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ON A SUBCLASS OF BAZILEVIČ FUNCTIONS

D. K. THOMAS

Department of Mathematics and Computer Science
University College of Swansea
Singleton Park
Swansea SA2 8PP
Wales, U. K.

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ABSTRACT. Let $B(\alpha)$ be the class of normalised Bazilevič functions of type $\alpha > 0$ with respect to the starlike function g. Let $B_1(\alpha)$ be the subclass of $B(\alpha)$ when $g(z) \equiv z$. Distortion theorems and coefficient estimates are obtained for functions belonging to $B_1(\alpha)$.

KEY WORDS AND PHRASES. Bazilevic functions, subclasses of S, functions whose derivative has positive real part, close-to-convex functions, coefficient and length-area estimates.
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1. INTRODUCTION.

Let S be the class of normalised functions f which are regular and univalent in the unit disc $D = \{z : |z|<1\}$. Let S^* be the subclass of S consisting of functions which are starlike, and denote by P, the class of functions which are regular in D and satisfy there the conditions $p(0) = 1$, $\text{Re } p(z) > 0$ for $p \in P$.

Bazilevič [1] showed that if α and β are real numbers, with $\alpha > 0$, then functions f, regular in D, and having the representation

$$f(z) = [(\alpha+i\beta) \int_0^z p(t)g(t)^\alpha t^{i\beta-1}dt]^{1/\alpha+i\beta} \quad ..... \tag{1.1}$$

for $g \in S^*$, $p \in P$ and $z \in D$, also form a subclass of S, denoted by $B(\alpha,\beta)$, which contains both S^* and the class of close-to-convex functions. (Powers in (1.1) are principal values). When $\beta = 0$, we write $B(\alpha,\beta) = B(\alpha)$. Zamorski [2] and the author [3] gave proofs of the Bieberbach conjecture for $f \in B(1/N)$, N a positive integer, and more recently Leach [4] has shown that the conjecture is true for $f \in B(\alpha)$, $0 \le \alpha \le 1$.

Singh [5] considered the subclass $B_1(\alpha)$ of $B(\alpha)$, obtained by taking the starlike function $g(z) \equiv z$, and gave sharp estimates for the modules of the coefficients a_2, a_3, and a_4, where for $z \in D$,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad ..... \tag{1.2}$$

We note that $B_1(1)$ is the subclass of S which consists of functions f for which $\mathrm{Re}\, f'(z) > 0$ for $z \in D$ [6].

In this paper, we shall obtain some distortion theorems for $f \in B_1(\alpha)$ and give sharp estimates for the coefficients a_n in (2) when $f \in B_1(1/N)$, N is a positive integer.

2. DISTORTION THEOREMS.

THEOREM 1. Let $f \in B_1(\alpha)$ and be given by (1.2). Then with $z = re^{i\theta}$, $0 \le r < 1$,

(i) $Q_2(r)^{1/\alpha} \le |f(z)| \le Q_1(r)^{1/\alpha}$,

(ii) If $0 < \alpha \le 1$,

$$r^{\alpha-1}\, Q_2(r)^{\frac{1-\alpha}{\alpha}}\, \frac{1-r}{1+r} \le |f'(z)| \le r^{\alpha-1} Q_1(r)^{\frac{1-\alpha}{\alpha}}\, \frac{1+r}{1-r},$$

and if $\alpha \ge 1$

$$r^{\alpha-1} Q_1(r)^{\frac{1-\alpha}{\alpha}}\, \frac{1-r}{1+r} \le |f'(z)| \le r^{\alpha-1} Q_2(r)^{\frac{1-\alpha}{\alpha}}\, \frac{1+r}{1-r},$$

where

$$Q_1(r) = \alpha \int_0^r \rho^{\alpha-1} \left(\frac{1+\rho}{1-\rho}\right) d\rho,$$

and

$$Q_2(r) = \alpha \int_0^r \rho^{\alpha-1} \left(\frac{1-\rho}{1+\rho}\right) d\rho.$$

Equality holds in all cases for the function f_ϕ, defined by

$$f_\phi(z) = \left(\alpha \int_0^z t^{\alpha-1}\left(\frac{1+te^{i\phi}}{1-te^{i\phi}}\right)dt\right)^{1/\alpha} \quad ..... \tag{2.1}$$

where $\phi = 0$ or π.

PROOF.

(i) Taking $\beta = 0$ and $g(z) \equiv z$ in (1.1), it follows that f satisfies the equation

$$z^{1-\alpha} f'(z) = f(z)^{1-\alpha} p(z) \quad ..... \tag{2.2}$$

for $z \in D$ and $p \in P$. Thus

$$f(z)^\alpha = \alpha \int_0^z t^{\alpha-1} p(t)dt,$$

and since $|p(z)| \le \frac{1+r}{1-r}$ for $z \in D$ [7], we have at once $|f(z)| \le Q_1(r)^{1/\alpha}$.

To obtain the left-hand inequality in (i), we observe that, since $\mathrm{Re}\, p(z) > 0$ for $z \in D$, $\mathrm{Re}\, p(z) \ge \frac{1-r}{1+r}$ [5], and so from (2.2)

$$\left|\frac{d}{dz}[f(z)]^\alpha\right| \ge \alpha r^{\alpha-1}\left(\frac{1-r}{1+r}\right) \quad ...... \tag{2.3}$$

Now let z_1, $|z_1| = r$ be chosen so that $|f(z_1)^\alpha| \le |f(z)^\alpha|$ for all z with $|z| = r$.

Then, writing $w = f_1(z)^\alpha$, it follows that the line segment λ from $w = 0$ to $w = f(z_1)^\alpha$ lies entirely in the image of D. Let L be the pre-image of λ, then by (2.3) we have

$$|f(z_1)|^\alpha = \int_\lambda |dw| = \int_L \left|\frac{dw}{dz}\right| |dz|$$

$$\geq \alpha \int_0^r \rho^{\alpha-1} \left(\frac{1-\rho}{1+\rho}\right) d\rho = Q_2(r),$$

which is the left-hand inequality in (i).

(ii) The proof follows at once from (2.2) and (i) on noting that for $p \in P$,

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r} \text{ [7]}.$$

Equality is attained in (i) for f_0 and in (ii) for f_o when $0 < \alpha \leq 1$ and for f_π when $\alpha \geq 1$.

We remark that as $\alpha \to 0$, the results of Theorem 1 should in some way correspond to the classical distortion theorems for regular starlike (univalent) functions [7]. The following shows that the bounds in Theorem 1 are asymptotic to the classical distortion theorems as $\alpha \to 0$.

THEOREM 2. Let $Q_1(r)$ and $Q_2(r)$ be defined as in Theorem 1. Then for $0 \lesssim r \lesssim 1$, as $\alpha \to 0$

(i) $Q_1(r)^{1/\alpha} \sim \dfrac{r}{(1-r)^2}$,

(ii) $Q_2(r)^{1/\alpha} \sim \dfrac{r}{(1-r)^2}$,

(iii) $Q_1(r) \sim Q_2(r) \sim 1$.

PROOF.

We prove (i), since (ii) and (iii) are similar. As $\alpha \to 0$,

$$Q_1(r)^{1/\alpha} = \left(\alpha \int_0^r \rho^{\alpha-1}\left(\frac{1+\rho}{1-\rho}\right)d\rho\right)^{1/\alpha} = r\left(1+2\alpha r^{-\alpha} \int_0^r \frac{\rho^\alpha}{1-\rho}\, d\rho\right)^{1/\alpha}$$

$$\sim r(1-2\alpha r^{-\alpha}\log(1-r))^{1/\alpha} \sim re^{-2\log(1-r)} = \frac{r}{(1-r)^2}$$

COROLLARY. Suppose that $f(z) \neq w$ for $z \in D$, then

$$|w| \geq Q_2(1)^{1/\alpha} \sim \tfrac{1}{4} \text{ as } \alpha \to 0.$$

PROOF. Let $\alpha > 0$, and w be a point on the boundary of $f(D)$ closest to the orgin. Let L_1 denote the straight line from 0 to w, and L its pre-mage in D. Then $|w| > |f(z)|$ for $z \in L \cap D$. Since the circle $|z| = r$, for each $0 \leq r < 1$, intersects L at least once, Theorem 1 (i) gives $|w| \geq Q_2(r)^{1/\alpha}$ and so $|w| > Q_2(1)^{1/\alpha}$ $\sim \frac{1}{4}$ as $\alpha \to 0$ (from Theorem 2 (ii)).

3. A COEFFICIENT THEOREM.

NOTATION. $\sum_{n=0}^{\infty} a_n z^n \ll \sum_{n=0}^{\infty} \beta_n z^n$ means $|\alpha_n| \leq |\beta_n|$ for $n \geq 0$.

THEOREM 3. Let $f \in B_1(1/N)$, with N a positive integer, and be given by (1.2). Suppose also that for $z \in D$,

$$f_0(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n,$$

where f_0 is given by (2.1). Then

(i) $f(z) << f_0(z)$,

(ii) $\gamma_n \sim (\frac{2}{N})^N (\frac{N}{n}) (\log n)^{N-1}$ as $n \to \infty$.

PROOF. (i) We first note that if $|\alpha_n| \le |\beta_n|$, then for $m = 1,2,...$

$$(\sum_{n=1}^{\infty} \alpha_n z^n)^m << (\sum_{n=1}^{\infty} \beta_n z^n)^m.$$

To see this, let

$$(\sum_{n=1}^{\infty} \alpha_n z^n)^m = \sum_{n=0}^{\infty} A_n^{(m)} z^n \quad \text{and} \quad (\sum_{n=0}^{\infty} \beta_n z^n)^m = \sum_{n=0}^{\infty} B_n^{(m)} z^n,$$

so that

$$A_n^{(k)} = \sum_{\nu=1}^{n} A_\nu^{(k-1)} \alpha_{n-\nu}, \quad B_n^{(k)} = \sum_{\nu=1}^{n} B_\nu^{(k-1)} \beta_{n-\nu}.$$

We now use induction on k to show that for $n \ge 1$, $|A_n^{(k)}| \le B_n^{(k)}$. Clearly for $n = 1,2,...,$ $|A_n^{(1)}| = |\alpha_n| \le \beta_n = B_n^{(1)}$. Suppose now that $|A_n^{(k)}| \le B_n^{(k)}$ for $n = 1,2,...$ and $k = 1,2,...,j$. Then for $n = 1,2,...$

$$|A_n^{(j+1)}| \le \sum_{\nu=1}^{n} |A^{(j)}||\alpha_{n-\nu}| \le \sum_{\nu=1}^{n} B_n^{(j)} \beta_{n-\nu} = B_n^{(j+1)}.$$

Thus (i) now follows at once, since from (2.2) we can write

$$f(z) = z \{ 1+ \frac{1}{N} \sum_{k=1}^{\infty} \frac{p_k z^k}{k+1/N} \}^N,$$

where $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$, and since $|p_k| \le 2$ [7] we have

$$f(z) << z[1+ \frac{2}{N} \sum_{k=1}^{\infty} \frac{z^k}{k+1/N}]^N = f_0(z).$$

(ii) When $\alpha = 1/N$, (2.1) gives

$$f_0(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n = z[1+ \frac{2}{N} \sum_{n=1}^{\infty} \frac{z^n}{n+1/N}]^N$$

$$= z \sum_{\nu=0}^{\infty} \binom{N}{\nu} (\frac{2}{N})^\nu (\sum_{n=1}^{\infty} \frac{z^n}{n+1/N})^\nu$$

Now trivially,

$$(\sum_{n=1}^{\infty} \frac{z^n}{n+1})^\nu << (\sum_{n=1}^{\infty} \frac{z^n}{n+1/N})^\nu << (\sum_{n=1}^{\infty} \frac{z^n}{n})^\nu.$$

Write these three series as

$$\sum_{n=\nu}^{\infty} C_n^{(\nu)} z^n, \quad \sum_{n=\nu}^{\infty} D_n^{(\nu)} z^n \quad \text{and} \quad \sum_{n=\nu}^{\infty} E_n^{(\nu)} z^n \quad \text{respectively.}$$

Then

$$\sum_{n=\nu}^{\infty} E_n^{(\nu)} z^n = z^\nu (\sum_{n=0}^{\infty} \frac{z^n}{n+1})^\nu$$

Now a result of Littlewood [8, p. 193], states that if ν is a fixed positive integer and

$$\left(\sum_{n=0}^{\infty} \frac{z^n}{n+1}\right)^{\nu} = \sum_{n=0}^{\infty} \phi_n^{(\nu)} z^n,$$

then $\phi_n^{(\nu)} \sim \frac{\nu}{n}(\log n)^{\nu-1}$ as $n \to \infty$.

Thus

$$E_n^{(\nu)} = \phi_{n-\nu}^{(\nu)} \sim \frac{\nu}{n}(\log n)^{\nu-1} \quad \text{as} \quad n \to \infty.$$

Also

$$\sum_{n=\nu}^{\infty} C_n^{(\nu)} z^n = \left(\sum_{n=0}^{\infty} \frac{z^n}{n+1} - 1\right)^{\nu} \quad \text{and so}$$

$$C_n^{(\nu)} = \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^{\nu-j} \phi_n^{(j)}$$

$$\sim \frac{\nu}{n}(\log n)^{\nu-1} \quad \text{as} \quad n \to \infty .$$

Thus $D_n^{(\nu)} \sim \frac{\nu}{n}(\log n)^{\nu-1}$ and so

$$\gamma_n \sim \sum_{\nu=0}^{N} \binom{N}{\nu} \left(\frac{2}{N}\right)^{\nu} \nu D_n^{(\nu)} \sim \left(\frac{2}{N}\right)\left(\frac{N}{n}\right)(\log n)^{N-1}$$

as $n \to \infty$.

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