual L^p-space. The space $L(p,\infty)$ is also known as the weak L^p-space.

2. MAIN RESULTS.

The main result can be stated as follows.

THEOREM A. Let $1 < p < \infty$. If $f \in B^p$ then the maximal operator defined by $Tf(x) = \sup_n |S_n(f,x)|$ maps B^p into $L(p,1)$ boundedly, that is, $\|Tf\|_{p1} \leq M\|f\|_{B^p}$ where M is a positive absolute constant, and $S_n(f,x)$ being the n^{th}-sum of the Fourier series of f.

PROOF. First of all we notice that the operator $T_a f = f^a$ where $f^a(x) = f(x-a)$ maps B^p into B^p continuously, so that we just need to prove this result for $f_h(t) = -\frac{1}{(2h)^{1/p}}\chi_{[-h,0)}(t) + \frac{1}{(2h)^{1/p}}\chi_{[0,h]}(t)$, $h > 0$ which will follow easily from the estimate for $g(t) = \chi_{[0,h]}(t)$. In fact let $S_n(g,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)D_n(x-t)dt$ where $D_n(t) = \frac{\sin(n+1/2)t}{2 \sin t/2}$ is the Dirichlet Kernel. Therefore we have $S_n(g,x)= \frac{1}{\pi}\int_{x-h}^{x} D_n(t)dt$, we now use the elementary inequality $|D_n(t)| \leq \frac{c}{|t|}$, $|t| \leq \pi$, satisfied by the Dirichlet Kernels D_n, where c is an absolute constant. Thus

$$\left|S_n(g,x)\right| \leq \int_{x-h}^{x} \left|D_n(t)\right| dt \leq \int_{x-h}^{x} \frac{1}{|t|} dt \leq \frac{ch}{x-h} \leq \frac{2ch}{x}$$

for $x > 2h$ and $\left|S_n(g,x)\right| \leq \frac{2ch}{-x}$ for $x < -2h$.

I recall that $\int_0^x D_n(t)dt$ is uniformly bounded in n and x, that is, $\left|\int_0^x D_n(t)dt\right| < A$ where A is an absolute constant see [7], Volume 1, page 57.

Consequently we have i) $Tg(x) \leq A \ \forall x$ and ii) $Tg(x) \leq \frac{2ch}{|x|}$ for $|x| > 2h$.

We now evaluate $\|Tg\|_{p1}$ using the definition of $L(p,1)$ norm. (See definition of $L(p,1)$ given before) we have

$$\|Tg\|_{p1} = \frac{1}{p}\int_0^\infty (Tg)^*(t)t^{\frac{1}{p}-1} dt = \frac{1}{p}\int_0^{2h}(Tg)^*(t)t^{\frac{1}{p}-1} dt + \frac{1}{p}\int_{2h}^\infty (Tg)^*(t)t^{\frac{1}{p}-1} dt$$

$$\leq \frac{A}{p}\int_0^{2h} t^{\frac{1}{p}-1} dt + \frac{2ch}{p}\int_{2h}^\infty t^{\frac{1}{p}-2} dt = A(2h)^{\frac{1}{p}} + \frac{c}{p-1}(2h)^{\frac{1}{p}}$$

for $1 < p < \infty$, therefore $\|Tg\|_{p1} < M(2h)^{\frac{1}{p}}$ where $M = A + \frac{c}{p-1}$.

Now for $f_h(t) = \frac{-1}{(2h)^{1/p}}\chi_{[-h,0)}(t) + \frac{1}{(2h)^{1/p}}\chi_{[0,h]}(t)$ we get $\|Tf_h\|_{p1} \leq M$ and so if $f(t) = \sum_{n=0}^\infty c_n b_n(t)$ where the b_n are special p-atoms and $\sum_{n=0}^\infty |c_n| < \infty$ we have $\|Tf\|_{p1} \leq M \sum_{n=0}^\infty |c_n|$ which implies $\|Tf\|_{p1} \leq M\|f\|_{B^p}$. The proof is complete.

COROLLARY. Let $1 < p < \infty$. If $f \in B^p$ then $S_n(f,x)$ converges to $f(x)$ almost everywhere, where $S_n(f,x)$ is the n^{th}-sum of the Fourier series of f.

PROOF. Let f in B^p for $1 < p < \infty$, then $f(t) = \sum_{k=0}^\infty c_k b_k(t)$ where $\sum_{k=0}^\infty |c_k| < \infty$ and the b_k are special p-atoms. Then we define the function $w_f(x)$ by $w_f(x) = \limsup_{\substack{n\to\infty \\ k\geq n \\ \ell\geq n}} \left|S_k(f,x) - S_\ell(f,x)\right|$ so that, $S_n(f,x)$ converges to f almost everywhere if and only if $w_f(x) = 0$ almost everywhere.

Observe now that $w_f(x) \leq 2Tf(x)$. Therefore $w_f \in L(p,1)$. On the other hand we see that $w_f(x) = w_{f-f_n}(x)$ where $f_n \to f$ in the B^p-norm, namely take $f_n(t) = \sum_{k=0}^n c_k b_k(t)$ a finite linear combination of special p-atoms b_k, then $w_f(x) = w_{f-f_n}(x) \leq 2T(f-f_n)(x)$ and consequently theorem A, that is, the boundedness of T, implies that $\|w_f\|_{p1} < 2\|T(f-f_n)\|_{p1} < 2M\|f-f_n\|_{B^p}$, so that as $\|f - f_n\|_{B^p} \to 0$ for $n \to \infty$, we get $\|w_f\|_{p1} = 0$, thus $w_f(x) = 0$ almost everywhere, which implies

$S_n(f,x) \to f(x)$ almost everywhere. The proof is complete.

We would like to point out that other direct proofs of the almost everywhere convergence for functions in B^p are also available. However we prefer this one because it is a consequence of the boundedness of the maximal operator and this boundedness could be useful in other contexts, as for example in interpolation of operators.

Another way of proving the almost everywhere convergence is by observing that if f is in B^p then

$$\int_0^{2\pi}\int_0^{2\pi} \left| \frac{f(x) - f(y)}{x - y} \right| dxdy < \infty.$$

The proof of this assertion is elementary and will be left to the interested reader, however we point out that we just need to prove it for functions of the type $f(x) = \frac{1}{h^{1/p}} \chi_{[0,h]}(x)$.

A consequence of the boundedness of the above integral is that any f in B^p satisfies Dini's condition and therefore the almost everywhere convergence is readily established.

One of the important features of the spaces B^p for $1 \leq p < \infty$ is that it can be identified with the space of analytic functions F on the disk $D = \{z \in C: |z| < 1\}$ satisfying

$$\int_0^1 \int_0^{2\pi} |F'(re^{i\theta})|(1-r)^{1/p-1}\, d\theta dr < \infty$$

where the dash means derivative.

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