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ON k-TRIAD SEQUENCES

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ABSTRACT. At the conference of the Indian Mathematical Society held at Allahabad in December 1981, S. P. Mohanty and A. M. S. Ramasamy pointed out that the three numbers 1, 2, 7, have the following property: the product of any two of them increased by 2 is a perfect square. They then showed that there is no fourth integer which hsares this property with all of them. They used Pell's equation and the theory of quadratic residues to prove their statement. In this paper, we show that their statement holds for a very large set of triads and our proof of the statement is very simple.

KEY WORDS AND PHRASES. Pell's equation, congruence, Fibonacci, sequence.
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1. INTRODUCTION.

DEFINITION. Given any integer k, three numbers a_1, a_2, a_3 are said to form a k-triad, if the numbers

$$a_1a_2 + k, \quad a_1a_3 + k \quad \text{and} \quad a_2a_3 + k \tag{1.1}$$

are all perfect squares.

An ascending sequence of integers

$$a_1, a_2, a_3, \ldots, a_n, \ldots\ldots$$

is said to be a k-triad sequence if every three consecutive elements of the sequence form a k-triad.

Evidently, if (1.1) is a k-triad sequence, then

$$a_2, a_3, a_4, \ldots, a_n, \ldots\ldots \tag{1.2}$$

is also a k-triad sequence.
This elementary statement will prove useful to us in our work.

2. CONSTRUCTION OF SEQUENCES (1.1).

In what follows, small letters denote integers and c_i's are positive.

Given any integer k, we can choose two integers a_1 and a_2, $a_2 > a_1$, such that $a_1a_2 + k$ is a perfect square:

$$a_1a_2 + k = c_1^2 . \tag{2.1}$$

The problem of constructing the sequence (1.1) then reduces to finding a number a_3 such that both

$$a_1a_3 + k \text{ and } a_2a_3 + k \text{ will be squares.}$$

Let
$$a_1a_3 + k = x^2; \quad a_2a_3 + k = y^2 . \tag{2.2}$$

Set
$$x = a_1 + c_1, \quad y = a_2 + c_1 .$$

Then from (2.2) we have

$$\begin{aligned}(a_2 - a_1)a_3 &= (y - x)(y + x)\\ &= (a_2 - a_1)(a_1 + 2c_1 + a_2);\end{aligned}$$

so that
$$a_3 = a_1 + a_2 + 2c_1 . \tag{2.3}$$

We assert that this value of a_3 actually satisfies our requirements. In fact, we have

$$\begin{aligned}a_1a_3 + k &= a_1(a_1 + 2c_1 + a_2) + k\\ &= a_1^2 + 2a_1c_1 + (a_1a_2 + k)\\ &= a_1^2 + 2a_1c_1 + c_1^2\\ &= (a_1 + c_1)^2 .\end{aligned}$$

Similarly

$$a_2a_3 + k = (a_2 + c_1)^2 . \tag{2.4}$$

Writing $c_2^2 = a_2a_3 + k$, we have

$$c_2 = a_2 + c_1 . \tag{2.5}$$

Notice that (2.3) provides a formula for writing the third element of a k-triad sequence interms of a_1, a_2 and c_1.

Applying this procedure to (1.2), we get

$$\begin{aligned}a_4 &= a_2 + a_3 + 2c_2\\ &= a_2 + (a_1 + a_2 + 2c_1) + 2(a_2 + c_1)\\ &= a_1 + 4a_2 + 4c_1 .\end{aligned} \tag{2.6}$$

Treating this as a formula for the fourth element of a k-triad sequence and applying it to (1.2), we get

$$\begin{aligned}a_5 &= a_2 + 4a_3 + 4c_2\\ &= 4a_1 + 9a_2 + 12c_1 .\end{aligned} \tag{2.7}$$

The process can be repeated until the desired number of elements of (1.1) has been obtained.

Then assuming that

$$a_j = u_j a_1 + v_j a_2 + w_j c_1 ;$$

we obtain

$$\begin{aligned} a_j &= u_j a_2 + v_j a_3 + w_j c_2 \\ &= u_j a_2 + v_j (a_1 + a_2 + 2c_1) + w_j (a_2 + c_1) \\ &= v_j a_1 + (u_j + v_j + w_j) a_2 + (2v_j + w_j) c_1 . \end{aligned} \tag{2.8}$$

This gives us the following recurrence relations:

$$u_{j+1} = v_j, \quad v_{j+1} = u_j + v_j + w_j, \quad w_{j+1} = 2v_j + w_j .$$

Identically:

$$\begin{bmatrix} u_{j+1} \\ v_{j+1} \\ w_{j+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} u_j \\ v_j \\ w_j \end{bmatrix} \tag{2.9}$$

or

$$\begin{bmatrix} u_j \\ v_j \\ w_j \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{j+1} \\ v_{j+1} \\ w_{j+1} \end{bmatrix} . \tag{2.10}$$

Using the same technique, from (2.5), we obtain

$$\begin{aligned} c_3 &= a_3 + c_2 \\ &= (a_1 + a_2 + 2c_1) + (a_2 + c_1) \\ &= a_1 + 2a_2 + 3c_1 . \end{aligned}$$

In general,

$$c_j = a_j + c_{j-1} . \tag{2.11}$$

Assuming that

$$c_j = r_j a_1 + s_j a_2 + t_j c_1$$

there is no difficulty in obtaining the recurrence relations:

$$\begin{bmatrix} r_{j+1} \\ s_{j+1} \\ t_{j+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} r_j \\ s_j \\ t_j \end{bmatrix} \tag{2.12}$$

and

$$\begin{bmatrix} r_j \\ s_j \\ t_j \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{j+1} \\ s_{j+1} \\ t_{j+1} \end{bmatrix} . \tag{2.13}$$

While relations (2.9) and (2.12) enable us to find expressions for a_j's and c_j's as j takes value in ascending order of magnitude, the relations (2.10) and (2.13) enable us to extend them in the opposite direction.

Our k-triad sequence can thus be defined for all integral values of the subscript; therefore, the sequence of c_i's is defined for all integral values of the subscript.

3. IDENTIFICATION OF THE COEFFICIENTS u, v, w AND r, s, t.

We hardly would expect that the Fibonacci sequence has anything to do with the coefficients u, v, w and r, s, t introduced in the preceding section. But the unexpected happens.

Recall that the Fibonacci sequence is defined by the recurrence relations:

$$f_0 = 0, \quad f_1 = 1, \quad f_m = f_{m-2} + f_{m-1}; \quad m \geq 2 . \tag{3.1}$$

This definiton can be extended to negative integral subscripts by noting that

$$f_{-m} = (-1)^{m+1} f_m . \tag{3.2}$$

From results (2.3), (2.6) and (2.7), it will be seen that for the values j = 3, 4 and 5;

$$a_j = f_{j-2}^2 a_1 + f_{j-1}^2 a_2 + 2f_{j-2} f_{j-1} c_1 ;$$

so that for these values of j,

$$u_j = f_{j-2}^2 , \quad v_j = f_{j-1}^2 , \quad w_j = 2f_{j-2} f_{j-1} .$$

From (2.9), we will now have

$$u_{j+1} = f_{j-1}^2 ; \quad v_{j+1} = f_{j-2}^2 + f_{j-1}^2 + 2f_{j-2} f_{j-1} = (f_{j-2} + f_{j-1})^2 = f_j^2 ;$$

and

$$w_{j+1} = 2f_{j-1}^2 + 2f_{j-2} f_{j-1} = 2f_{j-1} (f_{j-1} + f_{j-2}) = 2f_{j-1} f_j .$$

We now leave it to the reader to complete the induction and show that

$$a_j = f_{j-2}^2 a_1 + f_{j-1}^2 a_2 + 2f_{j-2} f_{j-1} c_1 \tag{3.3}$$

for all integral values of j.

It is no more difficult to prove that for all integral values of j

$$c_j = f_{j-2}f_{j-1}a_1 + f_{j-1}f_j a_2 + (f_j^2 - f_{j-2}f_{j-1})c_1 . \tag{3.4}$$

4. SUM OF CONSECUTIVE a's.

From (2.11), we have

$$c_j - c_{j-1} = a_j .$$

Hence we have an interesting summation formula

$$\sum_{j=m+1}^{n} a_j = c_n - c_m ; \quad n > m . \tag{4.1}$$

This provides a good check on the values of a_j's and c_i's.

5. A GENERALIZATION OF THE STATEMENT OF MOHANTY AND RAMASAMY.

Our generalization can be stated in the form of the following.

THEOREM. If a_1, a_2, a_3 is a k-triad and $k \equiv 2 \pmod 4$, then there is no integer a for which

$$a_1a + k, \quad a_2a + k, \quad a_3a + k$$

are all perfect squares.

We need two lemmas for the proof of our theorem.

LEMMA 1. Only two of the three numbers a_1, a_2, a_3 are odd.

PROOF. If a_1 and a_2 are both even, let

$$a_1a_2 + k = c_1^2 .$$

This implies that c_1 is even. Modulo 4, we have

$$2 \equiv 0 \pmod 4) .$$

This is impossible. Hence both a_1 and a_2 cannot be even.
If a_1 and a_2 are of opposite parity, then since

$$a_3 = a_1 + a_2 + 2c_1 ;$$

a_3 must be odd, and our lemma holds.
If a_1 and a_2 are both odd, then a_3 is even.
This completes the proof of our lemma.

LEMMA 2. The difference of the two odd elements of the given k-triad is congruent to 2 (mod 4).

PROOF. First let a_1 and a_2 be the odd elements of the k-triad. Then $a_2 - a_1 \equiv 0$ or $2 \pmod 4$.
But $a_2 - a_1$ cannot be congruent to 0 (mod 4). Suppose $a_2 - a_1 \equiv 0 \pmod 4$ and let

$$a_2 = a_1 + 4d \qquad \text{for some integer } d.$$

Since $a_1a_2 + k = c_1^2$, c_1 must be odd.

As $$a_1a_2 = a_1(a_1 + 4d) \equiv a_1^2 \equiv 1 \pmod 4) ,$$

we will have

$$1 + 2 \equiv c_1^2 \equiv 1 \pmod 4).$$

This is impossible. Hence

$$a_2 - a_1 \equiv 2 \pmod 4) .$$

Next let a_1 and a_3 be the two odd elements of the k-triad. Then a_2 is necessarily even. Again since $a_1a_2 + k = c_1^2$, c_1 must be even. We must, therefore, have

$$a_2 + 2 \equiv 0 \pmod 4) .$$

Hence $$a_2 \equiv 2 \pmod 4) .$$

Now $$a_3 = a_1 + a_2 + 2c_1 \quad \text{and} \quad c_1 \quad \text{is even.}$$

Hence $$a_3 - a_1 \equiv 2 \pmod 4).$$

The case in which a_2 and a_3 are the odd elements, can be dealt with in the same manner and the lemma is proven.

PROOF OF THE THEOREM. Since the existence or non-existence of the number a does not depend on the order in which the three expressions are taken, we can assume that a_1 is the even and a_2 and a_3 the odd elements of the given k-triad.

For some positive integers x, y, z, let

$$a_1a + k = x^2, \qquad \text{(i)}$$

$$a_2a + k = y^2, \qquad \text{(ii)}$$

$$a_3a + k = z^2. \qquad \text{(iii)}$$

From (i) it is evident that x is even. Replace x by 2g, a_1 (which is even) by 2h, and k (which is congruent to 2 (mod 4)) by 2q where q is odd. Then, (i) takes the form

$$ha + q = (2g)^2 . \qquad \text{(iv)}$$

Since q is odd and the right-hand side is even, h and a must both be odd. This implies that y and z are both odd. Now substracting (ii) and (iii), we have

$$(a_3 - a_2)a = z^2 - y^2 \equiv 0 \pmod 4). \qquad \text{(v)}$$

Since $a_3 - a_2 \equiv 2 \pmod 4$, (v) implies that a is even. This contradicts the earlier statement that a is odd. Hence a does not exist and we are through.

j	-4	-3	-2	-1	0	1	2	3	4	5	6	7
a_j	-67	-25	-10	-3	-1	2	5	15	38	101	263	690
c_j	41	16	6	3	2	4	9	24	62	163	426	1116
f_j	-3	2	-1	1	0	1	1	2	3	5	8	13
u_j	64	25	9	4	1	1	0	1	1	4	9	25
v_j	25	9	4	1	1	0	1	1	4	9	25	64
w_j	-80	-30	-12	-4	-2	0	0	2	4	12	30	80
r_j	-40	-15	-6	-2	-1	0	0	1	2	6	15	40
s_j	-15	-6	-2	-1	0	0	1	2	6	15	40	104
t_j	49	19	7	3	1	1	1	3	7	19	49	129

$k = 6, \ a_1 = 2, \ a_2 = 5, \ c_1 = 4 .$

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