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ON CERTAIN CLASSES OF p-VALENT FUNCTIONS

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ABSTRACT. Let $V_k^\lambda(\alpha,b,p)(k \geq 2$, $b \neq o$ is any complex number, $o \leq \alpha < p$ and $|\lambda| < \pi/2)$ denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ analytic in $U = \{z: |z| < 1\}$ having (p-1) critical points in U and satisfying

$$\limsup_{r\to 1^-} \int_0^{2\pi} \left|\frac{\mathrm{Re}\{e^{i\lambda}[p+\frac{1}{b}(1+\frac{zf''(z)}{f'(z)}-p)]-\alpha\cos\lambda\}}{p-\alpha}\right| d\theta \leq k\pi\cos\lambda.$$

In this paper we generalize both those functions $f(z)$ which are p-valent convex of order α, $o \leq \alpha < p$, with bounded boundary rotation and those p-valent functions $f(z)$ for which $zf'(z)/p$ is λ-spirallike of order α, $o \leq \alpha < p$.

KEY WORDS AND PHRASES. p-Valent, starlike, convex, spirallike functions, functions with bounded boundary rotation.
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1. INTRODUCTION.

Let $M_k (k \geq 2)$ denote the lower class of real-valued functions $m(t)$ of bounded variation on $[o,2\pi]$ which satisfy the conditions $\int_0^{2\pi} dm(T) = 2$ and $\int_0^{2\pi} |dm(t)| \leq k$. Let A_p, where p is a positive integer, denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ which are analytic in $U = \{z: |z| < 1\}$. For $f \in A_p$, we say that f belongs to the class $V_k^\lambda(\alpha,b,p)(k \geq 2$, $b \neq o$ is any complex number, $o \leq \alpha < p$ and $|\lambda| < \pi/2)$ if there exists $\delta > o$ such that

$$\int_0^{2\pi} \mathrm{Re}\left\{1+\frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta = 2p\pi \quad (1-\delta<r<1) \tag{1.1}$$

and

$$\limsup_{r \to 1^-} \int_0^{2\pi} \left|\frac{\mathrm{Re}\{e^{i\lambda}[p+\frac{1}{b}(1+\frac{zf''(z)}{f'(z)}-p)]-\alpha\cos\lambda\}}{p-\alpha}\right| d\theta \leq k\pi\cos\lambda. \tag{1.2}$$

condition (1.1) implies that f has $(p-1)$ critical points in U.

It is noticed that, by giving specific values to k,α,b,p and λ in $V_k^\lambda(\alpha,b,p)$, we obtain the following important subclasses studied by various authors in earlier papers:

(1) $V_k^\lambda(o,1,p) = V_k^\lambda(p)$. is the class of p-valent functions investigated by Silvia [1].

(2) $V_k^o(o,1,1) = V_k$, is the class of bounded boundary rotation introduced by Löwner [2] and Paatero [3, 4] $V_k^o(\alpha,1,1,) = V_k(\alpha)$, $o \le \alpha < 1$, is the class of functions $f(z) \in A_1$ studied by Padmanabhan and Parvatham [5], $V_k^\lambda(o,1,1) = V_k{}^\lambda$, is the class of functions $f(z) \in A_1$ investigated by Moulis [6] and Silvia [7], $V_k^\lambda(\alpha,1,1) = V_k^\lambda(\alpha)$, $o \le \alpha < 1$, is the class of functions $f(z) \in A_1$ investigated by Moulis [8], $V_k^o(o,b,1) = V_k(b)$, is the class of functions $f(z) \in A_1$ introduced by Nasr [9] and $V_k^\lambda(\alpha,b,1)= V_k^\lambda(\alpha,b)$, is the class of functions $f(z) \in A_1$ investigated by Lakshma[10, 11].

(3) $V_2^o(o,1,p) = C(p)$, is the class of p-valent convex functions considered by Goodman [12], $V_2^o(\alpha,1,p) = C(\alpha,p)$, is the class of p-valent convex functions of order α, $o \le \alpha < p$, $V_2^\lambda(o,1,p) = C^\lambda(p)$, is the class of functions $f(z) \in A_p$ for which $zf'(z)/p$ is λ-spirallike in U, $V_2^\lambda(\alpha,1,p) = C^\lambda(\alpha,p)$, is the class of functions $f(z) \in A_p$ for which $zf'(z)/p$ is λ-spirallike of order α, $o \le \alpha < p$, $V_2^o(o,b,p) = C(b,p)$, is the class of p-valent functions satisfying

$$\mathrm{Re}\ [p + \frac{1}{b}\ (1+ \frac{zf''(z)}{f'(z)} - p)] > o, z \in U.$$

The class $C(b,p)$ introduced by Aouf [13] and finally for k=2 any function $f(z) \in V_2^\lambda(\alpha,b,p) = C^\lambda(\alpha,b,p)$ if and only if

$$\mathrm{Re}\ e^{i\lambda}\ [p + \frac{1}{b}\ (1+ \frac{zf''(z)}{f'(z)} -p)] > \alpha \cos\lambda \ , \ z \in U.$$

Also from $V_k^\lambda(\alpha,b,p)$, we can obtain the following important subclasses:

(4) $V_k^o(\alpha,1,p) = V_k(\alpha,p)$, is the class of p-valent functions $f(z) \in A_p$ satisfying

$$\lim_{r\to 1^-} \sup \int_0^{2\pi} \left| \frac{\mathrm{Re}\ \{1+ \frac{zf''(z)}{f'(z)}\} -\alpha}{p - \alpha} \right| d\theta \le k\pi,$$

i.e., $f(z) \in V_k(\alpha,p)$ if and only if

$$f'(z) = pz^{p-1}\{\exp\ -(p-\alpha) \int_0^{2\pi} \log(1-ze^{-it})\ dm(t)\}\ ,\ m(t) \in M_k\ .$$

(5) $V_k^\lambda(\alpha,1,p) = V_k^o(\alpha,\cos e^{-i\lambda},p) = V_k^\lambda(\alpha,p)$, $|\lambda| < \frac{\pi}{2}$, is the class of p-valent functions $f(z) \in A_p$ satisfying

$$\lim_{r\to 1^-} \sup \int_0^{2\pi} \left| \frac{\mathrm{Re}\ \{ e^{i\lambda}(1+ \frac{zf''(z)}{f'(z)})\} - \alpha \cos\lambda}{p - \alpha} \right| d\theta \le k\pi \cos\lambda$$

i.e., $f(z) \in V_k^\lambda(\alpha,p)$ if and only if

$$f'(z) = pz^{p-1} \exp \{ -(p-\alpha)\cos\lambda\ e^{-i\lambda} \int_0^{2\pi} \log(1-ze^{-it})\ dm(t)\},\ m(t) \in M_k.$$

(6) $V_k^o(o,b,p) = V_k(b,p)$, is the class of p-valent functions $f(z) \in A_p$ satisfying

$$\lim_{r\to 1^-} \sup \int_0^{2\pi} |\mathrm{Re}\ \{p + \frac{1}{b}\ (1+ \frac{zf''(z)}{f'(z)} - p)\}\ |\ d\theta \le k\pi p,$$

i.e., $f(z) \in V_k(b,p)$ if and only if

$$f'(z) = pz^{p-1} \exp \{-pb \int_0^{2\pi} \log(1-ze^{-it})\ dm(t)\},\ m(t) \in M_k.$$

(7) $V_k^o(\alpha,b,p) = V_k(\alpha,b,p)$, is the class of p-valent functions $f(z) \in A_p$ satisfying

$$\lim_{r\to 1^-} \sup \int_o^{2\pi} \left| \frac{\mathrm{Re}\ \{p+ \frac{1}{b}(1+ \frac{zf''(z)}{f'(z)} - p)\} - \alpha}{p - \alpha} \right| d\theta \le k\pi,$$

i.e., $f(z) \in V_k(\alpha,b,p)$ if and only if

$$f'(z) = pz^{p-1} \exp \{-(p-\alpha)b \int_0^{2\pi} \log(1-ze^{-it})\ dm(t)\}\ ,\ m(t) \in M_k.$$

We state below some lemmas that are needed in our investigation.

LEMMA 1 [10]. $h(z) \in V_k^\lambda(\alpha,b)$ if and only if

(1) $h'(z) = \exp\{-(1-\alpha)b \cos\lambda\ e^{-i\lambda} \int_0^{2\pi} \log(1-ze^{-it})\ dm(t)\},\ m(t) \in M_k.$

(2) $h'(z) = [f'_1(z)]^{(1-\alpha)b \cos\lambda e^{-i\lambda}},\ f_1 \in V_k$.

(3) $h'(z) = [f'_2(z)]^{b \cos\lambda e^{-i\lambda}},\ f_2 \in V_k(\alpha)$.

(4) $h'(z) = [f'_3(z)]^{(1-\alpha)b},\ f_3 \in V_k^\lambda$.

(5) $h'(z) = [f'_4(z)]^b\ ,\ f_4 \in V_k^\lambda(\alpha)$.

(6) $h'(z) = [f'_5(z)]^{(1-\alpha)\cos\lambda\ e^{-i\lambda}},\ f_5 \in V_k(b)$.

(7) there exists two normalized starlike functions $s_1(z)$ and $s_2(z)$ such that

$$f'(z) = \left\{ \frac{[\frac{s_1(z)}{z}]^{\frac{k+2}{4}}}{[\frac{s_2(z)}{z}]^{\frac{k-2}{4}}} \right\}^{(1-\alpha)b \cos\lambda\ e^{-i\lambda}}$$

LEMMA 2[10]. If $h(z)$ belongs to $V_k^\lambda(\alpha,b)$, then $H(z)$ is defined by

$$H'(z) = \frac{h'(\frac{z+a}{1+\bar{a}z})}{h'(a)(1+\bar{a}z)^{2(1-\alpha)b\cos\lambda e^{-i\lambda}}}$$

$H(o) = o$, $|z| < 1$, $|a| < 1$ is also in the class $V_k^\lambda(\alpha,b)$.

LEMMA 3[10]. Suppose $h(z) = z + b_2z^2 + b_3z^z + \ldots$ belongs to $V_k^\lambda(\alpha,b)$ then

$$|b_2| \leq \frac{k}{2}(1-\alpha)\ |b|\ \cos\lambda\ ,$$

and

$$|b_3| \leq \frac{1}{3}(1-\alpha)\ |b|\ \cos\lambda[1-(1-\alpha)\ |b|\cos\lambda\ \frac{k^2}{2}]\ .$$

These bounds are sharp, with equality for the function $h(z) \in V_k^\lambda(\alpha,b)$ defined by

$$h'(z) = \left[\frac{(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}}\right]^{(1-\alpha)b\cos\lambda e^{-i\lambda}}.$$

LEMMA 4[1]. $g(z) \in V_k^\lambda(p)$, $p \geq 1$, if and only if $g'(z) = pz^{p-1}[f_3'(z)]^p$ for some $f_3 \in V_k^\lambda$.

2. REPRESENTATION FORMULAS FOR THE CLASS $V_k^\lambda(\alpha,b,p)$.

LEMMA 5. $f(z)$ $V_k^\lambda(\alpha,b,p)$ if and only if

$$f'(z) = pz^{p-1}[h'(z)]^{\frac{(p-\alpha)}{1-\alpha}} = pz^{p-1} + \sum_{n=p+1}^{\infty} n\,a_nz^{n-1} \tag{2.1}$$

for some $h \in V_k^\lambda(\alpha,b)$.

PROOF. Let $f'(z) = pz^{p-1}[h'(z)]^{(\frac{p-\alpha}{1-\alpha})} = pz^{p-1} + \sum_{n=p+1}^{\infty} n\,a_nz^{n-1}$ for $h(z) = z + \sum_{n=2}^{\infty} b_nz^n \in V_k^\lambda(\alpha,b)$, $z \in U$.

By direct computation, we obtain

$$\int_0^{2\pi} \left|\frac{\mathrm{Re}\,\{e^{i\lambda}[p + \frac{1}{b}(1 + \frac{zf''(z)}{f'(z)} - p)] - \alpha\cos\lambda\}}{p-\alpha}\right| d\theta =$$

$$\int_0^{2\pi} \left|\frac{\mathrm{Re}\,\{e^{i\lambda}[1 + \frac{1}{b}\frac{zh''(z)}{h'(z)}] - \alpha\cos\lambda\}}{1-\alpha}\right| d\theta$$

and the result follows from (1.2).

An immediate consequence of Lemmas 5 and 1 is

THEOREM 1. $f(z) \in V_k^\lambda(\alpha,b,p)$ if and only if

(1) $f'(z) = pz^{p-1}\exp\{-(p-\alpha)b\cos\lambda\ e^{-i\lambda}\int_0^{2\pi}\log(1-ze^{-it})dm(t)\}$, $m(t) \in M_k$.

(2) $f'(z) = pz^{p-1}[f_1'(z)]^{(p-\alpha)b\cos\lambda\, e^{-i\lambda}}$, $f_1 \in V_k$.

(3) $f'(z) = pz^{p-1}[f_2'(z)]^{(\frac{p-\alpha}{1-\alpha})\, b\cos\lambda\, e^{-i\lambda}}$, $f_2 \in V_k(\alpha)$.

(4) $f'(z) = pz^{p-1}[f_3'(z)]^{(p-\alpha)b}$, $f_3 \in V_k^\lambda$.

(5) $f'(z) = pz^{p-1}[f_4'(z)]^{(\frac{p-\alpha}{1-\alpha})b}$, $f_4 \in V_k(\alpha)$

(6) $f'(z) = pz^{p-1}[f_5'(z)]^{(p-\alpha)\cos\lambda\, e^{-i\lambda}}$, $f_5 \in V_k(b)$.

(7) there exists two normalized starlike functions $s_1(z)$ and $s_2(z)$ such that

$$f'(z) = pz^{p-1}\left[\frac{\left\{\frac{s_1(z)}{z}\right\}^{\frac{k+2}{4}}}{\left\{\frac{s_2(z)}{z}\right\}^{\frac{k-2}{4}}}\right]^{(p-\alpha)b\cos\lambda e^{-i\lambda}}$$

Also an immediate consequence of Theorem 1 (4) and Lemma 4 is

THEOREM 2. $f(z) \in V_k^\lambda(\alpha,b,p)$ if and only if

$$f'(z) = pz^{p-1}\left[\frac{g'(z)}{pz^{p-1}}\right]^{(\frac{p-\alpha}{p})b}, \quad g \in V_k^\lambda(p)$$

3. COEFFICIENT ESTIMATES FOR THE CLASS $V_k^\lambda(\alpha,b,p)$.

THEOREM 3. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in V_k^\lambda(\alpha,b,p)$, then

$$|a_{p+1}| \le [\frac{p(p-\alpha)}{p+1}]\, k\, |b| \cos\lambda , \tag{3.1}$$

$$|a_{p+2}| \le p(\frac{p-\alpha}{p+2})|b| \cos\lambda \{[1 - (1-\alpha)|b|\cos\lambda \frac{k^2}{2}] + (p-1)\, |b| \cos\lambda \frac{k^2}{2}\} \tag{3.2}$$

These bounds are sharp with equality for

$$f_*'(z) = pz^{p-1}\left[\frac{(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}}\right]^{(p-\alpha)b\cos\lambda\, e^{-i\lambda}} \tag{3.3}$$

PROOF. By Lemma 5, there exists an $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k^\lambda(\alpha,b)$ such that

$$f'(z) = pz^{p-1} + \sum_{n=p+1}^{\infty} n\, a_n z^{n-1} = pz^{p-1}[1 + \sum_{n=2}^{\infty} n\, b_n\, z^{n-1}]^{(\frac{p-\alpha}{1-\alpha})} \tag{3.4}$$

Expanding the right side of (3.4), we obtain

$$f'(z) = pz^{p-1} + 2p(\frac{p-\alpha}{1-\alpha})b_2 z^p + p\{3(\frac{p-\alpha}{1-\alpha})b_3 + 2(\frac{p-\alpha}{1-\alpha})(\frac{p-1}{1-\alpha})b_2{}^2\}\, z^{p+1} + \ldots \tag{3.5}$$

Equating coefficents from (3.5) and (3.4), we have

$$(p+1)a_{p+1} = 2p(\frac{p-\alpha}{1-\alpha})b_2,$$

$$(p+2)a_{p+2} = p\{3(\frac{p-\alpha}{1-\alpha})\ b_3 + 2(\frac{p-\alpha}{1-\alpha})(\frac{p-1}{1-\alpha})b_2{}^2\}\ .$$

Thus the result follows from lemma 3.

4. SHARP RADIUS OF CONVEXITY OF THE CLASS $V_k^\lambda(\alpha,b,p)$.

LEMMA 6. If $f \in V_k^\lambda(\alpha,b,p)$, then the transformation F_a satisfying

$$F_a'(z) = \frac{p\ a^{p-1} z^{p-1}\ f'(\frac{z+a}{1+\bar{a}z})}{f'(a)(z+a)^{p-1}(1+\bar{a}z)^{2(p-\alpha)b\ \cos\lambda\ e^{-i\lambda} - p+1}}\ ,\quad z \in U \tag{4.1}$$

and $F_a(o) = o$, is in $V_k^\lambda(\alpha,b,p)$ for all $|a| < 1$.

The proof of this lemma follows by using lemmas 5 and 2.

LEMMA 7. For $|z| \le r$ and f ranging over $V_k^\lambda(\alpha,b,p)$, the domain of values of $\frac{zf''(z)}{f'(z)}$ is the disc with center

$$\left(\frac{[2(p-\alpha)\mathrm{Re}\{b\ \cos\lambda e^{-i\lambda}\} - p+1]\ r^2+(p-1)}{1-r^2}\ ,\ \frac{2(p-\alpha)\mathrm{Im}\{b\ \cos\lambda e^{-i\lambda}\}\ r^2}{1-r^2}\right)$$

and radius $\frac{p(p-\alpha)k|b|\ \cos\lambda\ r}{1-r^2}$.

PROOF. Whenever

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in V_k^\lambda(\alpha,b,p),\quad \lim_{z\to o}\{\frac{f''(z)-p(p-1)z^{p-2}}{z^{p-1}}\} = p(p-1)a_{p+1}\ .$$

For $f \in V_k^\lambda(\alpha,b,p)$, let $F_a(z) = z^p + \sum_{n=p+1}^{\infty} A_n z^n \in V_k^\lambda(\alpha,b,p)$ be given by Lemma 6 for $|a| < 1$. By direct calculation we have

$$p(p+1)A_{p+1} = p(1-|a|^2)\frac{f''(a)}{f'(a)} - \frac{p[2(p-\alpha)b\ \cos\lambda\ e^{-i\lambda}-p+1]|a|^2+p(p-1)}{a}\ . \tag{4.2}$$

Combining (3.1) and (4.2), we obtain

$$\left|\frac{f''(a)}{f'(a)} - \frac{[2(p-\alpha)b\ \cos\lambda\ e^{-i\lambda}-p+1]|a|^2+(p-1)}{a(1-|a|^2)}\right| \le \frac{p(p-\alpha)k|b|\cos\lambda}{(1-|a|^2)} \tag{4.3}$$

From (4.3) it follows that for $|z| = r < 1$,

$$\left|\frac{zf''(z)}{f'(z)} - \frac{[2(p-\alpha)b\ \cos\lambda\ e^{-i\lambda}-p+1]r^2+(p-1)}{(1-r^2)}\right| \le \frac{p(p-\alpha)k|b|\cos\lambda r}{(1-r^2)} \tag{4.4}$$

and the proof is completed.

THEOREM 4. The sharp radius of convexity of the class $V_k^\lambda(\alpha,b,p)$ is

$$r_o = 2\{(p-\alpha)k|b|\cos\lambda + [(p-\alpha)^2k^2|b|^2\ \cos^2\lambda - 4(2(1-\frac{\alpha}{p})\mathrm{Re}\{b\cos\lambda e^{-i\lambda}\}-1)]^{\frac{1}{2}}\}^{-1} \tag{4.5}$$

PROOF. From (4.4), we have

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\left(\frac{1+(2(1-\frac{\alpha}{p})b\cos\lambda e^{-i\lambda}-1)r^2}{1-r^2}\right)\right| \leq \frac{p(p-\alpha)k|b|\cos\lambda\ r}{1-r^2}$$

which implies that

$$\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \geq p\left[\frac{1-(p-\alpha)k|b|\cos\lambda\ r+(2(1-\frac{\alpha}{p})\ \mathrm{Re}\{b\cos\lambda\ e^{-i\lambda}\}-1)r^2}{1-r^2}\right] .$$

Therefore $\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} > o$ for $|z| < r_o$ is given by (4.5). The function $f_x(z)$ defined by

$$f'_*(z) = pz^{p-1}\left[\frac{(1-ez)^{\frac{k}{2}-1}}{(1-\varepsilon z)^{\frac{k}{2}+1}}\right]^{(p-\alpha)b\cos\lambda\ e^{-i\lambda}}, \tag{4.6}$$

when $z = r$, $e = \dfrac{r+p\ e^{i\lambda}\sqrt{\frac{\bar b}{b}}}{1+rpe^{i\lambda}\sqrt{\frac{\bar b}{b}}}$ and $\varepsilon = \dfrac{r-pe^{i\lambda}\sqrt{\frac{\bar b}{b}}}{1-rpe^{i\lambda}\sqrt{\frac{\bar b}{b}}}$

shows that the bound in (4.5) is sharp.

5. DISTORTION AND ROTATION THEOREMS FOR THE CLASS $V_k^\lambda(\alpha,b,p)$.

In [11] Lakshma used variational method to solve the extremal problems for the class $V_k^\lambda(\alpha,b)$. He proved the following:

LEMMA 8[11]. Let $\xi \neq o$ be a given point in U and let $H(x_1,x_2,\ldots x_{n+1})$ be analytic in a neighbourhood of each point $H(h'(\xi),(h''(\xi),\ldots,h^{(n)}(\xi),\xi)$, $h \in V_k^\lambda(\alpha,b)$. Then the functional

$$J(h') = \mathrm{Re}\ H(h'(\xi),h''(\xi),\ldots,h^{(n)}(\xi),\xi)$$

attains its maximum (or minimum) in $V_k^\lambda(\alpha,b)$ only for a function of the form

$$h'(z) = \prod_{j=1}^{M}(1-\varepsilon_j z)^{\beta_j(1-\alpha)b\cos\lambda e^{-i\lambda}} \prod_{j=1}^{N}(1-e_j z)^{-\alpha_j(1-\alpha)b\cos\lambda\ e^{-i\lambda}} \tag{5.1}$$

where $M \leq n$, $N \leq n$, $|\varepsilon_j| = |e_j| = 1$, $\sum_{j=1}^{M}\beta_j \leq \frac{k}{2}-1$, $\sum_{j=1}^{N}\alpha_j \leq \frac{k}{2}+1$.

LEMMA 9[11]. If $h(z) \in V_k^\lambda(\alpha,b)$, then we have

$$\min_{|e|=|\varepsilon|=1}\left|\frac{(1-\varepsilon z)^{(\frac{k}{2}-1)A}}{(1-ez)^{(\frac{k}{2}+1)A}}\right| \leq |h'(z)| \leq \max_{|e|=|\varepsilon|=1}\left|\frac{(1-\varepsilon z)^{(\frac{k}{2}-1)A}}{(1-ez)^{(\frac{k}{2}+1)A}}\right| ,$$

where $A = (1-\alpha)b\cos\lambda\ e^{-i\lambda}$.

Bounds are sharp with properly selected e and ε with $M = N = 1$ in (5.1).

LEMMA 10[11]. If $h(z) \in V_k^\lambda(\alpha,b)$, then for $|z| = r < 1$, we obtain

$$|\arg h'(z)| \leq (1-\alpha)k|b|\cos\lambda\ \mathrm{arc}\sin|z| .$$

The proof of the following two theorems follows from Lemma 5 and the above bounds.

THEOREM 5. If $f \in V_k^{\lambda}(\alpha,b,p)$, then for $|z| = r < 1$, we obtain

$$|e|\overset{\min}{=}|\varepsilon| = 1\left|\frac{(1-\varepsilon z)^{(\frac{k}{2}-1)B}}{(1-ez)^{(\frac{k}{2}+1)B}}\right| \le \left|\frac{f'(z)}{pz^{p-1}}\right| \le |e|\overset{\max}{=}|\varepsilon| = 1\left|\frac{(1-\varepsilon z)^{(\frac{k}{2}-1)B}}{(1-ez)^{(\frac{k}{2}+1)B}}\right| ,$$

where $B = (p-\alpha)b\cos\lambda\, e^{-i\lambda}$.

Bounds are sharp with equality for $f'(z) = pz^{p-1}[h'(z)]^{(\frac{p-\alpha}{1-\alpha})}$, where $h(z)$ is defined by (5.1) with properly selected e and ε with $M = N = 1$.

THEOREM 6. If $f \in V_k^{\lambda}(\alpha,b,p)$, then for $|z| = r < 1$, we obtain

$$|\arg f'(z)| \le (p-1)^{\theta} + (p-\alpha)k\,|b|\cos\lambda \text{ arc sin } |z|.$$

6. HARDY CLASSES FOR THE CLASS $V_k^{\lambda}(\alpha,b)$.

In order to obtain the H^{μ} classes for the class $V_k^{\lambda}(\alpha,b)$, we will use the following lemmas.

LEMMA 11[7]. IF $f_3 \in V_k^{\lambda}$. Then for $|z|= r$, $|\arg f_3'(re^{i\theta})| \le k\cos\lambda$ arc sin r.

LEMMA 12[14]. If $f' \in H^{\mu}(o < \mu < 1)$ then $f \in H^{\frac{\mu}{1-\mu}}$ where, for $\mu=1$, H^{∞} is the class of bounded functions.

LEMMA 13[15]. If $f(z) \in H^{\mu}(o < \mu < 1)$ and $f(z) = \sum_{n=o}^{\infty} a_n z^n$, then $a_n = o(n^{\frac{1}{\mu}-1})$.

LEMMA 14[1]. If $f_3 \in V_k^{\lambda}$, then $f_3' \in H^{\mu}$ for all $\mu < \dfrac{2}{\cos^2\lambda(k+2)}$ and $f_3 \in H^{\eta}$ for $\eta < \dfrac{2}{[\cos^2\lambda\,(k+2)-2]}$, $\cos^2\lambda > \dfrac{2}{(k+2)}$. Furthermore, if f_3' is not of the form $f_3'(z) = [f_o'(z)]^{\cos\lambda\, e^{-i\lambda}}$ where f_o is given by

$$f_o'(z) = (1-ze^{-it_o})^{-(\frac{k}{2}+1)} \exp\{\int_o^{2\pi} -\log(1-ze^{-it})dm(t)\} \qquad (6.1)$$

($m(t)$ a probability measure on $[o,2\pi]$), then there exists $\delta = \delta(f_3) > o$ and $\varepsilon = \varepsilon(f_3) > o$ such that

$$f_3'(z) \in H^{\frac{(2+\delta)}{\cos^2\lambda\,(k+2)}} \quad\text{and}\quad f_3 \in H^{\frac{(2+\varepsilon)}{[\cos^2\lambda\,(k+2)-2]}} \quad\text{for}\quad \cos^2\lambda > \frac{2}{(k+2)} .$$

THEOREM 7. If $h \in V_k^{\lambda}(\alpha,b)$, $Re\{b\} > o$, then $h' \in H^{\mu}$ for all

$$\mu < \frac{2}{(1-\alpha)Re\{b\}\cos^2\lambda(k+2)} \quad\text{and}\quad h \in H^{\eta} \text{ for } \eta < \frac{2}{[(1-\alpha)Re\{b\}\cos^2\lambda(k+2)-2]} ,$$

where $\cos^2\lambda > \dfrac{2}{(1-\alpha)\mathrm{Re}\{b\}(k+2)}$. Furthermore, if h' is not of the form $h'(z) = [f_3'(z)]^{(1-\alpha)b} = [f_o'(z)]^{(1-\alpha)b\cos\lambda e^{-i\lambda}}$ where $f_o(z)$ is given by (6.1), then there exists $\varepsilon = \varepsilon(h) > o$ and $\delta = \delta(h) > o$ such that

$$h' \in H^{\varepsilon + \frac{2}{[(1-\alpha)\mathrm{Re}\{b\}\cos^2\lambda(k+2)]}} \quad \text{and} \quad h \in H^{\delta + \frac{2}{[(1-\alpha)\mathrm{Re}\{b\}\cos^2\lambda(k+2)-2]}}$$

for $\cos^2\lambda > \dfrac{2}{(1-\alpha)\mathrm{Re}\{b\}(k+2)}$.

PROOF. If $h \in V_k^\lambda(\alpha,b)$, then it follows from Lemma 1(4) that

$$h'(z) = [f_3'(z)]^{(1-\alpha)b}, \quad f_3 \in V_k^\lambda .$$

Then

$$|h'(z)|^\mu = |f_3'(z)|^{\mu(1-\alpha)\mathrm{Re}\{b\}} \cdot \exp\{-\mu(1-\alpha)\mathrm{Im}\{b\}\arg f_3'(z)\}.$$

By Lemma 11, the exponential factor is bounded. Thus the result follows from Lemmas 12 and 14. From Theorem 7 and Lemma 13, we obtain a growth estimate for the Taylor's coefficients of $h \in V_k^\lambda(\alpha,b)$.

COROLLARY 1. If $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k^\lambda(\alpha,b)$ and $\cos^2\lambda > \dfrac{2}{(1-\alpha)\mathrm{Re}\{b\}(k+2)}$, then

$$b_n = o\left(n^{\frac{[(1-\alpha)\mathrm{Re}\{b\}\cos^2\lambda(k+2)-4]}{2}}\right).$$

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REFERENCES

1. SILVIA, E.M., A note of special classes of p-valent functions, Rocky Mt. J. Math. 9(2)(1979), 365-370.

2. LOWNER, K., Untersuchungen uber die Verzerrung bei konformen Abbildungen Einheitskreises |z| < 1, die durch Funktionen mit nicht verschwindender Ableitung geliefert werden, Ber. Köngil. Sachs. Ges. Wiss. Leipzig 69 (1917), 89-106.

3. PAATERO, V., Über die konforme Abbildung von Gebieten deren Rander von beschrankter Drehung sind, Ann. Acad. Sci. Fenn. Ser. A. (33) 9 (1931), 1-77.

4. PAATERO, V., Üner Gebiete von beschränkter Raddrehung, Ann. Acad. Sci. Fenn. Ser. A. 37 (1933), 1-20.

5. PADMANBHAN, K.S. and PARVATHAM, R., Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math. 31 (3) (1975), 311-323.

6. MOULIS, E.J. Jr., A generalization of univalent functions with bounded boundary rotation, Trans. Amer. Math. Soc., 174(1972), 369-381.

7. SILVIA, E.M., A variational method on certain classes of functions, Rev. Roum. Math.Pures et Appl., 21(5) (1979), 549-557.

8. MOULIS, E.J. Jr., Generalizations of the Robertson functions, Pacific J. Math. 81(1) (1979), 167-174.

9. NASR, M.A., On a class of functions with bounded boundary rotation, Bull. of the Institute of Math., Academia Sinica 5(1977), 27-36.

10. LAKSHMA, R., On certain classes of functions with bounded boundary rotation, Indian J. Pure and Appl. Math., 13(2) (1982), 195-204.

11. LAKSHMA, R., A variational method on certain classes of analytic functions, Indian J. Pure and Appl. Math., 14(5) (1983), 601-613.

12. GOODMAN, A.W., On the SchwarzpChristofell transformation and p-valent functions, Trans. Amer. Math. Soc., 68(1950), 204-223.

13. AOUF, M.K., p-Valent classes related to convex functions of complex order, to appear, Rocky Mt. J. Math.

14. HARDY, G.H., and LITTLEWOOD, J.E., Some properties of fractional integrals II, Math. Z., 34(1932), 403-439.

15. DUREN, P.L., Theory of H^p spaces, Pure and Appl. Math. 38, Academic Press, New York, 1970.