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ON A FIXED POINT THEOREM OF GREGUŠ

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ABSTRACT. We consider two selfmaps T and I of a closed convex subset C of a Banach space X which are weakly commuting in X, i.e.

$$||TIx - ITx|| \le ||Ix - Tx|| \text{ for any } x \text{ in } X,$$

and satisfy the inequality

$$||Tx - Ty|| \le a||Ix - Iy|| + (1 - a) \max \{||Tx - Ix||, ||Ty - Iy||\}$$

for all x,y in C, where $0 < a < 1$. It is proved that if I is linear and non-expansive in C and such that IC contains TC, then T and I have a unique common fixed point in C.

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1. INTRODUCTION.

The second author [1], generalizing a result of Das and Naik [2], defined two mappings T and I of a metric space (X,d) into itself to be weakly commuting if

$$d(TIx,ITx) \le d(Ix,Tx) \tag{1.1}$$

for all x in X. Two commuting mappings clearly satisfy (1.1) but the converse is not generally true as is shown with the following example:

EXAMPLE 1. Let $X = [0,1]$ with the Euclidean metric and define T and I by

$$Tx = x/(x+4), \quad Ix = x/2$$

for all x in X. Then

$$d(TIx, ITx) = \frac{x}{x+8} - \frac{x}{2x+8} = \frac{x^2}{2(x+8)(x+4)}$$

$$\le \frac{x^2 + 2x}{2(x+4)} = \frac{x}{2} - \frac{x}{x+4} = d(Ix,Tx)$$

for all x in X but for any $x \neq 0$:

$$TIx = x/(x+8) > x/(2x+8) = ITx.$$

From now on, C denotes a closed convex subset of a Banach space X. In a recent paper Gregus [3] proved the following theorem:

THEOREM 1. Let T be a mapping of C into itself satisfying the inequality

$$||Tx - Ty|| \le a||x - y|| + b||Tx - x|| + c||Ty - y|| \tag{1.2}$$

for all x,y in C, where $0 < a < 1$, $b \ge 0$, $c \ge 0$ and $a + b + c = 1$. Then T has a unique fixed point.

Mappings satisfying inequality (1.2) with $a = 1$ and $b = c = 0$ are called nonexpansive and were considered by Kirk [4].

Wong [5] studied mappings satisfying inequality (1.2) with $a = 0$ and $b = c = \frac{1}{2}$.

2. MAIN RESULTS.

We now prove the following generalization of Theorem 1:

THEOREM 2. Let T and I be two weakly commuting mappings of C into itself satisfying the inequality

$$||Tx - Ty|| \le a||Ix - Iy|| + (1 - a) \max \{||Tx - Ix|| , ||Ty - Iy||\} \tag{2.1}$$

for all x,y in C, where $0 < a < 1$. If I is linear, nonexpansive in C and such that IC contains TC, then T and I have a unique common fixed point in C.

PROOF. Let $x = x_0$ be an arbitrary point in C and choose points x_1, x_2, x_3 in C such that

$$Ix_1 = Tx,\ Ix_2 = Tx_1,\ Ix_3 = Tx_2.$$

This can be done since IC contains TC. Then for $r = 1,2,3$ we have on using inequality (2.1)

$$\begin{aligned}||Tx_r - Ix_r|| &= ||Tx_r - Tx_{r-1}|| \\ &\le a||Ix_r - Ix_{r-1}|| + (1-a) \max \{||Tx_r - Ix_r||, ||Tx_{r-1} - Ix_{r-1}||\} \\ &= a||Tx_{r-1} - Ix_{r-1}|| + (1-a) \max \{||Tx_r - Ix_r||, ||Tx_{r-1} - Ix_{r-1}||\}\end{aligned}$$

and so

$$||Tx_r - Ix_r|| \le ||Tx_{r-1} - Ix_{r-1}||.$$

It follows that

$$||Tx_r - Ix_r|| \le ||Tx - Ix|| \tag{2.2}$$

for $r = 1,2,3$.

Further

$$\begin{aligned}||Tx_2 - Tx|| &\le a||Ix_2 - Ix|| + (1-a) \max \{||Tx_2 - Ix_2||, ||Tx - Ix||\} \\ &\le a(||Tx_1 - Ix_1|| + ||Tx - Ix||) + (1-a)\, ||Tx - Ix|| \\ &\le (1+a)\, ||Tx - Ix||\end{aligned}$$

on using inequality (2.2). Thus

$$||Tx_2 - Ix_1|| \leq (1+a)\ ||Tx - Ix||. \tag{2.3}$$

We will now define a point z by

$$z = \frac{1}{2} x_2 + \frac{1}{2} x_3.$$

Since C is convex the point z is in C and being I linear, we have

$$Iz = \frac{1}{2} Ix_2 + \frac{1}{2} Ix_3 = \frac{1}{2} Tx_1 + \frac{1}{2} Tx_2.$$

It follows that

$$\begin{aligned}
||Tz - Iz|| &\leq \frac{1}{2} ||Tz - Tx_1|| + \frac{1}{2}||Tz - Tx_2|| \\
&\leq \frac{1}{2} [a||Iz - Ix_1|| + (1-a) \max \{||Tz - Iz||, ||Tx_1 - Ix_1\}] \\
&\quad + \frac{1}{2} [a||Iz - Ix_2|| + (1-a) \max \{||Tz - Iz||, ||Tx_2 - Ix_2||\}] \\
&= \frac{1}{2} a(||Iz - Ix_1|| + ||Iz - Ix_2||) + (1-a) \max \{||Tz - Iz||, ||Tx - Ix||\}
\end{aligned}$$

on using inequalities (2.1) and (2.2). Now

$$\begin{aligned}
||Iz - Ix_1|| &\leq \frac{1}{2} ||Ix_2 - Ix_1|| + \frac{1}{2} ||Ix_3 - Ix_1|| \\
&= \frac{1}{2} ||Tx_1 - Ix_1|| + \frac{1}{2} ||Tx_2 - Ix_1|| \\
&\leq (1 + \frac{1}{2} a)\ ||Tx - Ix||
\end{aligned}$$

from inequalities (2.2) and (2.3) and

$$||Iz - Ix_2\ || = \frac{1}{2} ||Ix_3 - Ix_2|| = \frac{1}{2} ||Tx_2 - Ix_2|| \leq \frac{1}{2} ||Tx - Ix||.$$

It follows that

$$||Tz - Iz|| \leq \frac{1}{4} a(3 + a)\ ||Tx - Ix|| + (1-a) \max \{||Tz - Iz||, ||Tx - Ix||\}$$

and so

$$||Tz - Iz\ || \leq \lambda.||Tx - Ix||$$

where

$$\lambda = (4 - a + a^2)/4 < 1.$$

We therefore have

$$\inf \{||Tz - Iz|| : z = \frac{1}{2} x_2 + \frac{1}{2} x_3\} \leq \lambda.\inf \{||Tx - Tx|| : x \in C\}$$

and since we obviously have

$$\inf \{||Tz - Iz|| : z = \frac{1}{2} x_2 + \frac{1}{2} x_3\} \geq \inf \{||Tx - Ix|| : x \in C\},$$

it follows that

$$\inf \{||Tx - Ix|| : x \in C\} = 0.$$

Each of the sets

$$K_n = \{x \in C : ||Tx - Ix|| \le 1/n\},\ H_n = \{x \in C : ||Tx - Ix|| \le (a + 1)/an\}$$

(for $n = 1,2,\ldots$) must therefore be non-empty and obviously

$$K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \ \_\_\_ \ \ldots \quad .$$

Thus each of the sets $\overline{TK_n}$, where $\overline{TK_n}$ denotes the closure of TK_n, must be non-empty for $n = 1,2,\ldots$ and

$$\overline{TK_1} \supseteq \overline{TK_2} \supseteq \ldots \ \overline{TK_n} \supseteq \ldots \quad .$$

Further, for arbitrary x, y in K_n,

$$\begin{aligned} ||Tx - Ty|| &\le a\,||Ix - Iy|| + (1-a)\max\{||Tx - Ix||, ||Ty - Iy||\} \\ &\le a(||Tx - Ix|| + ||Tx - Ty|| + ||Ty - Iy||) + (1-a)/n \\ &\le (a+1)/n + a||Tx - Ty|| \end{aligned}$$

and so

$$||Tx - Ty|| \le \frac{a + 1}{(1-a)n} .$$

Thus

$$\lim_{n\to\infty} \operatorname{diam} (TK_n) = \lim_{n\to\infty} \operatorname{diam} (\overline{TK_n}) = 0 \quad .$$

It follows, by a well known result of Cantor (see, for example [6], p. 156), that the intersection $\bigcap_{n=1}^{\infty} \overline{TK_n}$ contains exactly one point w.

Now let y be an arbitrary point in $\overline{TK_n}$. Then for arbitrary $\varepsilon > 0$ there exists a point y' in K_n such that

$$||Ty' - y|| < \varepsilon \tag{2.4}$$

and so, using the weak commutativity of T and I and the nonexpansiveness of I, we have from (2.1) and (2.4):

$$\begin{aligned} ||Ty - Iy|| &\le ||Ty - TIy'|| + ||TIy' - ITy'|| + ||ITy' - Iy|| \\ &\le a||Iy - I^2y'|| + (1-a)\max\{||Ty - Iy||, ||TIy' - I^2y'||\} \\ &\qquad + ||Iy' - Ty'|| + ||Ty' - y|| \\ &\le a||y - Iy'|| + (1-a)\max\{||Ty - Iy||, ||TIy' - ITy'|| + ||Ty' - Iy'||\} \\ &\qquad + 1/n + \varepsilon \\ &\le a(||y - Ty'|| + 1/n) + (1-a)\max\{||Ty - Iy||, 2/n\} + 1/n + \varepsilon \\ &\le (1+a)\,\varepsilon + (a+1)/n + (1-a)\max\{||Ty - Iy||, 2/n\} \quad . \end{aligned}$$

Since ε is arbitrary it follows that

$$||Ty - Iy|| \le (a+1)/n + (1-a)\max\{||Ty - Iy||, 2/n\} \quad . \tag{2.5}$$

If $||Ty - Iy|| \le 2/n$, then we have

$$||Ty - Iy|| \le 2/n < (a+1)/an.$$

If $||Ty - Iy|| > 2/n$, (2.5) implies

$$||Ty - Iy|| \le (a+1).n + (1-a).||Ty - Iy||.$$

So in both cases y lies in H_n. Thus $\overline{TK_n} \subseteq H_n$ and so the point w must be in H_n for $n = 1,2,\ldots$ It follows that

$$||Tw - Iw|| \le (a+1)/an$$

for $n = 1,2,\ldots$ and so $Tw = Iw$.

Since (1.1) holds, we also have $ITw = TIw$. Thus

$$||T^2w - Tw|| \le a||ITw - Iw|| + (1-a) \max\{||T^2w - ITw||, ||Tw - Iw||\}$$

$$= a||T^2w - Tw||$$

and it follows that $Tw = w'$ is a fixed point of T since $a < 1$. Further $Iw' = ITw + TIw = TTw = Tw' = w'$ and so w' is also a fixed point of I. Now suppose that T and I have a second common fixed point w''. Then

$$\begin{aligned} ||w' - w''|| &= ||Tw' - Tw''|| \\ &\le a||Iw' - Iw''|| + (1-a) \max\{||Tw' - Iw'||, ||Tw'' - Iw''||\} \\ &\le a||w' - w''|| \end{aligned}$$

and the uniqueness of the common fixed point follows since $a < 1$. This completes the proof of the theorem.

EXAMPLE 2. Let $X = R$ and $C = [0,1]$ with the usual norm. Let T and I be as in example 1. I is clearly linear and nonexpansive and further

$TC = [0,1/5] \subset [0,1/2] = IC$.

Thus

$$||Tx - Ty|| = \frac{4||x - y||}{(x+4)(y+4)} \le \frac{1}{2} \cdot \frac{||x - y||}{2} = \frac{1}{2}||Ix - Iy||$$

for all x,y in C and inequality (2.1) is satisfied for $a = 1/2$.

So all the assumptions of Theorem 2 hold and $w = 0$ is the unique common fixed point of T and I.

Letting I be the identity mapping in Theorem 2, we have the following corollary which extends Theorem 1:

COROLLARY. Let T be a mapping of C into itself satisfying the inequality

$$||Tx - Ty|| \le a||x - y|| + (1-a) \max\{||Tx - x||, ||Ty - y||\}$$

for all x,y in C, where $0 < a < 1$. Then T has a unique fixed point.

The result of this corollary was given in [7].

We note that the weak commutativity in Theorem 2 is a necessary condition. It suffices to consider the following example:

EXAMPLE 3. Let $X = R$ and let $C = [0,1]$ with the usual norm.

Define T and I by $Tx = 1/3$, $Ix = x/2$ for any x in C.

It is easily seen that all the conditions of Theorem 2 are satisfied except that of weak commutativity since with $x = 1/2$

$$||TI(1/2) - IT(1/2)|| = 1/6 > 1/12 = ||T(1/2) - I(1/2)|| .$$

However T and I do not have a common fixed point.

We conclude that although the mappings T and I in Theorem 2 have a unique common fixed point in C, it is possible for them to have other fixed points, as proved in the next example:

Example 4. Let $X = C = R^2$ with norm

$$||(x,y)|| = \max \{|x|, |y|\}$$

for all (x,y) in R^2. Define mappings T and I on R^2 by

$$T(x,y) = (0,y),\ I(x,y) = (x,-y)$$

for all (x,y) in R^2. Then for all $(x,y) \in R^2$, $(x',y') \in R^2$

$$||T(x,y) - T(x',y')|| = |y - y'|$$

and

$$\begin{aligned} a||I(x,y) - I(x',y')|| &+ (1-a) \max \{||T(x,y) - I(x,y)||, ||T(x',y') - I(x',y')||\} \\ &= a \max \{|x-x'|, |y-y'|\} + (1-a) \max \{|x|, 2|y|, |x'|, 2|y'|\} \\ &\geq a|y - y'| + 2(1-a) \max \{|y|, |y'|\} \\ &\geq a|y - y'| + (1-a) (|y| + |y'|) \\ &\geq |y-y'| \end{aligned}$$

if $0 < a < 1$. Since T commutes with I and I is a linear isometry, it follows that all the conditions of Theorem 2 are satisfied but T and I each have an infinite number of fixed points.

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