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NOTE ON THE ZEROS OF FUNCTIONS WITH UNIVALENT DERIVATIVES

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ABSTRACT. Let E denote the class of functions $f(z)$ analytic in the unit disc D, normalized so that $f(0) = 0 = f'(0) - 1$, such that each $f^{(k)}(z)$, $k \geq 0$ is univalent in D. In this paper we establish conditions for some functions to belong to class E.

KEY WORDS AND PHRASES. Univalent functions, close-to-convex functions, entire functions.
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1. INTRODUCTION.

Let E denote the class of functions analytic in the unit disc D, normalized so that $f(0) = 0 = f'(0) - 1$, such that $f^{(k)}(z)$, $k \geq 0$ is univalent in D. For a survey of E see [1]. In [2] Shah and Trimble proved the following result:

THEOREM A. Let

$$f(z) = ze^{\beta z}(1 - z/z_1). \tag{1.1}$$

Suppose

$$0 < \beta \leq 1/2, \; 0 < z_1 \leq 2 \tag{1.2}$$

and

$$\frac{2+\beta}{1+\beta} \leq z_1 \leq \frac{2-4\beta+\beta^2}{\beta(2-\beta)} . \tag{1.3}$$

Then $f(z)$ and all of its derivatives are close-to-convex in D. In particular $f \in E$. For $\beta = 0.29$,

$$1.7751 \leq z_1 \leq 1.8634.$$

2. MAIN THEOREMS.

In this paper we prove the following:

THEOREM 1. Let $f(z)$ be defined by (1.1), suppose that (1.2) holds and $\beta z_1 < 1$. Then;

1 - $f'(z)$ is univalent in $|z| < \rho$ $(0 < \rho \leq 1)$ if and only if

$$z_1 \leq \frac{2+\beta^2\rho^2-4\rho\beta}{\beta(2-\beta\rho)} . \tag{2.1}$$

2 - Let F be the class of functions which are derivatives of univalent functions of the from (1.1). For a fixed β, the radius of univalence of F, ρ_F, is equal to

$$\frac{2}{\beta} - \frac{\phi(\beta) + \sqrt{\phi(\beta)^2+8}}{2\beta}$$

where

$$\phi(\beta) = \frac{\beta(2+\beta)}{1+\beta}$$

THEOREM 2. Let $f(z)$ be defined by (1.1) and suppose that (1.2) holds. If

$$\frac{2+\beta}{1+\beta} \le z_1 \le \frac{-6+\sqrt{8(6-\beta^2)}}{\beta} \tag{2.2}$$

then $f(z)$, $f''(z)$, $f'''(z)$,... are close-to-convex and consequently univalent in D. In particular if $\beta = 0.4766$, $1.6781 \le z_1 \le 1.6791$. In addition, if $\frac{2+\beta^2-4\beta}{\beta(2-\beta)} < \frac{2+\beta}{1+\beta}$ then $f'(z)$ is not univalent in D.

THEOREM 3. Let

$$f(z) = ze^{\beta z}(1 - z^2/z_1^2) . \tag{2.3}$$

Suppose $0 < \beta \le 0.4$ and

$$\frac{6+\beta^2+6\beta}{\beta^2+2\beta} \le z_1^2 \le \frac{2-6\beta+3\beta^2}{\beta^2} \tag{2.4}$$

Then $f(z)$ and all of its derivatives are close-to-convex and consequently univalent in D . In particular for $\beta = 0.2314$, $3.79664 \le z_1 \le 3.7978$

3. PROOFS.

PROOF OF THEOREM 1. Proof of sufficiency. The function $g(z) = \frac{e^{\beta z}-1}{\beta}$, β as in (1.2), is convex in D. If we can show that $\mathrm{Re}\{\frac{f''(z)}{g'(z)}\} \le 0$ for $|z| \le \rho$ then $f'(z)$ will be close-to-convex in $|z| \le \rho$ and consequently univalent there (see [3]).

If $\phi_\rho(x)$ denotes the real part of $\frac{f''(z)}{g'(z)}$ on $|z| = \rho$, where $x = \mathrm{Re}z$, then

$$\phi_\rho(x) = \{2(\beta - \frac{1}{z_1}) + \frac{\beta^2}{z_1}\rho^2\} + \beta(\beta - \frac{4}{z_1})\, x - \frac{2\beta^2}{z_1}x^2 .$$

By the maximum principle it suffices to prove that $\phi_\rho(x) \le 0$ for x in $[-1,1]$. For simplicity we write $\phi_\rho(x) = ax^2+bx+c$. Observe that $b^2-4ac > 0$. Thus $\phi_\rho(x)$ has two real roots, and we will be done if we can show that

$$-\rho \ge \frac{-b-\sqrt{b^2-4ac}}{2a} \tag{3.1}$$

(The larger root of $\phi_\rho(x)$ is $\frac{-b-\sqrt{b^2-4ac}}{2a}$. See figure 1).

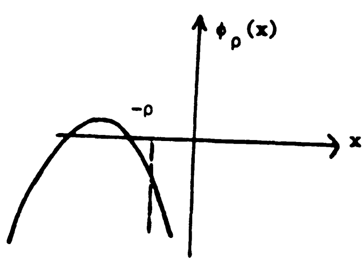

Figure 1.

Since $a<0$, (3.1) is equivalent to

$$\sqrt{b^2-4ac} \le 2a\rho-b. \tag{3.2}$$

From the definition of a and b we have

$$2a\rho-b = \beta[\frac{4}{z_1} - \beta(1+\frac{4\rho}{z_1})] = \beta\,\frac{4-\beta z_1-4\rho\beta}{z_1} \ge \frac{4-1-2}{z_1} = \frac{1}{z_1} > 0$$

since $\rho \le 1$ and (1.2) holds.

Squaring both sides of (3.2) and simplifying we get

$$4a\rho(a\rho-b) \ge -4ac.$$

Divide by 4a which is negative to get

$$\rho(b-a\rho) \ge c.$$

Using the definitions of a, b and c, this becomes

$$z_1\beta(\beta\rho-2) \ge 4\beta\rho - \beta^2\rho^2-2.$$

From this, noting that $\beta\rho-2 < 0$, we conclude that (3.2) is equivalent to

$$z_1 \le \frac{2+\beta^2\rho^2 - 4\beta\rho}{\beta(2-\beta\rho)},$$

which is (2.1).

Proof of necessity. We show that if

$$z_1 > \frac{2+\beta^2\rho^2 -4\beta\rho}{\beta(2-\rho\beta)} \tag{3.3}$$

then $f''(z)$ has a root in $|z| < \rho$, which means that $f'(z)$ is not univalent there. The equation, $f''(z) = 0$, that is

$$\frac{-\beta^2}{z_1} z^2 + (\frac{-4\beta}{z_1} + \beta^2)z + 2\beta - \frac{2}{z_1} = 0$$

has two negative roots given by

$$\frac{\beta-4/z_1 \pm \sqrt{\beta^2+8/z_1^2}}{\frac{2\beta}{z_1}}$$

The smaller root lies in the disc $|z| < \rho$ if

$$\left|\frac{\beta-4/z_1 + \sqrt{\beta^2+8/z_1^2}}{2\beta/z_1}\right| < \rho. \tag{3.4}$$

Since the roots are negative (3.4) is equivalent to

$$-(\beta - \frac{4}{z_1}) - \sqrt{\beta^2 + 8/z_1^2} < \frac{2\beta\rho}{z_1}$$

or

$$- \beta + \frac{4}{z_1} - \frac{2\beta\rho}{z_1} < \sqrt{\beta^2 + 8/z_1^2}\,. \tag{3.5}$$

But

$$-\beta + \frac{4}{z_1} - \frac{2\beta\rho}{z_1} = \frac{4-\beta z_1 - 2\beta\rho}{z_1} \ge \frac{2}{z_1} > 0$$

by (1.2) and $\rho\le 1$. Squaring both sides of (3.5) and simplifying we get (3.3).

This proves the first part of the theorem. To prove the second part note that by definition, ρ_F is the largest number such that $g(\rho_F z)$ is univalent for all $g\epsilon F$ in D. Let $g\epsilon F$. Then $g = f'$ for some f of the form (1.1). In [2] it is shown that f is univalent in D, given (1.2), if and only if $z_1 \ge \frac{2+\beta}{1+\beta}$. ρ_g, the radius of univalence of g, is non zero because $f''(0) \neq 0$. Therefore, by the first part of the theorem, the condition

$$\frac{2+\beta}{1+\beta} \leq z_1 \leq \frac{2+\beta^2\rho_g - 4\rho_g\beta}{\beta(2-\beta\rho_g)} \tag{3.6}$$

is the necessary and sufficient condition for $f(z)$ and $g(\rho_g z)$ to be univalent in D. Let $x = 2-\rho_g\beta$. It follows form (3.6) that

$$x^2 - \phi(\beta)\,x - 2 \geq 0$$

which is true if and only if

$$x \geq \frac{\phi(\beta) + \sqrt{\phi(\beta)^2 + 8}}{2}$$

or if

$$\rho_g \leq \frac{2}{\beta} - \frac{\phi(\beta) + \sqrt{\phi(\beta)^2+8}}{2\beta} \tag{3.7}$$

The case of equality in (3.7) corresponds to the case where both inequalities in (3.6) are equalities. That is the radius of univalence of the g for which $z_1 = \frac{2+\beta}{1+\beta}$ is precisely the expression on the right of (3.7). This proves the second part of the theorem.

Note that $\rho_F = 1$ corresponds to $\beta \doteq .29$. Also if $\beta = 0.4746$, $\rho_F \doteq 0.2793$.

PROOF OF THEOREM 2. We will show that $\mathrm{Re}\{\frac{f'(z)}{e^{\beta z}}\} \geq 0$ and $\mathrm{Re}\{\frac{f^{(n)}(z)}{e^{\beta z}}\} \leq 0$ for $n \geq 3$ and z in D. This will show that $f(z)$ and $f^{(n)}(z)$, $n \geq 2$ are close-to-convex in D. In [2] it was shown that, if (1.2) holds and $\frac{2+\beta}{1+\beta} \leq z_1 \leq 2$, then $\mathrm{Re}\{\frac{f'(z)}{e^{\beta z}}\} \geq 0$ in D. Thus we need only show that $\mathrm{Re}\{\frac{f^{(n)}(z)}{e^{\beta z}}\} \leq 0$ for $n \geq 3$ in D.

If we denote the real part of $\frac{f^{(n)}(z)}{e^{\beta z}}$ on the unit circle for $n \geq 3$ by $\phi_n(x)$ where $x = \mathrm{Re}\,z$, it will be sufficient to show that $\phi_n(x) \leq 0$ for x in $[-1,1]$. Henceforth we assume that $n \geq 3$ and note that

$$\phi_n(x) = n\beta^{n-2}(\beta - \frac{n-1}{z_1}) + \frac{\beta^n}{z_1} + \beta^{n-1}(\beta - \frac{2n}{z_1})\,x - \frac{2\beta^n}{z_1}\,x^2.$$

The quadratic $\phi_n(x)$ will be nonpositive for all $x \in [-1,1]$ if its discriminant is nonpositive. (We may note that the case when $\phi_n(x)$ has two real roots is not of interest). Thus we have

$$\beta^2 z_1^{\,2} + 8\beta^2 \leq 4n[n-(2+\beta z_1)], \qquad n \geq 3.$$

This inequality will be satisfied if it is satisfied for n = 3, that is, if

$$\beta^2 z_1^{\,2} + 12\beta z_1 + 8\beta^2 - 12 \leq 0.$$

This holds when $z_1 \leq \frac{-6 + \sqrt{8(6-\beta^2)}}{\beta}$, which is true by (2.2).

Letting $\beta = 0.4746$, calculations show that (2.2) implies $1.6781 \leq z_1 \leq 1.6791$.

Finally if $\frac{2+\beta^2-4\beta}{\beta(2-\beta)} < \frac{2+\beta}{1+\beta}$, then $z_1 > \frac{2+\beta^2-4\beta}{\beta(2-\beta)}$ by (2.2). Thus if $\beta z_1 < 1$, then by the first part of Theorem 1 $f'(z)$ is not univalent in D. But if $\beta z_1 = 1$, then $f''(0) = 0$ and $f'(z)$ is not univalent in D.

PROOF OF THEOREM 3. Note that $\frac{3-\sqrt{3}}{3} \doteq 0.4226$ is the smaller zero of $2-6\beta + 3\beta^2$. Thus $\beta \leq 0.4$ guarantees that the rightmost expression in (2.4) is positive. Let $a = \frac{1}{z_1^{\,2}}$ and

$\phi_n(x) = \mathrm{Re}\{\frac{f^{(n)}(z)}{e^{\beta z}}\}$ on the unit circle where $x = \mathrm{Re}z$. We will prove the theorem by showing that $\phi_1(x) \geq 0$, $\phi_2(x) \geq 0$ and $\phi_n(x) \leq 0$ for $n \geq 3$ and x in $[-1,1]$.
First observe that

$$\phi_1(x) = -4a\beta x^3 - 6ax^2 + (3a\beta+\beta)\, x + 1 + 3a$$

and

$$\phi_1'(x) = -12a\beta x^2 - 12ax + 3a\beta + \beta.$$

We will have $\phi_1(-1) \geq 0$ and $\phi_1(1) \geq 0$ if $\frac{1-\beta}{3-\beta} \geq a$ and $\frac{\beta+1}{3+\beta} \geq a$, respectively.
But both inequalities are true; this follows from (2.4) and the fact that, for $\beta \leq 0.4$, $\frac{\beta+1}{\beta+3} > \frac{1-\beta}{3-\beta} > \frac{2\beta+\beta^2}{6+\beta^2+6\beta}$. $\phi_1'(x)$ has one positive and one negative root. Also, since

$$\phi_1'(-x) = -9a\beta + 12a + \beta = a(12-9\beta) + \beta > 0,$$

the negative root of $\phi_1'(x)$ lies to the left of -1. (See Figure 2).

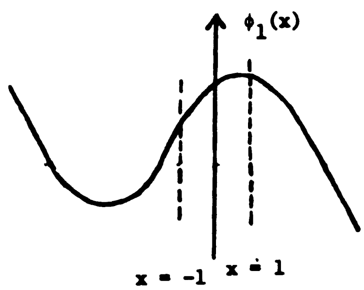

Figure 2.

Thus $\phi_1(x) \geq 0$ for x in $[-1,1]$. Next note that

$$\phi_2(x) = -4a\beta^2 x^3 - 12a\beta x^2 + (3a\beta^2 - 6a+\beta^2)x + 2\beta + 6a\beta.$$

Because of (2.4) and the fact that $\frac{\beta^2}{2-6\beta+3\beta^2} > \frac{\beta^2}{3(2-\beta^2)}$ for $\beta \leq .4$, the coefficient of x in $\phi_2(x)$ is negative. If follows from (2.4) that if $x\varepsilon[0,1]$ we have,

$$\phi_2(x) \geq -4a\beta^2 - 12a\beta + 3a\beta^2 - 6a+\beta^2 + 2\beta + 6a\beta = \beta^2 + 2\beta - a(6+\beta^2+6\beta) > 0.$$

Similarly for x in $[-1,0)$

$$\phi_2(x) > -12a\beta + 2\beta + 6a\beta = 2\beta(1-3a).$$

But $1-3a > 0$; this follows from $\frac{1}{3} > \frac{2\beta+\beta^2}{6+\beta^2+6\beta}$ and (2.4). Consequently $\phi_2(x) \geq 0$ for x in $[-1,1]$.

From now on we assume that $n \geq 3$. Note that

$$\beta^{3-n}\phi_n(x) = -4a\beta^3x^3 - 6an\beta^2x^2 + (3a\beta^3-3a\beta n(n-1)+\beta^3)x + n\beta^2 - an(n-1)(n-2) + 3an\beta^2,$$

and

$$\beta^{2-n}\phi_n'(x) = -12a\beta^2x^2 - 12an\beta x + 3a\beta^2 - 3an(n-1)+\beta^2$$

Since $3a\beta^2 - 3an(n-1) + \beta^2 < 0$, $\phi_n'(x)$ has two negative roots. Let t denote the larger of the roots. If we can show that $\phi_n(-1) \le 0$ and $-1 \ge t$, then the graph of ϕ_n will be as in Figure 3, and accordingly $\phi_n(x) \le -1$ for x in $[-1,1]$.

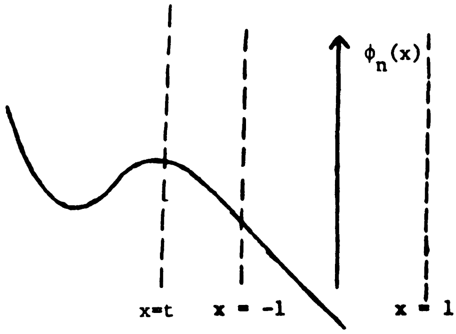

Figure 3.

But

$$\beta^{3-n}\phi_n(-1) = \beta^3(a-1) + n\,[-3a\beta^2 + \beta^2 + 3a\beta(n-1)-a(n-1)(n-2)].$$ The expression inside the bracket above will be negative for $n > 3$ if it is non positive for $n = 3$, that is, if

$$a(2-6\beta + 3\beta^2) \ge \beta^2. \tag{3.8}$$

But (3.8) is a consequence of (2.4), if we note that $2-6\beta + 3\beta^2 > 0$ for $\beta \le 0.4$. Moreover $a < 1$. Thus $\phi_n(-1) \le 0$.

Now the inequality $-1 \ge t$ is equivalent to

$$-1 \ge \frac{-6an\beta + \sqrt{36a^2n\beta^2 + 36a^2\beta^4 + 12a\beta^4}}{12a\beta^2}$$

which is equivalent to

$$6an\beta - 12a\beta^2 \ge \sqrt{36a^2n\beta^2 + 36a^2\beta^4 + 12a\beta^4}\ . \tag{3.9}$$

Note that the left hand side of (3.9) is positive. Squaring both sides of (3.9) and simplifying, we see that (3.9) is equivalent to

$$3an(n-4\beta-1) \ge \beta^2(1-9a).$$

This inequality will hold for $n \ge 3$ if it holds fo $n = 3$, that is, if

$$a \ge \frac{\beta^2}{9(\beta^2-4\beta+2)}\ . \tag{3.10}$$

But from (2.4) and the fact that $\beta \le 0.4$, we have that $\frac{\beta^2}{2-6\beta+3\beta^2} > \frac{\beta^2}{9(\beta^2-4\beta+2)}$ and (3.10) follows from this.

Finally, letting $\beta = .2314$, calculations show that (2.4) implies that $3.7964 \le z_1 \le 3.9798$.

4. REMARKS.

(i) It follows from the proof of the first part of Theorem 1 that if (1.2) holds and $\beta z_1 < 1$ then the inequality $z_1 \le \frac{2-4\beta+\beta^2}{\beta(2-\beta)}$ is the necessary and sufficient condition for $f'(z)$ to be close-to-convex in D. This along with the fact that, given (1.2), $f(z)$ is close-to-convex if and only if $z_1 \ge \frac{2+\beta}{1+\beta}$ implies that if (1.2) holds and $\beta z_1 < 1$, (1.3) is the necessary and sufficient condition for $f(z)$ and $f'(z)$ to be close-to-convex in D.

(ii) If in Theorem 3 we have $z_1^2 = \frac{6+\beta^2+6\beta}{\beta^2+2\beta}$ then $f''(1) = 0$ in which case $f'(z)$ is not univalent in a disc larger than D.

(iii) In [4] I have showed that if

$$f(z) = z\, e^{\beta z}(1-z/z_1)(1-z/z_2)$$

and if

$$0 < \beta < 1/3, \quad \beta \le b \le 1,$$

$$\frac{2b - 2\beta + 4\beta b - \beta^2 + b\beta^2}{\beta^2 + 6\beta + 6} \ge a,$$

$$a \ge \frac{b\beta}{1-3\beta},$$

$$b - 2\beta - 3a\beta + b\beta - 3a + 1 \ge 0,$$

where $a = \frac{1}{z_1 z_2}$ and $b = \frac{1}{z_1} + \frac{1}{z_2}$, then $f(z)$ and all of its derivatives are close-to-convex in D. If $z_2 > z_1$, and $\beta = 0.01$ then calculations show that $z_1 = 2.05$ and $z_2 = 94.9298$ satisfy the above inequalities. If $z_1 = z_2$ and $\beta = 0.08$ then $z_1 = 4.3478$ will satisfy the above inequalities.

(iv) Let $f(z) = z\, e^{\beta z}(1-z/z_1)$, where $z_1 = x_1 + iy_1$, $x_1 \ge 3/2$ and $0 < \beta \le 0.29$. We can show that $f(z)$ and all of its derivative are close-to-convex in D if

$$(\beta+2)\, x_1 + 2\,|y_1|\,(1+\beta) \le (1+\beta)|z_1|^2$$

and

$$\beta[x_1(4-\beta) + (2-\beta)\,|z_1|^2 + 2(2-\beta)|y_1|] \le 2x_1.$$

When $y_1 \ge 0$ the region in which z_1 lies is the shaded region in Figure 4. (The case $y_1 \le 0$ is completely symmetric).

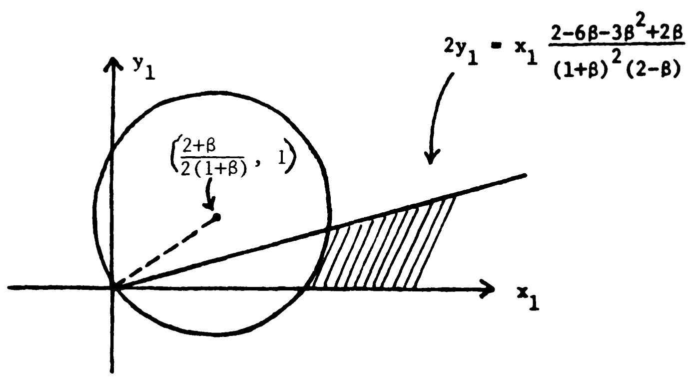

Figure 4.

As we see from the picture, the smallest value of $|z_1|$ is obtained when $y_1 = 0$ in which case the above inequalities reduce to (1.3).

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