Internat. J. Math. & Math. Sci.
Vol. 8 No. 1 (1985) 201-203

A CHARACTERIZATION OF CLOSED MAPS USING THE WHYBURN CONSTRUCTION

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(Received May 22, 1984)

ABSTRACT. In this paper we modify the Whyburn construction for a continuous function $f : X \to Y$. If the range is first countable, we get a characterization of closed maps; namely, the constructions are the same if and only if the map is closed.

KEY WORDS AND PHRASES. *Closed maps, first countable, Whyburn construction.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. Primary 54A10, 54C10.

1. INTRODUCTION.

Let $f : X \to Y$ be continuous and let X and Y be Hausdorff. In [3] Whyburn defined the unified space Z to be the disjoint union of X and Y with a set Q open in Z if and only if $Q \cap X$ is open in X, $Q \cap Y$ is open in Y, and for any compact $K \subset Q \cap Y$, $f^{-1}(K) - Q$ is compact. In this paper we modify the topology on $X \cup Y$ by defining Q to be open if and only if $Q \cap X$ is open in X, $Q \cap Y$ is open in Y, and for any point $p \in Q \cap Y$, $f^{-1}(p) - Q$ is compact. We denote the modified Whyburn space by W.

It is obvious that any set open in Z is open in W. We will show that if f is closed, the topologies are in fact the same and if Y is first countable, then Z and W being the same implies that f is closed. This will yield the following corollary:

COROLLARY. Any continuous function from a Hausdorff space into the reals (or any metric space) is closed if and only if the Whyburn space and the modified Whyburn space are the same.

2. PRELIMINARIES.

Arguments similar to those of Whyburn's show that W is a T_1 topological space containing X as an open subspace and Y as a closed subspace. However, just as in the Whyburn space, W need not be Hausdorff. Whyburn showed that Z is Hausdorff if X is locally compact. Asking when W is Hausdorff led to the following definitions and propositions:

DEFINITION 2.1. Let $f : X \to Y$ be continuous. Then $A \subset X$ is fiber compact if and only if A is closed and for all $y \in f(A)$, $f^{-1}(y) \cap A$ is compact. Also, X is locally fiber compact if every point has a neighborhood whose closure is fiber compact.

PROPOSITION 2.2. If A is fiber compact in X, then $W - A$ is open.

PROOF. Since A is closed in X, $(W - A) \cap X$ is open in X; also $(W - A) \cap Y = Y$ is open in Y. Now let p be any point in Y. Then $f^{-1}(p) - (W - A) = f^{-1}(p) \cap A$ which is compact since A is fiber compact.

PROPOSITION 2.3. If X is locally fiber compact, then W is Hausdorff.

PROOF. The only interesting case is when p is in X and $f(p) = q$. Since X is locally fiber compact, there exists a U open in X such that p is in U and $\overline{U}$ is fiber compact. Hence U is open in W and $W - \overline{U}$ is a neighborhood of q by Proposition 2.2.

We define, as did Whyburn, a retraction $r : W \to Y$ to be f on X and the identity on Y. The following results parallel those of Whyburn's for $r : Z \to Y$. The proof is omitted.

PROPOSITION 2.4. The map $r : W \to Y$ is continuous, has compact fibers and is closed (open) if f is.

The next proposition shows that some of the properties mentioned above actually characterize the modified Whyburn construction. This proposition is similar to a theorem about the Whyburn construction proved by Dickman [1].

PROPOSITION 2.5. Let $r : S \to Y$ be a retraction with compact fibers from a Hausdorff space onto a regular subspace. Let $X = S - Y$ and $f = r_{|X}$. If fiber compact subsets of X are closed in S, then the modified Whyburn space for $f : X \to Y$ is homeomorphic to S.

PROOF. Let W be the modified Whyburn space for $f : X \to Y$. If V is open in S and p is any point in $V \cap Y$, then $r^{-1}(p)$ is compact. But $r^{-1}(p) - V = f^{-1}(p) - V$ and therefore V is open in W.

Now let Q be open in W and let $x \in Q$. If $x \in X$, then $Q \cap X$ is an open set in S and is contained in Q. Suppose $x \in Q \cap Y$. Then since Y is regular, we can find a neighborhood V of x such that $x \in V \subset \overline{V} \subset Q \cap Y$. Let $f^{-1}(\overline{V}) - Q = B$. Then B is fiber compact and so $S - B$ is open in S. Let $U = (S - B) \cap r^{-1}(V)$. Then U contains x, is open in S, and is contained in Q.

3. MAIN THEOREM.

We now state and prove the major theorem of this paper which allows us to determine when W and Z are the same.

THEOREM 3.1. Let $f : X \to Y$ be continuous, X and Y be Hausdorff, and let Y be first countable. Then Z and W are equal if and only if f is a closed mapping.

PROOF. Assume f is closed, Q is open in W and K is any compact subset of $Q \cap Y$. Let $f^{-1}(K) - Q = A$. Then A is closed and $f(A)$ is a closed subset of K and hence is compact. Then $f_{|A} : A \to f(A)$ is a continuous, closed surjection with compact fibers and therefore is a perfect map. By [2, Theorem 5.3] A is compact. Hence Q is open in Z.

Now assume that Z and W are equal. Let A be a closed set in X. Suppose that y is a limit point of $f(A)$. Since Y is first countable and Hausdorff, there exists a sequence of distinct points $\{y_n\} \subset f(A)$ which converges to y. So we may choose a sequence $\{x_n\}$ in A such that $f(x_n) = y_n$.

Let $B = \{x_n\}$. Now suppose B has no limit points. Then B is closed in X. Since for any $y_n \in f(B)$, $f^{-1}(y_n) \cap B = \{x_n\}$, B is fiber compact and thus $W - B$ is open in W by Proposition 2.2. Since $Z = W$, $Z - B$ is open in Z. Now $K = \{y_n\} \cup \{y\}$ is a compact subset of $Y \cap (Z - B)$; therefore, $f^{-1}(K) - (Z - B) = B$ is compact. Since B is also infinite it must have a limit point, contradicting our assumption. Hence B has a limit point, say x.

Suppose that $f(x) = z \neq y$. Then we can find disjoint neighborhoods V of z and U of y. Since $\{y_n\}$ converges to y, there exists an N such that for every $n \geq N$, $y_n \in U$. However, since x is a limit point of B, we have an integer $m > N$ such that $x_m \in f^{-1}(V)$. Hence $f(x_m) = y_m$ is in both U and V which is impossible. Hence $f(x) = y$. Since $B \subset A$ and A is closed, $x \in A$ and therefore $f(A)$ is closed.

Notice that Y being first countable is a necessary hypothesis for the preceding theorem. The following is an example to illustrate this.

Let $X_i = [0,1)$ for all $i = 1,2,3,\ldots$. Then let X be the disjoint union of these X_i's. Let $Y = X \cup p$ where p is not in X. Define $V \subset Y$ to be open if and only if

1) V is an open set contained in X or

2) If $p \in V$, then there exists a finite set of indices such that if $i \in \{i_1,\ldots,i_n\}$ then $X_i - V = X_i$ and if $i \notin \{i_1,\ldots,i_n\}$ then $X_i - V$ is compact.

The inclusion map from X to Y is not closed, Y is not first countable at p and yet W and Z are the same.

REFERENCES

1. DICKMAN, R. F., JR., Unified Spaces and Singular Sets for Mappings of Locally Compact Spaces, Fundamenta Mathematicae, 62 (1968), 103-123.

2. DUGUNDJI, J., Topology. Boston, Allyn and Bacon (1966).

3. WHYBURN, G. T., A Unified Space for Mappings, Trans. Amer. Math. Soc. 74 (1953), 344-350.