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ON LOCALLY CONFORMAL KÄHLER SPACE FORMS

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ABSTRACT. An m-dimensional locally conformal Kähler manifold (l.c.K-manifold) is characterized as a Hermitian manifold admitting a global closed 1-form α_λ (called the Lee form) whose structure $(F_\mu{}^\lambda, g_{\mu\lambda})$ satisfies

$$\nabla_\nu F_{\mu\lambda} = -\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu},$$

where ∇_λ denotes the covariant differentiation with respect to the Hermitian metric $g_{\mu\lambda}$, $\beta_\lambda = -F_\lambda{}^\varepsilon \alpha_\varepsilon$, $F_{\mu\lambda} = F_\mu{}^\varepsilon g_{\varepsilon\lambda}$ and the indices $\nu,\mu,\ldots,\lambda$ run over the range $1,2,\ldots,m$.

For l.c.K-manifolds, I.Vaisman [4] gave a typical example and T.Kashiwada ([1], [2],[3]) gave a lot of interesting properties about such manifolds.

In this paper, we shall study certain properties of l.c.K-space forms. In §2, we shall mainly get the necessary and sufficient condition that an l.c.K-space form is an Einstein one and the Riemannian curvature tensor with respect to $g_{\mu\lambda}$ will be expressed without the tensor field $P_{\mu\lambda}$. In §3, we shall get the necessary and sufficient condition that the length of the Lee form is constant and the sufficient condition that a compact l.c.K-space form becomes a complex space form. In the last §4, we shall prove that there does not exist a non-trivial recurrent l.c.K-space form.

KEY WORDS & PHRASES: L.c.K-manifolds, Lee form, l.c.K-space forms, hybrid, recurrent l.c.K-space form.
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1. INTRODUCTION.

This paper is directed to specialist readers with background in the area and appreciative of its relation of this area of study.

Let $M(F_\mu{}^\lambda, g_{\mu\lambda}, \alpha_\lambda)$ be an l.c.K-manifold. Then, by the definition, at any point of M there exists a neighborhood in which a conformal metric $g^* = e^{-2\rho}g$ is a Kähler one, i.e.,

$$\nabla^*_\nu(e^{-2\rho}F_{\mu\lambda}) = 0, \qquad d\rho = \alpha,$$

where ∇^*_λ denotes the covariant differentiation with respect to g^*. Then we have

$$\nabla_\nu F_{\mu\lambda} = -\alpha_\mu F_{\nu\lambda} + \alpha^\varepsilon F_{\varepsilon\lambda} g_{\nu\mu} + \alpha_\lambda F_{\nu\mu} + \alpha^\varepsilon F_{\mu\varepsilon} g_{\nu\lambda}. \tag{1.1}$$

The following proposition was proved by T.Kashiwada [1]

PROPOSITION 1.1. A Hermitian manifold $M(F_\mu{}^\lambda, g_{\mu\lambda})$ is an l.c.K-manifold if and only if there exists a global closed 1-form α_λ satisfying (1.1).

In an l.c.K-manifold M, we define a tensor field $P_{\mu\lambda}$ as follows;

$$P_{\mu\lambda} = -\nabla_\mu \alpha_\lambda - \alpha_\mu \alpha_\lambda + \frac{1}{2}\|\alpha\|^2 g_{\mu\lambda}, \tag{1.2}$$

where $\|\alpha\|$ denotes the length of the Lee form α_λ with respect to $g_{\mu\lambda}$.

In an m-dimensional l.c.K-manifold M, we know the following formula;

$$R_{\mu\varepsilon} F_\lambda{}^\varepsilon + R_{\lambda\varepsilon} F_\mu{}^\varepsilon - (m-2)(P_{\mu\varepsilon} F_\lambda{}^\varepsilon + P_{\lambda\varepsilon} F_\mu{}^\varepsilon) = 0, \tag{1.3}$$

where $R_{\mu\lambda}$ denotes the Ricci tensor with respect to $g_{\mu\lambda}$ [1]. Thus we have

PROPOSITION 1.2. In an m-dimensional ($m \neq 2$) l.c.K-manifold M, the tensor field $P_{\mu\lambda}$ is hybrid, i.e.,

$$P_{\mu\varepsilon} F_\lambda{}^\varepsilon + P_{\lambda\varepsilon} F_\mu{}^\varepsilon = 0, \tag{1.4}$$

if and only if the Ricci tensor $R_{\mu\lambda}$ is hybrid.

From now on in this paper, we assume that the tensor field $P_{\mu\lambda}$ is hybrid.

REMARK. In an m-dimensional ($m \neq 2$) Einstein l.c.K-manifold, the tensor field $P_{\mu\lambda}$ is hybrid, identically.

An l.c.K-manifold M is called an l.c.K-space form if the holomorphic sectional curvature of the section $\{X, FX\}$ at each point of M has the constant value. Let $M(H)$ be an l.c.K-space form with constant holomorphic sectional curvature H. Then the Riemannian curvature tensor $R_{\omega\nu\mu\lambda}$ with respect to $g_{\mu\lambda}$ can be written as

$$\begin{aligned} 4R_{\omega\nu\mu\lambda} &= H(g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda} + F_{\omega\lambda} F_{\nu\mu} - F_{\omega\mu} F_{\nu\lambda} - 2F_{\omega\nu} F_{\mu\lambda}) + 3(P_{\omega\lambda} g_{\nu\mu} - P_{\omega\mu} g_{\nu\lambda} \\ &+ g_{\omega\lambda} P_{\nu\mu} - g_{\omega\mu} P_{\nu\lambda}) - \{\tilde{P}_{\omega\lambda} F_{\nu\mu} - \tilde{P}_{\omega\mu} F_{\nu\lambda} + F_{\omega\lambda} \tilde{P}_{\nu\mu} - F_{\omega\mu} \tilde{P}_{\nu\lambda} - 2(\tilde{P}_{\omega\nu} F_{\mu\lambda} \\ &+ F_{\omega\nu} \tilde{P}_{\mu\lambda})\}, \end{aligned} \tag{1.5}$$

where $\tilde{P}_{\mu\lambda} = P_\mu{}^\varepsilon F_{\varepsilon\lambda}$ [1].

2. L.C.K-SPACE FORMS.

In this section, we shall consider the necessary and sufficient condition that an l.c.K-space form becomes an Einstein one. Next, we shall get an expression of the Riemannian curvature $R_{\omega\nu\mu\lambda}$ that does not include the tensor field $P_{\mu\lambda}$.

Let $M(H)$ ba an m-dimensional l.c.K-space form with constant holomorphic sectional curvature H. Then we have (1.5). Transvecting (1.5) with $g^{\omega\lambda}$, we have from the straightfoward calculation

$$4R_{\mu\lambda} = \{(m+2)H + 3P\} g_{\mu\lambda} + 3(m-4) P_{\mu\lambda}, \tag{2.1}$$

where $P = P_{\mu\lambda} g^{\mu\lambda}$ and it can be written as

$$P = -\nabla_\varepsilon \alpha^\varepsilon + \frac{1}{2}(m-2)\|\alpha\|^2. \tag{2.2}$$

Thus we have

PROPOSITION 2.1. A 4-dimensional l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid is an Einstein one and then the scalar field P is constant.

We have from (2.2) and the Green's theorem [5]

PROPOSITION 2.2. A compact m-dimensional l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid has a non-negative P.

Next, we shall prove the following ;

THEOREM 2.3. An m-dimensional ($m \neq 4$) l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid is an Einstein one if and only if the tensor field $P_{\mu\lambda}$ is proportional to $g_{\mu\lambda}$.

PROOF. If the tensor field $P_{\mu\lambda}$ is proportional to $g_{\mu\lambda}$, then the tensor field $P_{\mu\lambda}$ can be written as

$$P_{\mu\lambda} = \frac{P}{m} g_{\mu\lambda}. \tag{2.3}$$

Thus we have from (2.1) and (2.3)

$$R_{\mu\lambda} = \{(m + 2)H + \frac{6(m - 2)}{m}P\}g_{\mu\lambda}.$$

The inverse is trivial, so we omit its proof.

COROLLARY 2.4. An m-dimensional ($m \neq 4$) Einstein l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid is a complex space form if $P = 0$.

Transvecting (2.1) with $g^{\mu\lambda}$, we have

$$4R = m(m + 2)H + 6(m - 2)P, \tag{2.4}$$

where R denotes the scalar curvature with respect to $g_{\mu\lambda}$. By virtue of (2.1) and (2.4), we can easily see that

$$3P_{\mu\lambda} = \frac{4}{m - 4}R_{\mu\lambda} - \frac{(m - 4)(m + 2)H + 4R}{2(m - 2)(m - 4)} g_{\mu\lambda}, \tag{2.5}$$

$$\tilde{P}_{\mu\lambda} = \frac{4}{3(m - 4)}\tilde{R}_{\mu\lambda} - \frac{(m - 4)(m + 2)H + 4R}{6(m - 2)(m - 4)} F_{\mu\lambda}, \tag{2.6}$$

where $\tilde{R}_{\mu\lambda} = R_{\mu}{}^{\varepsilon}F_{\varepsilon\lambda}$. Substituting (2.5) and (2.6) into (1.5), we obtain

$$\begin{aligned} R_{\omega\nu\mu\lambda} = &- \frac{(m - 4)H + R}{(m - 2)(m - 4)}(g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda}) + \frac{(m - 4)(m - 1)H + R}{3(m - 2)(m - 4)}(F_{\omega\lambda}F_{\nu\mu} \\ &- F_{\omega\mu}F_{\nu\lambda} - 2F_{\omega\nu}F_{\mu\lambda}) + \frac{1}{(m - 4)}(R_{\omega\lambda}g_{\nu\mu} - R_{\omega\mu}g_{\nu\lambda} + g_{\omega\lambda}R_{\nu\mu} - g_{\omega\mu}R_{\nu\lambda}) \\ &+ \frac{1}{3(m - 4)}\{\tilde{R}_{\omega\lambda}F_{\nu\mu} - \tilde{R}_{\omega\mu}F_{\nu\lambda} + F_{\omega\lambda}\tilde{R}_{\nu\mu} - F_{\omega\mu}\tilde{R}_{\nu\lambda} - 2(\tilde{R}_{\omega\nu}F_{\mu\lambda} + F_{\omega\nu}\tilde{R}_{\mu\lambda})\}. \end{aligned} \tag{2.7}$$

Thus we have

PROPOSITION 2.5. In an m-dimensional ($m \neq 2,4$) l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid, the Riemannian curvature tensor $R_{\omega\nu\mu\lambda}$ can be written as (2.7) without $P_{\mu\lambda}$.

3. COMPACT L.C.K-SPACE FORMS.

In this section, we shall mainly deal with compact l.c.K-space form.

Let $M(H)$ be an m-dimensional l.c.K-space form with constant holomorphic sectional curvature H. If we assume that the scalar curvature R is constant, then by virtue of (2.4) all of the scalar fields R, H and P are constant. Under this assumption, differentiating (2.1) covariantly, we get

$$4\nabla_{\omega}R_{\nu\mu} = 3(m - 4)\nabla_{\omega}P_{\nu\mu}. \tag{3.1}$$

Substituting (1.2) into the above equation, we have

$$4\nabla_{\omega}R_{\nu\mu} = 3(m - 4)\{-\nabla_{\omega}\nabla_{\nu}\alpha_{\mu} - (\nabla_{\omega}\alpha_{\nu})\alpha_{\mu} - \alpha_{\nu}\nabla_{\omega}\alpha_{\mu} + \frac{1}{2}(\nabla_{\omega}\|\alpha\|^2)g_{\nu\mu}\}. \tag{3.2}$$

By virtue of the Ricci identity [5] and the assumption $\nabla_{\mu}\alpha_{\lambda} = \nabla_{\lambda}\alpha_{\mu}$, the equation (3.2) implies

$$4(\nabla_\omega R_{\nu\mu} - \nabla_\nu R_{\omega\mu}) = 3(m-4)\{R_{\omega\nu\mu}{}^{\varepsilon}\alpha_\varepsilon + \alpha_\omega(\nabla_\nu\alpha_\mu) - \alpha_\nu(\nabla_\omega\alpha_\mu) + \frac{1}{2}(\nabla_\omega\|\alpha\|^2 g_{\nu\mu} - \nabla_\nu\|\alpha\|^2 g_{\omega\mu})\}.$$

Transvecting the above equation with $g^{\nu\mu}$ and taking account of the formula $2\nabla_\varepsilon R_\lambda{}^\varepsilon = \nabla_\lambda R$ [5], we obtain

$$R_\omega{}^\varepsilon\alpha_\varepsilon + (\nabla_\varepsilon\alpha^\varepsilon)\alpha_\omega + \frac{1}{2}(m-2)\nabla_\omega\|\alpha\|^2 = 0. \tag{3.3}$$

Substituting (2.1) into (3.3), we obtain

$$\{(m+2)H + 3\|\alpha\|^2 + \nabla_\varepsilon\alpha^\varepsilon\}\alpha_\omega + \frac{m-4}{2}\nabla_\omega\|\alpha\|^2 = 0. \tag{3.4}$$

Thus we have

THEOREM 3.1. In an m-dimensional ($m \neq 2,4$) l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid and the scalar curvature R is constant, the length $\|\alpha\|$ of the Lee form α_λ is non-zero constant if and only if

$$(m+2)H + 3\|\alpha\|^2 + \nabla_\varepsilon\alpha^\varepsilon = 0. \tag{3.5}$$

By virtue of (3.5) and the Green's theorem, we have

COROLLARY 3.2. In a compact m-dimensional ($m \neq 2,4$) l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid and the scalar curvature R is constant, if the length $\|\alpha\|$ of the Lee form α_λ is non-zero constant, then there exists the following relation between the holomorphic sectional curvature H and the length $\|\alpha\|$ of the Lee form α_λ;

$$(m+2)H + 3\|\alpha\|^2 = 0. \tag{3.6}$$

COROLLARY 3.3. There does not exist a compact m-dimensional ($m \neq 2,4$) l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid and the holomorphic sectional curvature H is positive if the length $\|\alpha\|$ of the Lee form α_λ and the scalar curvature R are constant. Especially, if $H = 0$, then the manifold M must be locally Euclidean, that is, the Riemannian curvature tensor $R_{\omega\nu\mu\lambda}$ is identically zero.

The following proposition was proved by T.Kashiwada [1];

PROPOSITION 3.4. In a compact m-dimensional ($m \neq 2$) l.c.K-manifold M, if

$$\tilde{H}_\varepsilon{}^\varepsilon - R \geqslant 0 \tag{3.7}$$

holds good, then the manifold M is a Kähler manifold, where $\tilde{H}_{\mu\lambda} = \frac{1}{2}R_\mu{}^\varepsilon{}_{\delta\gamma}F^{\delta\gamma}F_{\varepsilon\lambda}$. The inequality $\geqslant$ in this case is naturally reduced to $=$.

Now, let $M(H)$ be a compact m-dimensional ($m \neq 2,4$) l.c.K-space form. Then transvecting (2.5) with $F^{\omega\nu}F^{\mu\lambda}$, we get

$$\frac{1}{2}R_{\omega\nu\mu\lambda}F^{\omega\nu}F^{\mu\lambda} = \frac{-m(m+2)H + R}{3}. \tag{3.8}$$

By virtue of (2.4) and (3.8), we obtain

$$H_\varepsilon{}^\varepsilon - R = \frac{m(m+2)H - 4R}{3}. \tag{3.9}$$

Thus we have from PROPOSITION 3.4 and (3.9)

THEOREM 3.5. In a compact m-dimensional ($m \neq 2,4$) l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid, if the inequlity $m(m+2)H \geqslant 4R$ holds good, then the manifold M is a complex space form.

4. RECURRENT L.C.K-SPACE FORMS.

A Riemannian manifold M is said to be recurrent if the Riemannian curvature tensor

$R_{\omega\nu\mu\lambda}$ satisfies

$$\nabla_\kappa R_{\omega\nu\mu\lambda} = \theta_\kappa R_{\omega\nu\mu\lambda} \tag{4.1}$$

for a certain non-zero vector field θ_κ. For a recurrent Riemannian manifold, it is trivial that

$$\nabla_\nu R_{\mu\lambda} = \theta_\nu R_{\mu\lambda}, \qquad \nabla_\lambda R = \theta_\lambda R. \tag{4.2}$$

Now, let $M(H)$ be an m-dimensional ($m \neq 2,4$) recurrent l.c.K-space form which the tensor field $P_{\mu\lambda}$ is hybrid. Then we have (2.7) and (4.1). Differentiating (2.7) covariantly and taking account of (4.1) and (4.2), we have

$$\begin{aligned}
&\frac{H}{m-2}\theta_\kappa(g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda}) - \frac{(m-1)H}{3(m-2)}\theta_\kappa(F_{\omega\lambda}F_{\nu\mu} - F_{\omega\mu}F_{\nu\lambda} - 2F_{\omega\nu}F_{\mu\lambda}) \\
&+ \frac{(m-4)(m-1)H + R}{3(m-2)(m-4)}\{(g_{\kappa\mu}F_{\nu\lambda} - g_{\kappa\lambda}F_{\nu\mu} + 2g_{\kappa\nu}F_{\mu\lambda})\beta_\omega + (g_{\kappa\lambda}F_{\omega\mu} - g_{\kappa\mu}F_{\omega\lambda} \\
&- 2g_{\kappa\omega}F_{\mu\lambda})\beta_\nu + (g_{\kappa\nu}F_{\omega\lambda} - g_{\kappa\omega}F_{\nu\lambda} + 2g_{\kappa\lambda}F_{\omega\nu})\beta_\mu + (g_{\kappa\omega}F_{\nu\mu} - g_{\kappa\nu}F_{\omega\mu} - 2g_{\kappa\mu}F_{\omega\nu})\beta_\lambda \\
&+ (F_{\kappa\mu}F_{\nu\lambda} - F_{\kappa\lambda}F_{\nu\mu} + 2F_{\kappa\nu}F_{\mu\lambda})\alpha_\omega + (F_{\kappa\lambda}F_{\omega\mu} - F_{\kappa\mu}F_{\omega\lambda} - 2F_{\kappa\omega}F_{\mu\lambda})\alpha_\nu \\
&+ (F_{\kappa\nu}F_{\omega\lambda} - F_{\kappa\omega}F_{\nu\lambda} + 2F_{\kappa\lambda}F_{\kappa\nu})\alpha_\mu + (F_{\kappa\omega}F_{\nu\mu} - F_{\kappa\nu}F_{\omega\mu} - 2F_{\omega\nu}F_{\kappa\mu})\alpha_\lambda\} \\
&- \frac{1}{3(m-4)}[\{R_\omega{}^\varepsilon g_{\kappa\mu}F_{\nu\lambda} - R_\omega{}^\varepsilon g_{\kappa\lambda}F_{\nu\mu} - R_\nu{}^\varepsilon g_{\kappa\mu}F_{\omega\lambda} + R_\nu{}^\varepsilon g_{\kappa\lambda}F_{\omega\mu} + 2(R_\omega{}^\varepsilon g_{\kappa\nu}F_{\mu\lambda} \\
&+ R_\mu{}^\varepsilon g_{\kappa\lambda}F_{\omega\mu})\}\beta_\varepsilon + \{R_\omega{}^\varepsilon F_{\kappa\mu}F_{\nu\lambda} - R_\omega{}^\varepsilon F_{\kappa\lambda}F_{\nu\mu} - R_\nu{}^\varepsilon F_{\kappa\mu}F_{\omega\lambda} + R_\nu{}^\varepsilon F_{\kappa\lambda}F_{\omega\mu} \\
&+ 2(R_\omega F_{\kappa\nu}F_{\mu\lambda} + R_\mu F_{\kappa\lambda}F_{\omega\nu})\}\alpha_\varepsilon + (g_{\kappa\mu}\tilde{R}_{\nu\lambda} - g_{\kappa\lambda}\tilde{R}_{\nu\mu} + 2g_{\kappa\nu}\tilde{R}_{\mu\lambda})\beta_\omega + \{g_{\kappa\lambda}\tilde{R}_{\omega\mu} \\
&- g_{\kappa\mu}\tilde{R}_{\omega\lambda} - 2(F_{\mu\lambda}R_{\omega\kappa} + g_{\kappa\omega}\tilde{R}_{\mu\lambda})\}\beta_\nu + (g_{\kappa\nu}\tilde{R}_{\omega\lambda} - F_{\nu\lambda}R_{\omega\kappa} + F_{\omega\lambda}R_{\kappa\nu} - g_{\kappa\omega}\tilde{R}_{\nu\lambda} \\
&+ 2g_{\kappa\lambda}\tilde{R}_{\omega\nu})\beta_\mu + \{F_{\nu\mu}R_{\omega\kappa} - g_{\kappa\nu}\tilde{R}_{\omega\mu} + g_{\kappa\omega}\tilde{R}_{\nu\mu} - F_{\omega\mu}R_{\kappa\nu} - 2(g_{\omega\nu}\tilde{R}_{\kappa\mu} + F_{\omega\nu}R_{\kappa\mu})\}\beta_\lambda \\
&+ (F_{\kappa\mu}\tilde{R}_{\nu\lambda} - F_{\kappa\lambda}\tilde{R}_{\nu\mu} + 2F_{\kappa\nu}\tilde{R}_{\mu\lambda})\alpha_\omega + \{F_{\kappa\lambda}\tilde{R}_{\omega\mu} - F_{\kappa\mu}\tilde{R}_{\omega\lambda} - 2(F_{\mu\lambda}\tilde{R}_{\kappa\omega} + F_{\kappa\omega}\tilde{R}_{\mu\lambda})\}\alpha_\nu \\
&+ (F_{\kappa\nu}\tilde{R}_{\omega\lambda} - F_{\nu\lambda}\tilde{R}_{\kappa\omega} + F_{\omega\lambda}\tilde{R}_{\kappa\nu} - F_{\kappa\omega}\tilde{R}_{\nu\lambda} + 2F_{\kappa\lambda}\tilde{R}_{\omega\nu})\alpha_\mu + \{F_{\nu\mu}\tilde{R}_{\kappa\omega} - F_{\kappa\nu}\tilde{R}_{\omega\mu} \\
&+ F_{\kappa\omega}\tilde{R}_{\nu\mu} - F_{\omega\mu}\tilde{R}_{\kappa\nu} - 2(F_{\kappa\mu}\tilde{R}_{\omega\nu} + F_{\omega\nu}\tilde{R}_{\kappa\mu})\}\alpha_\lambda] = 0.
\end{aligned} \tag{4.3}$$

Transvecting (4.3) with $F^{\omega\lambda}$, we get

$$\begin{aligned}
&\frac{(m+2)H}{3}\theta_\kappa F_{\nu\mu} = \frac{(m+2)\{(m-4)(m-1)H + R\}}{3(m-4)(m-2)}(g_{\kappa\nu}\beta_\mu - g_{\kappa\mu}\beta_\nu - F_{\kappa\mu}\alpha_\nu + F_{\kappa\nu}\alpha_\mu) \\
&\frac{1}{3(m-4)}[\{(m-1)R_\nu{}^\varepsilon F_{\kappa\mu} - 5R_\mu{}^\varepsilon F_{\kappa\nu}\}\alpha_\varepsilon + \{(m-1)R_\nu{}^\varepsilon g_{\kappa\mu} - 5R_\mu{}^\varepsilon g_{\kappa\nu}\}\beta_\varepsilon \\
&+ (RF_{\kappa\mu} + 5R_{\kappa\mu})\alpha_\nu - \{RF_{\kappa\nu} + (m-1)R_{\kappa\nu}\}\alpha_\mu + (Rg_{\kappa\mu} + 5R_{\kappa\mu})\beta_\nu \\
&- \{Rg_{\kappa\nu} + (m-1)R_{\kappa\nu}\}\beta_\mu].
\end{aligned}$$

From this, we obtain

$$H\theta_\kappa = 0. \tag{4.4}$$

Thus we have

THEOREM 4.1. An m-dimensional ($m \neq 2,4$) recurrent l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid is trivial, that is, the manifold is locally symmetric or of zero holomorphic sectional curvature.

Let $M(H)$ be a 4-dimensional recurrent l.c.K-space form. Then, by virtue of PROPOSITION 2.1, the manifold is Einstein. Thus we have from (2.1) and (4.2)

$$(2H + P)\theta_\kappa = 0. \tag{4.5}$$

Thus we have

THEOREM 4.2. A 4-dimensional recurrent l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid is trivial or the manifold has a property $2H + P = 0$.

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