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RESEARCH NOTES

THE SPACES O_M AND O_C ARE ULTRABORNOLOGICAL A NEW PROOF

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ABSTRACT. In [1] Laurent Schwartz introduced the spaces $\mathcal{O}_M$ and $\mathcal{O}'_C$ of multiplication and convolution operators on temperate distributions. Then in [2] Alexandre Grothendieck used tensor products to prove that both $\mathcal{O}_M$ and $\mathcal{O}'_C$ are bornological. Our proof of this property is more constructive and based on duality.

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We use C, N, R, and Z, resp., for the set of all complex, nonnegative integer, real, and integer numbers. For each $q \in N$, the space

$$L_q = \{f: R^n \to C;\ \|f\|_q^2 = \sum_{|\alpha+\beta| \le q} \int_{R^n} x^{2\alpha}|D^\beta f(x)|^2\, dx < +\infty\} \text{ is Hilbert.}$$

Here $D^\beta f$ stands for the Sobolev generalized derivative. We denote by L_{-q} the strong dual of L_q and by $\|\cdot\|_{-q}$ the standard norm on L_{-q}. Then the space $\mathcal{S}$ of rapidly decreasing functions, resp. its strong dual $\mathcal{S}'$, is the $\operatorname{proj\,lim}_{q \to \infty} L_q$, resp. $\operatorname{ind\,lim}_{q \to \infty} L_{-q}$.

It is convenient to introduce the weight-function $W(x) = (1 + |x|^2)^{\frac{1}{2}}$, $x \in R^n$. The mapping $T_k : f \mapsto W^k f: \mathcal{S}' \to \mathcal{S}'$, $k \in Z$, is injective. We denote by $W^k L_m$, $k, m \in Z$, the image of L_m under T_k and provide it with the topology which makes $T_k : L_m \to W^k L_m$ a topological isomorphism. Further, $\mathcal{O}_q$, $q \in N$, stands for the $\operatorname{ind\,lim}_{p \to \infty} W^p L_q$, and $\mathcal{O}_{-q}$ for its strong dual. It is proved in [7] that for each $q \in N$, the space $\mathcal{O}_q$ is reflexive and $\mathcal{O}_{-q} = \operatorname{proj\,lim}_{p \to \infty} W^{-p} L_{-q}$. Finally, the space $\mathcal{O}_M$ of multiplication operators on $\mathcal{S}'$ equals $\operatorname{proj\,lim}_{q \to \infty} \mathcal{O}_q$, see [6].

PROPOSITION 1. The strong dual $\mathcal{O}'_M$ of $\mathcal{O}_M$ equals $\operatorname{ind\,lim}_{q \to \infty} \mathcal{O}_{-q}$.

PROOF. The space $\mathcal{S}$ is dense in each L_q, $q \in N$. Hence $\mathcal{S} = W^p\mathcal{S}$ is dense in $W^p L_q$ for each $p \in N$. Then $\mathcal{S}$, and à fortiori its superset $\mathcal{O}_M$, are dense in each

$\mathcal{O}_q = \operatorname{ind\,lim}_{p\to\infty} W^p L_q$, $q \in N$. By [3, ch. IV, 4.4], the dual of $\mathcal{O}_M$, equipped with the Mackey topology, equals $\operatorname{ind\,lim}_{q\to\infty} \mathcal{O}_{-q}$. The Mackey and strong topologies on $\mathcal{O}'_M$ coincide since $\mathcal{O}_M$, as a projective limit of reflexive spaces $\mathcal{O}_q$, is semireflexive, see [3, ch. IV, 5.5].

PROPOSITION 2. $\mathcal{O}_M$ is the strong dual of $\operatorname{ind\,lim}_{q\to\infty} \mathcal{O}_{-q}$.

PROOF. By [3, ch. IV, 4.5], the topology τ of $\mathcal{O}_M = \operatorname{proj\,lim}_{q\to\infty} \mathcal{O}_q$ is consistent with the duality $< \mathcal{O}'_M, \mathcal{O}_M >$. Hence τ is coarser than the strong topology $\beta(\mathcal{O}_M, \mathcal{O}'_M)$. On the other hand, it is proved in [5, Prop. 4] that τ is finer than $\beta(\mathcal{O}_M, \mathcal{O}'_M)$.

THEOREM 1. The space $\mathcal{O}_M$ is reflexive and $\mathcal{O}'_M$ is the strong dual.

LEMMA 1. Let $r = 1 + [\frac{1}{2} n]$, $q \in N$. Then $W^{-r}L_q \subset L_q$ and every set bounded in $W^{-r}L_q$ is relatively compact in L_q.

PROOF. Let B be an absolutely convex, bounded, and closed, set in $W^{-r}L_q$. Then B is weakly compact as a polar of a neighborhood in $W^r L_{-q}$. By [3, Ch. IV, 11.1, Cor 2], B is weakly sequentially compact and every sequence in B contains a subsequence $\{f_k\}$ which converges weakly to some $g \in B$. We may assume $g = 0$.

Since the set $\{W^{r+q}f;\ f \in B\}$ is bounded in $L^2(R^n)$, the set $\{W^q f;\ f \in B\}$ is bounded in $L^1(R^n)$ and for any $\alpha \in N^n$, $|\alpha| \le q$, the set $\{D^\alpha \mathcal{F}f;\ f \in B\}$, where $\mathcal{F}f$ is the Fourier transform of f, is uniformly bounded and locally equicontinuous on R^n. Hence $\{f_k\}$ contains a subsequence, let it be again $\{f_k\}$, such that $\{D^\alpha \mathcal{F} f_k(x)\}$ converges uniformly on R^n for all $\alpha \in N^n$, $|\alpha| \le q$.

Take a non-negative function $h \in \mathcal{S}$, $\int_{R^n} h(x)dx = 1$, and put $h_i(x) = i^n h(ix)$, $i \in N$. Then $f * h^i \to f$ as $i \to \infty$ in the topology of L_q uniformly on B. Given $\varepsilon > 0$, there is $i \in N$ such that $\|f - f * h_i\|_q < \varepsilon$ for any $f \in B$. We fix this i. For every $\alpha, \beta \in N^n$, $|\alpha + \beta| \le q$, the sequence $\{W^\alpha D^\beta(\mathcal{F}f_k \cdot \mathcal{F}h_i)\}$ converges uniformly to 0 on R^n as $k \to \infty$ and has an integrable majorant from $\mathcal{S}$. Hence $\mathcal{F}(f_k * h_i) \to 0$, and à fortiori $f_k * h_i \to 0$, both in the topology of L_q. If we choose $k_0 \in N$ so that $\|f_k * h_i\|_q < \varepsilon$ for $k > k_0$, then $\|f_k\|_q < 2\varepsilon$ for $k > k_0$.

LEMMA 2. Let $r = 1 + [\frac{1}{2} n]$, $q \in N$. Then $W^{-r}L_{-q} \subset L_{-q}$ and every set bounded in $W^{-r}L_{-q}$ is relatively compact in L_{-q}.

PROOF. Let B be an absolutely convex, bounded, and closed, set in $W^{-r}L_{-q}$. By the same argument as in Lemma 1, every sequence in B has a subsequence $\{f_k\}$ which converges weakly to some $g \in B$. We again assume $g = 0$.

Denote by $\|\cdot\|_{-r,-q}$, resp. $\|\cdot\|_{r,q}$, the norm in $W^{-r}L_{-q}$, resp. $W^r L_q$. Let A be the closed unit ball in L_q, B_0 the open unit ball in $W^r L_q$, and $a = \sup\{\|f\|_{-r,-q};\ f \in B\}$. Choose $\varepsilon > 0$. By Lemma 1, A is compact in the topology of $W^r L_q$. Since L_q is dense in $W^r L_q$, there exists a finite set $\{\varphi_i;\ i \in F\} \subset L_q$ such that $A \subset \cup\{\varphi_i + \varepsilon B_0;\ i \in F\}$. For any $\varphi \in A$, there exists φ_i such that $\|\varphi - \varphi_i\|_{r,q} < \varepsilon$ and for any $k \in N$ we have $|< \varphi, f_k >| \le |< \varphi - \varphi_i, f_k >| + |< \varphi_i, f_k >| \le \|\varphi - \varphi_i\|_{r,q} \cdot \|f_k\|_{-r,-q} + |< \varphi_i, f_k <| \le$

$\le \varepsilon a + |< \varphi_i, f_k >|$. If we choose $k_0 \in N$ so that $|< \varphi_i, f_k >| < \varepsilon$ for all $i \in F$ and $k > k_0$ and the sequence $\{f_k\}$ converges in L_{-q}.

PROPOSITION 3. For each $q \in N$, $\mathcal{O}_{-q}$ is a Schwartz space.

PROOF. By Lemma 2, for every $p \in N$ the closed unit ball is $W^{-r-p}L_{-q}$, where $r = 1 + [\frac{1}{2}n]$, is compact in $w^{-p}L_{-q}$. By [4, Ch. 3.15, Prop. 9], the space $\mathcal{O}_{-q} = \operatorname{proj}\lim_{p \to \infty} W^{-p}L_{-q}$ is Schwartz.

PROPOSITION 4. Let $E_1 \subset E_2 \subset \ldots$ be locally convex spaces with identity maps: $E_k \to E_{k+1}$, $k \in N$, continuous and $E = \operatorname{ind}\lim_{k \to \infty} E_k$ Hausdorff. Assume:

(1) every set bounded in E is bounded in some E_k,

(2) every E_k is a Schwartz space.

Then E is a Schwartz space.

Proposition 4 is slightly more general than Prop. 8 in [4, Ch. 3.15] and its proof requires only minor changes of the proof presented in [4].

THEOREM 2. $\mathcal{O}'_M$ is a Schwartz space.

PROOF. We have $\mathcal{O}'_M = \operatorname{ind}\lim_{q \to \infty} \mathcal{O}_{-q}$. Each space $\mathcal{O}_{-q}$ is Schwartz and Fríchet. Further, $\mathcal{O}'_M$ is reflexive, hence quasi-complete, which in turn implies fast completeness. By [8, Th. 1], the assumption (1) of Prop. 4 is satisfied and $\mathcal{O}'_M$ is a Schwartz space.

THEOREM 3. $\mathcal{O}'_M$ is complete.

PROOF. The space $\dot{\mathcal{B}}$ of C^∞ - functions, whose derivatives vanish at ∞ was introduced in [1]. We denote the space $W^m\dot{\mathcal{B}}$ by $\dot{\mathcal{B}}_m$ and provide it with the topology for which $f \mapsto W^m f : \dot{\mathcal{B}} \to \dot{\mathcal{B}}_m$ is a topological isomorphism. Then the strong dual $\mathcal{O}_C$ of $\mathcal{O}'_C$ equals $\operatorname{ind}\lim \dot{\mathcal{B}}_m$, see [2, Ch. 2, 4.4]. Also, $\mathcal{O}_C$ is isomorphic to $\mathcal{O}'_M$ via Fourier transformation. Hence it suffices to prove that $\operatorname{ind}\lim_{m \to \infty} \dot{\mathcal{B}}_m$ is complete.

Let F be a Cauchy filter on $\mathcal{O}_C$, G a filter of all 0-neighborhoods in $\mathcal{O}_C$, and H the filter with base $\{A + B;\ A \in F,\ B \in G\}$. By [4, Ch. 2.12, Lemma 3], there exists $m \in N$ such that H induces a filter H_m on $\dot{\mathcal{B}}_m$ which is Cauchy in the topology inherited from $\mathcal{O}_C$. On each ball $\{x \in R^n,\ |x| \le n\}$, $r > 0$, the filter H_m converges uniformly pointwise to a function $f \in \dot{\mathcal{B}}_m$. Then f adheres to H_m on the subset $\dot{\mathcal{B}}_m$ of $\mathcal{O}_C$ and by [4, Ch. 2.9, Prop. 1] the filter F converges to f.

THEOREM 4. The spaces $\mathcal{O}_M$ and $\mathcal{O}'_M$ are ultrabornological.

PROOF. By Exercise 9 in [4, Ch. 3.15], the strong dual of a complete Schwartz space is ultrabornological. Hence $\mathcal{O}_M$ is ultrabornological by Theorems 1, 2, and 3.

The space $\mathcal{O}'_M$ is ultrabornological as an inductive limit of Fréchet spaces $\mathcal{O}_{-q}$, $q \in N$.

THEOREM 5. The spaces $\mathcal{O}_C$ and its strong dual $\mathcal{O}'_C$ are both complete, reflexive, and ultrabornological spaces.

PROOF. The space $\mathcal{O}_M$ is complete as a strong dual of a bornological space. Since the Fourier transformations $\mathcal{F}\colon \mathcal{O}_M \to \mathcal{O}'_C$ and $\mathcal{F}\colon \mathcal{O}'_M \to \mathcal{O}_C$ are topological isomorphisms, Theorem 5 follows from Theorems 1, 3, and 4.

REFERENCES

1. SCHWARTZ, L. Théorie des Distributions, Hermann, Paris, 1966.

2. GROTHENDIECK, A. Produits, Tensoriels Topologiques et Espaces Nucléaires, Memoirs of the AMS 16 Providence (1955).

3. SCHAEFER, H. Topological Vector Spaces, Grad. Texts in Math., (3), 3rd printing, Springer 1971.

4. HORVÁTH, J. Topological Vector Spaces and Distributions, Addison-Wesley, Reading 1966.

5. KUCERA, J. On Multipliers of Temperate Distributions, Czech. Math. J. 21, (96), (1971), 610-618.

6. KUCERA, J., MCKENNON, K. Certain Topologies on the Space of Temperate Distributions and its Multipliers, Indiana Univ. Math. J. 24 (8), (1975), 773-775.

7. KUCERA, J., MCKENNON, K. The Topology on Certain Spaces of Multipliers of Temperate Distributions, Rocky Mountain J. of Math. 7 (2) (1977), 377-383.

8. KUCERA, J., BOSCH, C. Bounded Sets in Fast Complete Inductive Limits, Intern. J. Math. 7 (3) (1984), 615-617.