SCATTERING OF SURFACE WAVES BY A HALF IMMERSED
CIRCULAR CYLINDER IN FLUID OF FINITE DEPTH

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ABSTRACT. A train of surface waves is normally incident on a half immersed circular cylinder in a fluid of finite depth. Assuming the linearized theory of fluid under gravity an integral equation for the scattered velocity potential on the half immersed surface of the cylinder is obtained. It has not been found possible to solve this in closed form even for infinite depth of fluid. Our purpose is to obtain the asymptotic effect of finite depth "h" on the transmission and reflection coefficients when the depth is large. It is shown that the corrections to be added to the infinite depth results of these coefficients can be expressed as algebraic series in powers of a/h starting with $(a/h)^2$ where "a" is the radius of the circular cylinder. It is also shown that the coefficients of $(a/h)^2$ in these corrections do not vanish identically.

1.   INTRODUCTION.

This paper deals with the reflection and transmission of surface waves normally incident on an infinitely long cylinder with circular cross section of radius "a" and horizontal generators, half immersed in an ideal fluid of infinite horizontal extent but finite constant depth "h." The two-dimensional linearized theory of an ideal fluid under gravity is assumed to hold in the fluid region. The motion of fluid can then be taken as irrotational and can be described by velocity potential. The existence and uniqueness of solution for a partly immersed obstacle was discussed by

John [1] in fluid of both infinite and finite depth, although a closed form solution is apparently not obtainable.

Dean and Ursell [2] investigated the interaction of a fixed half immersed circular cylinder with a train of surface waves in a fluid of infinite depth. They expressed the potential in the fluid region as a sum of wave source potentials and multiple singularities at the center of the circle with unknown coefficients and finally obtained an infinite set of linear algebraic equations involving the unknown coefficients. This infinite set was then solved numerically for particular values of wave number, and the reflection and transmission coefficients were then computed numerically. The present problem is a generalization of the above where the depth of fluid is taken into account. By the use of Green's integral theorem the velocity potential at any point in the fluid region can be made to depend upon the scattered potential on the half-immersed surface of the cylinder. This potential can be shown to satisfy an integral equation of the second kind by another use of Green's theorem. We recall that in the plane vertical barrier problems in fluid of infinite depth, by the use of Green's theorem and applying boundary condition on the barrier, a singular integral equation of the first kind in the difference of potential across the barrier could be obtained and this could be solved in a closed form (see Goswami [3]). Then for finite depth of fluid a perturbation technique could be applied so as to obtain correction to the reflection and transmission coefficients for finite depth over their infinite depth values in the cases when the barrier is in the form of a submerged fixed vertical barrier (see Goswami [4]) and a submerged fixed vertical plate (see Goswami [5]). Mei and Black [6] considered scattering of surface waves by rectangular obstacles fixed at the bottom or at the free surface in a fluid of finite depth and applied variational techniques to obtain the reflection and transmission coefficients numerically. From their results it is not apparently possible to obtain the effect of finite depth. Macaskill [7] recently considered scattering of surface waves by a vertical barrier of arbitrary permeability in a fluid of finite depth. By the use of Green's theorem he reduced the problem to a set of integral equations which are then solved numerically by using a special decomposition of the finite depth source potential.

For obstacles of types other than vertical barriers, the corresponding problem is rather complicated. Even the apparently simple case of a half immersed circular cylinder in fluid of infinite depth can no longer be solved in closed forms, although asymptotic methods are available for both long and short waves. For short waves in a fluid of infinite depth Ursell [8] used an integral equation approach. In this case there is only one parameter to consider, viz. Ka where K is the infinite depth wave number and "a" is the radius of the cylinder. But in the finite depth case there are two independent parameters out of Ka, Kh, a/h instead of only one in the infinite depth case. In the present paper we take Ka to be moderate and consider h to be large so that Kh is large and a/h is small. If we want to deal with cases where K is very large or K is very small, then the present method will perhaps fail. Under this assumption, it is found that the finite depth corrections of transmission and reflection coefficients over their infinite depth values can be expressed

as algebraic series in a/h starting with $(a/h)^2$. It is also shown that the coeffi-
cients of $(a/h)^2$ in these series do not vanish identically. A similar conclusion
was arrived at by Goswami ([4], [5]) when the obstacle is in the form of a submerged
vertical barrier and plate in fluid of finite depth. However, no numerical result is
presented here.

2. FORMULATION OF THE PROBLEM.

We will use rectangular Cartesian coordinates with origin on the undisturbed free
surface and at the center of the circular cylinder of radius "a," the y-axis verti-
cally downwards, the z-axis horizontal along the generators of the cylinder and the
x-axis perpendicular to the yz-plane. Let "h" be the depth of water below the free
surface. Let a train of surface waves from positive infinity (x direction) be made
incident normally on the cylinder. The motion of the fluid is irrotational and can be
described by a velocity potential given by Re $\{\phi(x,y)e^{-i\sigma t}\}$ where σ is the angular
frequency, the incoming wave being described by Re $\{\phi_{inc}(x,y)e^{-i\sigma t}\}$, since the
motion is two-dimensional and independent of z. The time independent complex valued
potential function $\Phi(x,y)$ satisfies the Laplace's equation throughout the fluid
region, i.e.,

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \phi = 0 \quad , o < y < h \quad , |x| < \infty \quad , r < a, \quad (2.1)$$

where $r^2 = x^2 + y^2$. The linearized boundary condition on the undisturbed free
surface is

$$\frac{\partial \phi}{\partial y} + K \phi = 0, \quad y = 0, \quad |x| > a, \quad (2.2)$$

where $K = \sigma^2/g$, g being the acceleration due to gravity. On the bottom of the
fluid, the normal component of velocity is zero, so that

$$\frac{\partial \phi}{\partial y} = 0, \quad y = h. \quad (2.3)$$

Also since the cylinder is at rest, there is no flow across the surface, so that

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{on} \quad r = a, \quad y > 0. \quad (2.4)$$

At great distance from the cylinder, the incident wave train produces a transmitted
wave train moving in the negative x-direction for $x \to -\infty$ and a reflected wave train
moving in the positive x direction for $x \to +\infty$. These conditions can be mathemati-
cally expressed as

$$\phi(x,y) \to \phi_{inc}(x,y) + R\phi_{inc}(-x,y) \quad \text{as} \quad x \to +\infty \quad (2.5)$$

and $\qquad \phi(x,y) \to T \phi_{inc}(x,y) \qquad \text{as} \quad x \to -\infty \quad (2.6)$

where R and T are respectively the reflection and transmission coefficients which
are to be determined.

The incident field is given by

$$\phi_{inc}(x,y) = \frac{\cosh k_0(h-y)}{\cosh k_0 h} e^{-ik_0 x} \quad (2.7)$$

where k_0 is the positive root of the transcendental equation

$$K = k_0 \tanh k_0 h. \quad (2.8)$$

We note that, for large h,

$$k_0 \simeq K (1 + 2e^{-2 kh}), \quad (2.9)$$

so that k_0 and K differ by an exponentially small quantity for large h.

Let us write

$$\phi(x,y) = \phi(x,y) + \phi_{inc}(x,y),\qquad(2.10)$$

so that $\phi(x,y)$ is the scattered potential and it satisfies (2.1), (2.2), (2.3) together with

$$< a\frac{\partial}{\partial r}\phi(x,y) > = - < a\frac{\partial}{\partial r}\phi_{inc}(x,y) > \equiv v(\theta),\ \text{say},\ 0 < \theta < \pi\qquad(2.11)$$

where the angular bracket denotes the value at $r = a$, and $x = r\cos\theta$, $y = r\sin\theta$. Conditions (2.5) and (2.6) respectively reduce to

$$\phi(x,y) \rightarrow R\ \phi_{inc}(-x,y)\qquad \text{as } x \rightarrow +\infty\qquad(2.12)$$

$$\phi(x,y) \rightarrow (T-1)\phi_{inc}(x,y)\qquad \text{as } x \rightarrow -\infty\qquad(2.13)$$

which show that $\phi(x,y)$ behaves as an outgoing wave as $|x| \rightarrow \infty$.

3. INTEGRAL EQUATION FOR $\phi(x,y)$ ON THE CYLINDER.

Let $G(x,y;\ \xi,\ \eta)$ be the Green's function satisfying the Laplace equation

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\ G = 0\quad \text{except at } (\xi,\ \eta),\ 0 < y < h,\ |x| < \infty\qquad(3.1)$$

with the conditions

$$\frac{\partial G}{\partial y} + K\ G = 0,\quad y = 0,\quad |x| < \infty\qquad(3.2)$$

$$\frac{\partial G}{\partial y} = 0,\quad y = h,\quad |x| < \infty\qquad(3.3)$$

$$G \rightarrow \ln\rho\ \text{as}\ \rho \equiv \{(x-\xi)^2 + (y-\eta)^2\}^{\frac{1}{2}} \rightarrow 0,\qquad(3.4)$$

and

$$G \rightarrow\ \text{a multiple of}\ \frac{\cosh k_0(h-y)}{\cosh k_0 h}\ e^{ik_0|x-\xi|}\qquad \text{as } |x-\xi| \rightarrow \infty.\qquad(3.5)$$

Condition (3.5) is the radiation condition at infinity. Then G is given by (see Thorne [9])

$$G(x,y;\ \xi,\ \eta) = \ln\frac{\rho}{\rho'} -2\fint_0^\infty (\frac{e^{-kh}}{k}\sinh k\eta\ \sinh ky +$$

$$\frac{\cosh k(h-y)\ \cosh k(h-\eta)}{k\ \sinh kh - K\cosh kh})\ \frac{\cos k(x-\xi)}{\cosh kh}\ dk\qquad(3.6)$$

where $\rho' \equiv \{(x-\xi)^2 + (y+\eta)^2\}^{\frac{1}{2}}$, and the contour of integration is indented below the simple pole at $k = k_0$ to ensure the radiation condition (3.5) at infinity. For, as $|x-\xi| \rightarrow \infty$, it can be shown that

$$G(x,y;\ \xi,\ \eta) \rightarrow -2\pi\ iF(k_0h)\ \frac{\cosh k_0(h-y)\ \cosh k_0(h-\eta)}{\cosh^2 k_0 h}\ e^{ik_0|x-\xi|}\qquad(3.7)$$

where $F(k_0h) = \frac{2\cosh^2 k_0 h}{2 k_0 h + \sinh 2 k_0 h} \simeq 1 + 0\ (khe^{-kh})$ for large h.\qquad(3.8)

Applying Green's integral theorem to $\phi(x,y)$ and $G(x,y;\ a\cos\alpha,\ a\sin\alpha)$ in the fluid region with a small indentation at the point $(a\cos\alpha,\ a\sin\alpha)$ on the circle $r = a$, we obtain

$$-\pi\ \phi(\alpha) + \int_0^\eta \phi(\theta) < a\frac{\partial}{\partial r} G_1(r,\theta;\ a,\ \alpha) > d\theta = \int_0^\pi v(\theta) < G(r,\theta;\ a,\ \alpha) > d\theta\qquad(3.9)$$

where $G_1 (r, \theta; a, \alpha) = G (x,y; a \cos\alpha, a \sin\alpha) - \frac{1}{2} \ln[\{(x - a \cos\alpha)^2$
$+ (y - a \sin\alpha)^2\}/\{(x - a \cos\alpha)^2 + (y + a \sin\alpha)^2\}]$, and the angular brackets denote the values at $r = a$. Equation (3.9) provides an integral equation for $\phi(\theta)$ on the contour of the immersed portion of the cylinder. In deriving (3.9) we have made use of the fact that

$$\int_0^\pi < a \frac{\partial}{\partial r} \ln \frac{(r \cos\theta - a \cos\alpha)^2 + (r \sin\theta - a \sin\alpha)^2}{(r \cos\theta - a \cos\alpha)^2 + (r \sin\theta + a \sin\alpha)^2} > d\theta = 0.$$

The potential function $\phi(\xi, \eta)$ at a general point (ξ, η) of the fluid region can be obtained by applying Green's theorem in the fluid region to $\phi(x,y)$ and $G(x,y; \xi, \eta)$, excluding the point (ξ, η), and is given by

$$- 2 \pi \phi(\xi, \eta) = \int_0^\pi \phi(\theta) < a \frac{\partial}{\partial r} G (x,y; \xi, \eta) > d\theta$$

$$- \int_0^\pi v(\theta) < G (x,y; \xi, \eta) > d\theta. \quad (3.10)$$

Let the quantities with a superscript ∞ denote their infinite depth values. Then

$$G^\infty (x,y; \xi, \eta) = \ln \frac{\rho}{\rho'} - 2 \int_0^\infty \frac{e^{-k(y+\eta)}}{k - K} \cos k (x - \xi) dk \quad (3.11)$$

$$\phi_{inc}^\infty (x,y) = e^{-Ky - iKx}. \quad (3.12)$$

Let

$$G = G^\infty + G_D, \quad \phi_{inc} = \phi_{inc}^\infty + \phi_{inc}^D (x,y).$$

We note that $\phi_{inc}^D (x,y)$ is exponentially small for large h. Also, from (3.16) and (3.11), $G_D (x,y; \xi, \eta)$ can readily be shown to be equal to

$$-2 \int_0^\infty \frac{e^{-kh} (K\sinh ky - k \cosh ky)(K \sinh k\eta - k \cosh k\eta)}{k (k - K)(k \sinh kh - K \cosh kh)} \cos k (x - \xi) dk \quad (3.13)$$

where the contour is indented below the poles at $k = K$ and $k = k_0$. For infinite depth the integral equation (3.9) reduces to

$$-\pi \phi^\infty (\alpha) + \int_0^\pi \phi^\infty(\theta) < a \frac{\partial}{\partial r} G_1^\infty (r, \theta; a, \alpha) > d\theta$$

$$= \int_0^\pi v^\infty(\theta) G^\infty (a, \theta; a, \alpha) d\theta, 0 < \alpha < \pi. \quad (3.14)$$

Let

$$\phi(\theta) = \phi^\infty(\theta) + \phi_D(\theta), \quad v(\theta) = v^\infty(\theta) + v_D(\theta)$$

where obviously $v_D(\theta)$ is exponentially small for large h, and

$$v^\infty(\theta) = - < a \frac{\partial}{\partial r} \phi_{inc}^\infty (x,y) > = i Ka e^{-i Ka \cos\theta - Ka \sin\theta - i\theta}. \quad (3.15)$$

From (3.9) using (3.14) and neglecting exponentially small quantities for large h, we obtain the integral equation for $\phi_D(\theta)$ as

$$-\pi \phi_D(\alpha) + \int_0^\pi \phi_D(\theta) \{< a \frac{\partial}{\partial r} G_1^\infty (r, \theta; a, \alpha)> + < a \frac{\partial}{\partial r} G_D (r, \theta; a, \alpha) >\} d\theta$$

$$= \int_0^\pi v^\infty(\theta) < G_D (r, \theta; a, \alpha) > d\theta$$

$$- \int_0^\pi \phi^\infty(\theta) < a \frac{\partial}{\partial r} G_D (r, \theta; a, \alpha) > d\theta, \quad 0 < \alpha < \pi \tag{3.16}$$

where now $\phi^\infty(\theta)$ is supposed to be known being obtained as the solution of the integral equation (3.14).

4. REFLECTION AND TRANSMISSION COEFFICIENTS.

Using (3.7) in (3.10) we obtain as $\xi \to \infty$,

$$\phi(\xi, \eta) \to F(k_0 h) \, \phi_{inc}(-\xi, \eta) \int_0^\pi [\phi(\theta) \, ak_0 \cosh \{k_0(h - a \sin\theta) - i\theta\}$$

$$- i \, v(\theta) \cosh k_0(h - a \sin\theta)] \frac{e^{-ik_0 a \cos\theta}}{\cosh k_0 h} \, d\theta,$$

so that by (2.12) the reflection coefficient R is given by,

$$R = F(k_0 h) \int_0^\pi [\phi(\theta) \, ak_0 \cosh \{k_0(h - a \sin\theta) - i\theta\}$$

$$- i \, v(\theta) \cosh k_0(h - a \sin\theta)] \frac{e^{-ik_0 a \cos\theta}}{\cosh k_0 h} \, d\theta. \tag{4.1}$$

Again, similarly using (3.7) in (3.10) we obtain as $\xi \to -\infty$,

$$\phi(\xi, \eta) \to -F(k_0 h) \, \phi_{inc}(\xi, \eta) \int_0^\pi [\phi(\theta) \, ak_0 \cosh \{k_0(h - a \sin\theta) + i\theta\}$$

$$+ iv(\theta) \cosh k_0(h - a \sin\theta)] \frac{e^{ik_0 a \cos\theta}}{\cosh k_0 h} \, d\theta$$

so that by (2.13), we obtain the transmission coefficient T as

$$T = 1 - F(k_0 h) \int_0^\pi [\phi(\theta) \, ak_0 \cosh \{k_0(h - a \sin\theta) + i\theta\}$$

$$+ iv(\theta) \cosh k_0(h - a \sin\theta)] \frac{e^{ik_0 a \cos\theta}}{\cosh k_0 h} \, d\theta. \tag{4.2}$$

Making $h \to \infty$ in (4.1) we obtain

$$R^\infty = aK \int_0^\pi [\phi^\infty(\theta) \, e^{-Ka(\sin\theta + i \cos\theta)}] \, e^{-Ka(\sin\theta + i \cos\theta) - i\theta} \, d\theta. \tag{4.3}$$

Neglecting exponentially small quantities for large h we obtain

$$R - R^\infty \simeq a K \int_0^\pi \phi_D(\theta) \, e^{-Ka(\sin\theta + i \cos\theta) - i\theta} \, d\theta. \tag{4.4}$$

Similarly from (4.2)

$$T^\infty = 1 - aK \int_0^\pi [\phi^\infty(\theta) \, e^{i\theta} - e^{-Ka(\sin\theta - i \cos\theta) - i\theta}] \, e^{-Ka(\sin\theta - i \cos\theta)} d\theta, \tag{4.5}$$

and neglecting exponentially small quantities

$$T - T^\infty \simeq - a K \int_0^\pi \phi_D(\theta) \, e^{-Ka(\sin\theta - i \cos\theta) + i\theta} d\theta. \tag{4.6}$$

Thus the depth correction to the reflection and transmission coefficients over their infinite depth values are approximately given by (4.4) and (4.6) respectively for large h. It will be shown that these are algebraically small for large h and, in fact, can be expressed as algebraic series in a/h starting with $(a/h)^2$. The

quantity $\phi_D(\theta)$ in (4.4) and (4.6) satisfies the integral equation (3.16). Now $< G_D >$ and $< a \frac{\partial}{\partial r} G_D (r, \theta; a, \alpha) >$ in the right side of (3.16) can be expressed as a series in terms of a/h for large h starting with $(a/h)^2$ assuming Ka to be fixed. This suggests that $\phi_D(\theta)$ has also a similar type of expansion for large h. Thus the expressions (4.4) and (4.6) are algebraically small for large h. That the first terms in the expansion of (4.4) and (4.6) are nonvanishing will be shown later. These are done in the next sections.

5.   ASYMPTOTIC EXPANSION FOR LARGE h.

By (3.13)

$$G_D (x,y; \xi, \eta) = -2 \int_0^\infty \frac{e^{-kh}(K\sinh ky - k \cosh ky)(K\sinh k\eta - k \cosh k\eta)}{k(K-k)(K\cosh kh - k \sinh kh)} \cos k(x - \xi) \, dk. \tag{5.1}$$

In view of (3.16), we confine our attention to $|y| \le a$, $|\eta| \le a$, $|x| \le a$, $|\xi| \le a$. The integrand in (5.1) (excepting the term $\cos k(x - \xi)$) is of the order $e^{-2kh} e^{2ka}$, which is obviously exponentially small when $k > k^*$ where k^* is sufficiently large, and it is still exponentially small when we replace $|\cos k(x - \xi)|$ by the crude bound $\cos 2ka$. Thus we can expand the factors in

$$\frac{(K\sinh ky - k \cosh ky)(K \sinh k\eta - k \cosh k\eta)}{k (K - k)} \cos k (x - \xi) \tag{5.2}$$

by the power series expansion in k. Also write

$$\frac{e^{-kh}}{K\cosh kh - k\sinh kh} = \frac{e^{-kh}}{K\cosh kh} \sum_{m=0}^\infty \left(\frac{k}{K}\right)^m \tanh^m kh$$

$$= \frac{1}{K} (1 - \tanh kh) \sum_{m=0}^\infty \left(\frac{k}{K}\right)^m \tanh^m kh.$$

In (5.2), put $k = Ku$, then (5.2) becomes

$$\frac{(K\sinh uKy - K u \cosh uKy)(K \sinh uK\eta - K u \cosh Ku\eta) \cos u(Kx - K\xi)}{u K^2 (1 - u)}$$

$$= \sum_{s=1}^\infty u^s f_s (Ky, K\eta, K|x - \xi|), \text{ say,}$$

where $f_1 (Ky, K\eta, K|x - \xi|) = (Ky - 1)(K\eta - 1)$,
and other f_s's can be found.
Thus the integrand in (5.1) can be expanded as

$$\frac{e^{-kh}}{K\cosh kh} \left(\sum_{m=0}^\infty \left(\frac{k}{K}\right)^m \tanh^m kh \right) \left(\sum_{s=1}^\infty \left(\frac{k}{K}\right)^s f_s (Ky, K\eta, K|x - \xi|) \right).$$

Hence $G_D (x,y; \xi, \eta) = -2 \int_0^\infty \frac{e^{-u}}{K\cosh u} \left(\sum_{m=0}^\infty \frac{u^m \tanh^m u}{(Kh)^m} \right) \left(\sum_{s=1}^\infty \frac{u^s}{(Kh)^s} \right) \times$

$$f_s (Ky, K\eta, K|x - \xi|)) \frac{du}{h}$$

$$= - \frac{2}{Kh} \sum_{m=0}^\infty \sum_{s=1}^\infty \frac{1}{(Kh)^{m+s}} f_s (Ky, K\eta, K|x - \xi|) \alpha_{ms}$$

where

$$\alpha_{ms} = \int_0^\infty \frac{e^{-u}}{\cosh u} u^{m+s} \tanh^m u \, du.$$

Thus the first term in the expansion of $G_D(a \cos\theta, a \sin\theta; a \cos\alpha, a \sin\alpha)$ is

$$- \frac{2\alpha_{01}}{(Kh)^2} f_1 (Ka \sin\theta, Ka \sin\alpha, Ka|\cos\theta - \cos\alpha|) = - \left(\frac{a}{h}\right)^2 \frac{2\alpha_{01}}{(Ka)^2}(Ka \sin\theta - 1)(Ka \sin\alpha - 1)$$

and the first term in the expansion of $< a \frac{\partial}{\partial r} G_D$ (r cosθ, r sinθ; a cosα, a sinα) $>$ is

$$- \frac{2\alpha_{01}}{(Kh)^2} < a \frac{\partial}{\partial r} f_1 (Ky, Ka \sin\alpha, K|x - a \cos\alpha|) >$$

$$= - \frac{2Ka \, \alpha_{01}}{(Kh)^2} (Ka \sin\alpha - 1) \sin\theta = - \left(\frac{a}{h}\right)^2 \frac{2\alpha_{01}}{Ka} (Ka \sin\alpha - 1) \sin\theta,$$

where

$$\alpha_{01} = \int_0^\infty \frac{e^{-u}}{\cosh u} u \, du = \pi^2/24.$$

These suggest an asymptotic expansion of $\phi_D(\theta)$ for large "h" as

$$\phi_D (\theta) = \frac{1}{(Kh)^2} \psi_D (\theta) + \frac{1}{(Kh)^3} \psi_D^{(3)} (\theta) + \ldots, \qquad (5.3)$$

when $\psi_D(\theta)$ satisfies

$$- \pi\psi_D (\alpha) + \int_0^\pi \psi_D (\theta) < a \frac{\partial}{\partial r} G_1^\infty (r \cos\theta, r \sin\theta; a \cos\alpha, a \sin\alpha) > d\theta$$

$$= - 2 \alpha_{01} (Ka \sin\alpha - 1) \int_0^\pi \{v^\infty (\theta)(Ka \sin\theta - 1) - Ka \, \phi^\infty (\theta) \sin\theta\} \, d\theta \qquad (5.4)$$

where $\phi^\infty(\theta)$ in the right side is known in principle. Thus to a first approximation

$$(R - R^\infty)_1 = \left(\frac{a}{h}\right)^2 \frac{1}{Ka} \int_0^\pi \psi_D (\theta) e^{-Ka (\sin\theta + i \cos\theta) - i\theta} \, d\theta$$

and
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.5)$$

$$(T - T^\infty)_1 = - \left(\frac{a}{h}\right)^2 \frac{1}{Ka} \int_0^\pi \psi_D (\theta) e^{-Ka (\sin\theta - i \cos\theta) + i\theta} \, d\theta.$$

We will now show that the integrals in (5.4) do not vanish identically. These integrals can be regarded as functions of Ka regular for all values of Ka. As these integrals no longer involve h, we can waive the restriction to Ka originally imposed and instead we assume Ka to be small in the discussion of the following section.

6. APPROXIMATE VALUES FOR $\phi^\infty (\theta)$.

In the above we have stated that $\phi^\infty(\theta)$ is known in principle which is in fact the solution of the integral equation (3.14). As it is almost impossible to solve analytically this integral equation for any Ka, without any loss of generality we will assume that Ka is small, and then obtain an approximate value of $\phi^\infty(\theta)$. Using this we will show that the integrals in (5.4) do not vanish for small Ka, and hence these do not identically vanish for any Ka.

Following Ursell [8], we can show that

$$G^\infty (a\cos\theta, a \sin\theta, a \cos\alpha, a \sin\alpha) = \ln \left| \frac{\sin \frac{\theta - \alpha}{2}}{\sin \frac{\theta + \alpha}{2}} \right|$$

$$+ 2[\{(\gamma + \ln 2ka - \pi i) + \ln \sin \frac{\theta + \alpha}{2} \} < e^{-K (y+\eta)} \cos K (x - \xi) >$$

$$- \frac{\alpha - \theta}{2} < e^{-K(y+\eta)} \sin K(x - \xi) > + \sum_{m=1}^\infty (-1)^{m-1} \frac{(1 + \frac{1}{2} + \ldots + \frac{1}{m})}{m!} (2Ka \sin \frac{\theta + \alpha}{2})^m \cos m \frac{\alpha - \theta}{2}]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.1)$$

and

$$< a \frac{\partial}{\partial r} G_1^\infty (r \cos\theta, r \sin\theta; a \cos\alpha, a \sin\alpha) > = < e^{-K(y+\eta)} \cos K (x - \xi) > +$$

$$2[i(\gamma + \ln 2Ka - \pi i) + \ln \sin \frac{\theta+\alpha}{2}] < a \frac{\partial}{\partial r} e^{-K(y+\eta)} \cos K (x - \xi) >$$

$$- < \zeta a \frac{\partial}{\partial r} e^{-K(y+\eta)} \sin k(x - \xi) >$$

$$+ \sum_{m=1}^{\infty} (-1)^{m-1} \frac{(1 + \frac{1}{2} + \ldots + \frac{1}{m})}{m!} < a \frac{\partial}{\partial r} (K\rho)^m \cos m\zeta >] \qquad (6.2)$$

where $\xi = a \cos\alpha$, $\eta = a \sin\alpha$, $< \zeta > = \frac{\alpha-\theta}{2}$, $< \rho > = 2a \sin \frac{\theta+\alpha}{2}$, and the angular brackets denote the values for $r = a$.

The integral equation (3.14) can now be rewritten as

$$- \pi \phi^\infty (\alpha) + \int_0^\infty M(\theta, \alpha) \phi^\infty(\theta) d\theta = L(\alpha, Ka) - A(Ka) < e^{-k\eta} \cos K\xi >$$

$$- B(Ka) < e^{-k\eta} \sin K\xi >, \ 0 < \alpha < \pi, \qquad (6.3)$$

where

$$M(\theta, \alpha) = 2[\ln \sin \frac{\theta+\alpha}{2} < a \frac{\partial}{\partial r} e^{-k(y+\eta)} \cos K(x - \xi) > - < \zeta a \frac{\partial}{\partial r} e^{-K(y+\eta)} \sin K(x - \xi) >$$

$$+ \sum_{m=1}^{\infty} (-1)^{m-1} \frac{(1 + \frac{1}{2} + \ldots \frac{1}{m} + m)}{m!} < a \frac{\partial}{\partial r} (K\rho)^m \cos m\zeta >], \qquad (6.4)$$

$$L(\alpha, Ka) = i Ka \int_0^\pi e^{-Ka(\sin\theta + i \cos\theta) - i\theta} G^\infty (a \cos\theta, \sin\theta; a \cos\alpha, a \sin\alpha) d\theta. \qquad (6.5)$$

$$\begin{matrix} A \\ B \end{matrix} = \int_0^\pi \phi^\infty (\theta) < e^{-Ky} \begin{matrix} \cos \\ \sin \end{matrix} Kx > + 2(\gamma + \ln 2Ka - \pi i) \int_0^\pi \phi^\infty (\theta)$$

$$< a \frac{\partial}{\partial r} e^{-ky} \begin{matrix} \cos \\ \sin \end{matrix} Kx > d\theta. \qquad (6.6)$$

Neglecting terms of order $O((Ka)^2 \ln Ka)$ we see that

$$M(\theta, \alpha) = Ka M_1 (\theta, \alpha) + O (Ka)^2$$

where

$$M_1 (\theta, \alpha) = -2 \ln \sin \frac{\theta+\alpha}{2} \sin\theta -(\alpha - \theta) \cos\theta + (\sin\theta + \sin\alpha) \qquad (6.7)$$

so that $M_1 (\theta, \alpha)$ is bounded for all values of θ, α in $(0, \pi)$. Also

$$L (\alpha) = Ka L_1 (\alpha) + O((Ka)^2 \ln Ka)$$

where $L_1 (\alpha) = 4(\gamma + \ln 2Ka - \pi i) - 2 \ln 2 - 2 + 2 \ln \sin\alpha$

$$- 2 \cos\alpha \ln \tan \frac{\alpha}{2} - \pi i \cos\alpha. \qquad (6.8)$$

Thus $L_1 (\alpha)$ is bounded for all values of α in $(0, \pi)$. For the time being we assume that $A(Ka)$ and $B(Ka)$ are known. We now apply the following theorem due to Ursell [8] to the integral equation (6.3).

Theorem If (i) $q(\alpha)$ is bounded in $(0, \pi)$

(ii) $\frac{1}{\pi} \int_0^\pi |L(\theta, \alpha)| d\theta \leq m < 1, 0 < \alpha < \pi,$

then the integral equation

$$\pi p(\alpha) - \int_0^\pi L(\theta, \alpha) p (\theta) d\theta = q (\alpha), 0 < \alpha < \pi,$$

may be solved by iteration, and

$$p(\alpha) = q(\alpha) + \sum_{s=1}^{\infty} q_s(\alpha)$$

where

$$q_1(\alpha) = {}_0\int^{\pi} L(\theta, \alpha) q(\theta) d\theta, \quad q_s(\alpha) = {}_0\int^{\pi} L(\theta, \alpha) q_{s-1}(\theta) d\theta, \quad s > 1.$$

The applicability of the theorem to the integral equation (6.3) is assured by the fact that by virtue of (6.7), the condition (ii) is satisfied for small Ka, and by virtue of (6.8), the condition (i) is satisfied for small Ka; $A(Ka)$ and $B(Ka)$ are assumed to be such that the whole of the right side of (6.3) satisfies the condition (i) of the theorem.

Now the integral equation (6.3) can symbolically be written as

$$-\pi(I - \frac{1}{\pi} M) \phi^{\infty}(\alpha) = L(\alpha) - A < e^{-Kn} \cos K\xi > - B < e^{-Kn} \sin K\xi >$$

so that

$$-\pi \phi^{\infty}(\alpha) = (I - \frac{1}{\pi} M)^{-1} L - A(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \cos K\xi >$$

$$-B(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \sin K\xi > \qquad (6.9)$$

where I denotes the identity operator. Neglecting terms of the order of $O((Ka)^2)$ $O((Ka)^2 \ln Ka)$, we see that

the first term in (6.9) $= Ka \, L_1(\alpha)$

the second term in (6.9) $= 1 - Ka \sin\alpha + \frac{1}{\pi} Ka (2 + 2 \ln 2 - 2\ln \sin\alpha + 2 \cos\alpha \ln \tan \frac{\alpha}{2})$

the third term in (6.9) $= Ka \cos\alpha$.

Thus

$$-\pi \phi^{\infty}(\alpha) = Ka \, L_1(\alpha) - A\{1 - Ka \sin\alpha + \frac{1}{\pi} Ka (2 + 2\ln2 - 2\ln \sin\alpha + 2 \cos\alpha \ln \tan \frac{\alpha}{2})\}$$

$$- B \, Ka \cos\alpha + O((Ka)^2, (Ka)^2 \ln Ka). \qquad (6.10)$$

Now the equations for the determination of A and B are obtained by using (6.9) in (6.6). These equations are

$$A[-\pi + \int_0^{\pi} \{(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \cos K\xi > \} < e^{-Ky} \cos Kx > d\theta$$

$$+ 2(\gamma + \ln 2Ka - \pi i) \int_0^{\pi} \{(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \cos K\xi > \} < a \frac{\partial}{\partial r} e^{-KY} \cos Kx > d\theta]$$

$$+ B[\int_0^{\pi} \{(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \sin K\xi > \} < e^{-Ky} \cos Kx > d\theta$$

$$+ 2(\gamma + \ln 2Ka - \pi i) \int_0^{\pi} \{(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \sin K\xi > \} < a \frac{\partial}{\partial r} e^{-Ky} \cos Kx > d\theta]$$

$$= \int_0^{\pi} \{(I - \frac{1}{\pi} M)^{-1} L(\alpha)\} < e^{-Ky} \cos Kx > d\theta$$

$$+ 2(\gamma + \ln 2Ka - \pi i) \int_0^{\pi} \{(I - \frac{1}{\pi} M)^{-1} L(\alpha)\} < a \frac{\partial}{\partial r} e^{-Ky} \cos Kx > d\theta, \qquad (6.11)$$

and

$$A[\int_0^\pi \{(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \cos K\xi >\} < e^{-Ky} \sin Kx > d\theta$$

$$+ 2(\gamma + \ln 2Ka - \pi i) \int_0^\pi \{(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \cos K\xi >\} < a \frac{\partial}{\partial r} e^{-Ky} \sin Kx > d\theta]$$

$$+ B[-\pi + \int_0^\pi \{(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \sin K\xi >\} < e^{-Ky} \sin Kx > d\theta$$

$$+ 2(\gamma + \ln 2Ka - \pi i) \int_0^\pi \{(I - \frac{1}{\pi} M)^{-1} < e^{-Kn} \sin K\xi >\} < a \frac{\partial}{\partial r} e^{-Ky} \sin Kx > d\theta$$

$$= \int_0^\pi \{(I - \frac{1}{\pi} M)^{-1} L(\alpha)\} < e^{-Ky} \sin Kx > d\theta$$

$$+ 2(\gamma + \ln 2Ka - \pi i) \int_0^\pi \{(I - \frac{1}{\pi} M)^{-1} L(\alpha)\} < a \frac{\partial}{\partial r} e^{-Ky} \sin Kx > d\theta \qquad (6.12)$$

where in the terms in all the second brackets α is to be replaced by θ.
From (6.11) we obtain

$$A \{-4 Ka (1 + \gamma + \ln Ka - \pi i) + 0 ((Ka)^2 \ln Ka, (Ka)^2)\} + B \{0 ((Ka)^2)\}$$
$$= \pi Ka \{4 (\gamma + \ln Ka - \pi i)\} + 0 ((Ka)^2)$$

so that

$$A(Ka) = A_0 + 0 (Ka \ln Ka) \qquad (6.13)$$

where $A_0 = -\pi + \dfrac{\pi}{1 + \gamma + \ln Ka - \pi i}$.

We note that it is sufficient to determine $A(Ka)$ up to this order. In (6.12), the coefficent of A is $0 ((Ka)^3)$ and the coefficient of B is

$$\pi \{-1 + \frac{(Ka)^2}{2} + (Ka)^2 (\gamma + \ln 2 Ka - \pi i) + 0 ((Ka)^3 \ln Ka)\}$$

while the right side of (6.12) is

$$-(Ka)^2 \frac{i\pi}{2} \{1 + 2(\gamma + \ln 2Ka - \pi i)\} + 0 ((Ka)^3 \ln Ka).$$

Hence $B(Ka) = 0 ((Ka)^2)$, and we neglect it compared to A.
Thus

$$-\pi \phi^\infty(\alpha) = Ka L_1 (\alpha) - A_0[1 + Ka\{-\sin\alpha + \frac{2}{\pi} (1 + \ln 2 - \ln \sin\alpha$$

$$+ \cos\alpha \ln \tan \frac{\alpha}{2})\}] + 0 ((Ka)^2 \ln Ka) \qquad (6.14)$$

where A_0 is given by (6.13). Hence $\phi^\infty (\alpha)$ is approximated for small Ka.

We next consider the integral equation (5.3) for $\psi_D (\alpha)$.
This can be rewritten as

$$-\pi\psi_D (\alpha) + \int_0^\pi \psi_D(\theta) M(\theta, \alpha) d\theta = L_D(\alpha) - A_D(Ka) < e^{-Kn} \cos K\xi >$$

$$- B_D(Ka) < e^{-Kn} \sin K\xi > \qquad (6.15)$$

where

$$L_D = -2Ka \, \alpha_{01} (Ka \sin \alpha - 1)[i \int_0^\pi e^{-Ka (\sin\theta + i \cos\theta)} - i\theta (Ka \sin\theta - 1) d\theta \qquad (6.16)$$

$$- \int_0^\pi \phi^\infty (\theta) \sin\theta \, d\theta],$$

$$\begin{matrix} A_D \\ B_D \\ D \end{matrix} = \int_0^\pi \psi_D (\theta) < e^{-Ky} \begin{matrix} \cos \\ \sin \end{matrix} Kx > d\theta +$$

$$2(\gamma + \ln 2Ka - i\pi) \int_0^\pi \psi_D (\theta) < a \frac{\partial}{\partial r} e^{-Ky} \begin{matrix} \cos \\ \sin \end{matrix} Kx > d\theta. \qquad (6.17)$$

We note that

$$L_D = - \frac{4 \alpha_{ol} Ka}{1 + \gamma + \ln Ka - \pi i} + 0 ((Ka)^2 \ln Ka). \qquad (6.18)$$

Proceeding as in the case of ϕ^∞, we obtain

$$A_D = A_{DO} + 0 (Ka \ln Ka)$$

$$B_D = 0((Ka)^2 \ln ka)$$

where

$$A_{DO} = \frac{\pi \alpha_{ol}}{(1 + \gamma + \ln Ka - \pi i)^2} \qquad (6.19)$$

and

$$-\pi\psi_D = - \frac{4 \alpha_{ol} Ka}{1 + \gamma + \ln Ka - \pi i} - A_{DO} \left[1 + \frac{0(Ka \ln Ka)}{1 + \gamma + \ln Ka - \pi i} \right]. \qquad (6.20)$$

Hence

$$Ka \int_0^\pi \psi_D e^{-Ka (\sin\theta + i \cos\theta) - i\theta} d\theta = \frac{2Ka}{i} \frac{\alpha_{ol}}{(1 + \gamma + \ln Ka - \pi i)^2} + 0((Ka)^2 \ln Ka).$$

Thus for small Ka,

$$\left. \begin{matrix} (R - R^\infty)_1 = - \frac{1}{(Kh)^2} \left[\frac{2i \alpha_{ol} Ka}{(1 + \gamma + \ln Ka - \pi i)^2} + 0((Ka)^2 \ln Ka) \right] \\ \\ (T - T^\infty)_1 = - \frac{1}{(Kh)^2} \left[\frac{2i \alpha_{ol} Ka}{(1 + \gamma + \ln Ka - \pi i)^2} + 0((Ka)^2 \ln Ka) \right] \end{matrix} \right\} \qquad (6.21)$$

Thus for small Ka, the coefficients $(R - R^\infty)_1$, $(T-T^\infty)_1$ of $(a/h)^2$ in the expansion of $R - R^\infty$ and $T - T^\infty$ respectively are not zero, and hence these do not vanish identically.

7. CONCLUSION.

The depth effect on the reflection and transmission coefficients for the case of a normally incident wave on a half immersed circular cylinder is not too small to be neglected. If however, the depth is assumed to be large (but not infinite), then within the framework of our above analysis, the depth corrections to these coefficients over their infinite depth values, can be expressed as algebraic series in powers of a/h commencing with $(a/h)^2$.

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