Internat. J. Math. & Math. Sci.
Vol. 8 No. 3 (1985) 545-548

SEMI-PERFECT AND F-SEMI-PERFECT MODULES

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(Received February 5, 1985)

ABSTRACT. A module is semi-perfect iff every factor module has a projective cover. A module M = A + B (for submodules A and B) is amply supplemented iff there exists a submodule A′ (called a supplement of A) of B such M = A + A′ and A′ is minimal with this property. If B = M then M is supplemented. Kasch and Mares [1] have shown that the first and last of these conditions are equivalent for projective modules. Here it is shown that an arbitrary module is semi-perfect iff it is (amply) supplemented by supplements which have projective covers, an extension of the result of Kasch and Mares [1]. Corresponding results are obtained for F-semi-perfect modules.

KEY WORDS AND PHRASES. F-semi-perfect module, projective cover, semi-perfect module, small submodule, supplement
1980 AMS MATHEMATICS SUBJECT CLASSIFICATION CODE. 16A51

1. INTRODUCTION.

A module is semi-perfect iff every factor module has a projective cover. A module M = A + B (for submodules A and B) is amply supplemented iff there exists a submodule A′ (called a supplement of A) of B such M = A + A′ and A′ is minimal with this property. If B = M then M is supplemented. Kasch and Mares [1] have shown that the first and last of these conditions are equivalent for projective modules. Here it is shown that an arbitrary module is semi-perfect iff it is (amply) supplemented by supplements which have projective covers, an extension of the result of Kasch and Mares [1]. Corresponding results are obtained for F-semi-perfect modules.

2. CONVENTIONS, NOTATION, AND TERMINOLOGY.

Unless otherwise stated, we use the following conventions, notation, and terminology.

All rings are associative, but not necessarily commutative. Every ring has a multiplicative identity element, denoted by 1, which is preserved by ring homomorphisms, inherited by subrings, and acts as the identity operator on modules.

We use the word map for module homomorphism. Maps are written on the side opposite to that of the scalars. Thus the order of writing map compositions depends on the side of the module.

If M and N are R-modules we usually write Hom(M,N) for $\mathrm{Hom}_R(M,N)$ when no confusion can arise.

The symbols < and > will be used to denote proper set theoretical inclusion and containment, respectively, as well as the usual order relationships. The symbols $\leq$ and $\geq$, respectively, are used for the preceeding if equality can occur.

A submodule S of a module M is defined to be small (or superfluous) iff whenever $S + M' = M$ for a submodule M' of M then we must have $M' = M$. A map with a small kernel is called a small map. It is easy to verify that the product of small surjective maps is small and that small submodules are small in overmodules. A module M covers or is a cover of a module N iff there is a small epimorphism from M to N. If M is projective it is called a projective cover.

Basic properties of projective covers can be found in Bass [2] and Kasch [3].

Following Kasch [3] we call a module semi-perfect iff every factor module has a projective cover. Kasch [3] contains basic facts about semi-perfect modules.

The following Lemma and its Corollary are easy to verify:

LEMMA. If $X \leq Y \leq Z$ are modules then:

Y is small in Z iff X is small in Z and Y/X is small in Z/X.

COROLLARY. Z covers Z/Y iff Z covers Z/X and Z/X covers Z/Y.

3. SUPPLEMENTS.

Let A, B, A' be submodules of a module M.

The submodule A' is called a supplement in B of the submodule A iff A' is contained in B, $M = A + A'$, and A' is minimal with respect to this last property. If $B = M$ we say that A' is a supplement of A.

We now have:

PROPOSITION 1. The submodule A' is a supplement in B of the submodule A iff:

(1) $M = A + A'$.

(2) The intersection of A and A' is a small submodule of A'.

(3) A' is a submodule of B.

PROOF. The "only if" part can be found in Kasch [3]. The "if" part is an easy modification of the preceeding.

COROLLARY. The submodule A' is a supplement of the submodule A iff $M = A + A'$ and the intersection of A and A' is small in A'.

4. (AMPLY) SUPPLEMENTED MODULES.

A module is defined to be supplemented iff every submodule has a supplement. The module M is defined to be amply supplemented iff M = A + B implies that A has a supplement in B.

PROPOSITION 2. Let X be a submodule of the module Y, and let h in Hom(Y,Y/X) be the canonical epimorphism. Also let P be any module, f an element of Hom(P,Y/X) and g an element of Hom(P,Y) such that f is g composed with h. If Im denotes image, then:

(1) The map f is an epimorphism iff Y = X + Im(g).

(2) X is small in Y iff f being an epimorphism is equivalent to g being an epimorphism for all such f,g, and P.

(3) The map f is a small epimorphism iff Im(g) is a supplement of X and g is a small map.

(4) If X is small in Y then: f is a small epimorphism iff g is a small epimorphism.

PROOF.

(1) and the "if" part of (2) are easy. For the "only if" part of (2), if Y = X + Z for some submodule Z of Y then the inclusion map g from Z to Y must be an epimorphism since the canonical map f from Z to Y/X is an epimorphism.

(3) If f is a small epimorphism then the image of its kernel under g, which is the intersection of Im(g) with X, is small in Im(g). This implies that Im(g) is a supplement of X since f is an epimorphism. Moreover the kernel of g must be small since it is contained in the kernel of f. Conversely, The smallness of both the kernel of g and the intersection of Im(g) with X implies the smallness of the kernel of f, while the fact that Im(g) is a supplement of X implies that f is an epimorphism.

(4) follows easily from (3).

We come now to our main result:

THEOREM 1. For any module M the following statements are equivalent:

(1) M is a semi-perfect module.

(2) M is amply supplemented by supplements which have projective covers.

(3) M is supplemented by supplements which have projective covers.

PROOF.

(1) implies (2): If M = A + B, let P be a projective cover of M/A, with epimorphism f. Since P is projective and M/A is isomorphic to B/D, where D is the intersection of A and B, the map f lifts to a map g from P to B. Since f is a small epimorphism, Im(g) is a supplement of the intersection of A and B in B, and g is a small map. Hence P is a projective cover of Im(g), which is clearly contained in B.

(2) implies (3) is clear.

(3) implies (1): If A is a submodule of M let A′ denote a supplement of A. Then if B denotes the intersection of A and A′, A′ is a cover of A′/B. Then any projective cover of A′ is a projective cover of A′/B, which is isomorphic to M/A.

5. F-SEMI-PERFECT MODULES.

Extending a definition of Jansen [4] we will call a finitely generated module F-semi-perfect iff every factor module by a finitely generated submodule has a projective cover.

We will call a module semi-supplemented iff every finitely generated submodule has a supplement, with a corresponding definition for amply semi-supplemented. With these definitions we have:

THEOREM 2. For any finitely generated module M the following statements are equivalent:

(1) M is a F-semi-perfect module.

(2) M is amply semi-supplemented by finitely generated supplements which have projective covers.

(3) M is semi-supplemented by finitely generated supplements which have projective covers.

PROOF. The proof is an easy modification of the proof of the preceeding Theorem.

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