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CERTAIN SUBCLASSES OF BAZILEVIČ FUNCTIONS OF TYPE α

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ABSTRACT. Certain subclasses $B(\alpha,\beta)$ and $B_1(\alpha,\beta)$ of Bazilevič functions of type α are introduced. The object of the present paper is to derive a lot of interesting properties of the classes $B(\alpha,\beta)$ and $B_1(\alpha,\beta)$.

KEY WORDS AND PHRASES. Bazilevič function, starlike function of order β, convex function of order β, subordination.

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1. INTRODUCTION.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. Let S be the subclass of A consisting of univalent functions in the unit disk U. A function f(z) belonging to the class A is said to be starlike of order β if and only if

$$\mathrm{Re}\{zf'(z)/f(z)\} > \beta \tag{1.2}$$

for some β $(0 \leqq \beta < 1)$, and for all $z \in U$. We denote by $S^*(\beta)$ the class of all functions in A which are starlike of order β. Throughout this paper, it should be understood that functions such as $zf'(z)/f(z)$, which have removable singularities at $z = 0$, have had these singularities removed in statements like (1.2). A function $f(z)$ belonging to the class A is said to be convex of order β if and only if

$$\mathrm{Re}\{1 + zf''(z)/f'(z)\} > \beta \tag{1.3}$$

for some β $(0 \leqq \beta < 1)$, and for all $z \in U$. Also we denote by $K(\beta)$ the class of all functions in A which are convex of order β.

We note that $f(z) \in K(\beta)$ if and only if $zf'(z) \in S^*(\beta)$. We also have $S^*(\beta) \subseteq S^*(0) \equiv S^*$, $K(\beta) \subseteq K(0) \equiv K$, and $K(\beta) \subset S^*(\beta)$ for $0 \leqq \beta < 1$.

The classes $S^*(\beta)$ and $K(\beta)$ were first introduced by Robertson [1], and were studied subsequently by Schild [2], MacGregor [3], Pinchuk [4], Jack [5], and others.

A function $f(z)$ of A is said to be in the class $B(\alpha,\beta)$ if and only if

$$\mathrm{Re}\{zf'(z)f(z)^{\alpha-1}/g(z)^{\alpha}\} > \beta \qquad (z \in U) \tag{1.4}$$

for some α $(\alpha > 0)$ and for some β $(0 \leqq \beta < 1)$, where $g(z) \in S^*$. Furthermore, we denote by $B_1(\alpha,\beta)$ the subclass of $B(\alpha,\beta)$ for which $g(z) \equiv z$.

Note that $B(0,0) = B_1(0,0) = S^*$, $B(0,\beta) = B_1(0,\beta) = B^*(\beta)$, and that $B_1(1,\beta)$ is the subclass of A consisting of functions for which $\mathrm{Re}\{f'(z)\} > \beta$ for $z \in U$.

The class $B(\alpha,0)$ when $\beta = 0$ was studied by Singh [6] and Obradović ([7], [8]). Since $B(\alpha,\beta) \subseteq B(\alpha,0)$ for $0 \leqq \beta < 1$, the class $B(\alpha,\beta)$ is the subclass of Bazilevic functions of type α (cf. [6]).

Let $f(z)$ and $g(z)$ be analytic in the unit disk U. Then a function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in the unit disk U satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. We denote by $f(z) \prec g(z)$ this relation. In particular, if $g(z)$ is univalent in the unit disk U the subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

The consept of subordination can be traced back to Lindelöf [9], but Littlewood [10] and Rogosinski [11] introduced the term and discovered the basic relations.

2. SOME PROPERTIES OF THE CLASS $B(\alpha,\beta)$.

We begin to state the following lemma due to Miller and Mocanu [12].

LEMMA 1. Let $M(z)$ and $N(z)$ be regular in the unit disk U with $M(0) = N(0) = 0$, and let β be real. If $N(z)$ maps U onto a (possibly many-sheeted) region which is starlike with respect to the origin then

$$\mathrm{Re}\{M'(z)/N'(z)\} > \beta \quad (z \in U) \implies \mathrm{Re}\{M(z)/N(z)\} > \beta \quad (z \in U), \tag{2.1}$$

and

$$\mathrm{Re}\{M'(z)/N'(z)\} < \beta \quad (z \in U) \implies \mathrm{Re}\{M(z)/N(z)\} < \beta \quad (z \in U). \tag{2.2}$$

Applying Lemma 1, we prove

LEMMA 2. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\beta)$, and let α and c be positive integers. Then the function $F(z)$ defined by

$$F(z)^{\alpha} = \frac{\alpha + c}{z^c} \int_0^z t^{c-1} f(t)^{\alpha} dt \qquad (z \in U)$$

is also in the class $S^*(\beta)$.

PROOF. Setting

$$\frac{\alpha z F'(z)}{F(z)} = \frac{z^c f(z)^{\alpha} - c\int_0^z t^{c-1} f(t)^{\alpha} dt}{\int_0^z t^{c-1} f(t)^{\alpha} dt} = \frac{M(z)}{N(z)}, \tag{2.4}$$

we have $M(0) = N(0) = 0$ and

$$\mathrm{Re}\{M'(z)/N'(z)\} = \alpha \mathrm{Re}\{zf'(z)/f(z)\} > \alpha\beta. \tag{2.5}$$

As N(z) is $(\alpha+1)$-valently starlike in the unit disk U, Lemma 1 shows that

$$\mathrm{Re}\{M(z)/N(z)\} = \alpha \mathrm{Re}\{zF'(z)/F(z)\} > \alpha\beta \tag{2.6}$$

which implies $F(z) \in S^*(\beta)$.

Now, we state and prove

THEOREM 1. Let the function f(z) defined by (1.1) be in the class $B(\alpha,\beta)$ for $g(z) \in S^*(\beta)$, where α is a positive integer and $0 \leq \beta < 1$. Then the function F(z) defined by (2.3) is also in the class $B(\alpha,\beta)$.

PROOF. It follows from (2.3) that

$$\frac{\alpha zF'(z)}{F(z)^{1-\alpha}} = \frac{\alpha + c}{z^c}\left\{z^c f(z)^\alpha - c\int_0^z t^{c-1} f(z)^\alpha dt\right\}. \tag{2.7}$$

Note that there exists a function g(z) belonging to the class $S^*(\beta)$ such that

$$\mathrm{Re}\{zf'(z)f(z)^{\alpha-1}/g(z)^\alpha\} > \beta. \tag{2.8}$$

Define the function G(z) by

$$G(z)^\alpha = \frac{\alpha + c}{z^c}\int_0^z t^{c-1} g(t)^\alpha dt. \tag{2.9}$$

Then, by using Lemma 2, we have $G(z) \in S^*(\beta)$. Combining (2.7) and (2.9), we observe that

$$\frac{\alpha zF'(z)}{F(z)^{1-\alpha}G(z)^\alpha} = \frac{z^c f(z)^\alpha - c\int_0^z t^{c-1} f(t)^\alpha dt}{\int_0^z t^{c-1} g(t)^\alpha dt}. \tag{2.10}$$

Setting

$$\frac{\alpha zF'(z)}{F(z)^{1-\alpha}G(z)^\alpha} = \frac{M(z)}{N(z)}, \tag{2.11}$$

(2.10) gives

$$\mathrm{Re}\{M'(z)/N'(z)\} = \alpha \mathrm{Re}\{zf'(z)f(z)^{\alpha-1}/g(z)^{\alpha}\} > \alpha\beta. \tag{2.12}$$

Consequently, with the help of Lemma 1, we conclude that

$$\mathrm{Re}\{zF'(z)F(z)^{\alpha-1}/G(z)^{\alpha}\} > \beta , \tag{2.13}$$

that is, that $F(z) \in B(\alpha,\beta)$. Thus we have Theorem 1.

COROLLARY 1. Let the function $f(z)$ defined by (1.1) be in the class $B(\alpha,0)$, where α is a positive integer. Then the function $F(z)$ defined by (2.3) is also in the class $B(\alpha,0)$.

THEOREM 2. The set of all points $\log\{z^{1-\alpha}f'(z)/f(z)^{1-\alpha}\}$, for a fixed $z \in U$ and $f(z)$ ranging over the class $B(\alpha,\beta)$, is convex.

PROOF. We employ the same manner due to Singh [6]. For the function $f(z)$ belonging to the class $B(\alpha,\beta)$, we define the function

$$h(z) = zf'(z)/f(z)^{1-\alpha}g(z)^{\alpha} , \tag{2.14}$$

where $g(z) \in S^*$. Then, it is clear that $\mathrm{Re}\{h(z)\} > \beta$ for $z \in U$. We denote by $P(\beta)$ the subclass of analytic functions $h(z)$ satisfying $\mathrm{Re}\{h(z)\} > \beta$ for $0 \leqq \beta < 1$ and $z \in U$. We note from (2.14) that

$$\log\{z^{1-\alpha}f'(z)/f(z)^{1-\alpha}\} = \log h(z) + \alpha \log\{g(z)/z\} . \tag{2.15}$$

Since, for a fixed $z \in U$, the range of $\log h(z)$, as $h(z)$ ranges over the class $P(\beta)$, is a convex set, and the range of $\log\{g(z)/z\}$, as $g(z)$ ranges over the class S^*, is a convex set, we complete the proof of Theorem 2.

Taking $\alpha = 0$ in Theorem 2, we have

COROLLARY 2. The set of all points $\log\{zf'(z)/f(z)\}$, for a fixed $z \in U$ and $f(z)$ ranging over the class $S^*(\beta)$, is convex.

Furthermore, taking $\alpha = 1$ in Theorem 2, we obtain

COROLLARY 3. The set of all points $\log\{f'(z)\}$, for a fixed $z \in U$ and $f(z)$ ranging over the class $C(\beta)$, is convex, where $C(\beta)$ is the class of analytic functions $f(z)$ which satisfy $\operatorname{Re}\{zf'(z)/g(z)\} > \beta$ for $g(z) \in S^*$

3. SOME PROPERTIES OF THE CLASS $B_1(\alpha,\beta)$.

In order to derive some properties of the class $B_1(\alpha,\beta)$, we shall recall here the following lemmas.

LEMMA 3 (Miller [13]). Let $\phi(u,v)$ be the complex function, $\phi: D \to C$, $D \quad C \times C$ (C-complex plane) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function ϕ satisfies the conditions:

(i) $\phi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $\operatorname{Re}\{\phi(1,0)\} > 0$;

(iii) $\operatorname{Re}\{\phi(iu_2,v_1)\} \leqq 0$ for all $(iu_2,v_1) \in D$ and such that $v_1 \leqq -(1 + u_2^2)/2$.

Let $p(z) = 1 + p_1 z + \cdots$ be regular in the unit disk U, such that $(p(z), zp'(z)) \in U$ for all $z \in U$. If $\operatorname{Re}\{\phi(p(z), zp'(z))\} > 0$ $(z \in U)$, then $\operatorname{Re}\{p(z)\} > 0$ for $z \in U$.

LEMMA 4 (Robertson [14]). Let $f(z) \in S$. For each $0 \leqq t \leqq 1$ let $F(z,t)$ be regular in the unit disk U, let $F(z,0) \equiv f(z)$ and $F(0,t) \equiv 0$. Let p be a positive real number for which

$$F(z) = \lim_{t \to +0} \frac{F(z,t) - F(z,0)}{zt^p}$$

exists. Let $F(z,t)$ be subordinate to $f(z)$ in U for $0 \leqq t \leqq 1$, then

$$\operatorname{Re}\{F(z)/f'(z)\} \leqq 0 \qquad (z \in U). \tag{3.1}$$

If in addition $F(z)$ is also regular in the unit disk U and $\operatorname{Re}\{F(0)\} \neq 0$, then

$$\operatorname{Re}\{F(z)/f'(z)\} < 0 \qquad (z \in U). \tag{3.2}$$

LEMMA 5 (MacGregor [15]). Let the function $f(z)$ be in the class $K(\beta)$. Then $f(z) \in S^*(\gamma(\beta))$, where

$$\gamma(\beta) = \begin{cases} \dfrac{2\beta - 1}{2(1 - 2^{1-2\beta})} & (\beta \neq 1/2) \\ \dfrac{1}{2\log 2} & (\beta = 1/2). \end{cases} \tag{3.3}$$

We begin with

LEMMA 6. Let the function $f(z)$ be in the class $B_1(\alpha,\beta)$, where α is a positive integer and $0 \leqq \beta < 1$. Then

$$\mathrm{Re}\{f(z)/z\}^{\alpha} > \beta \qquad (z \in U). \tag{3.4}$$

PROOF. For $f(z) \in B_1(\alpha,\beta)$, we have

$$\mathrm{Re}\{zf'(z)f(z)^{\alpha-1}/z^{\alpha}\} = \mathrm{Re}\left\{ \frac{df(z)^{\alpha}/dz}{dz^{\alpha}/dz} \right\} > \beta . \tag{3.5}$$

Applying Lemma 1, we can prove the assertion (3.4).

THEOREM 3. Let the function $f(z)$ be in the class $B_1(\alpha,\beta)$, where α is a positive integer and $0 \leqq \beta < 1$. Then the function $F_1(z)$ defined by

$$F_1(z)^{\alpha+\gamma} = z^{\gamma} f(z)^{\alpha} \tag{3.6}$$

belongs to the class $B_1(\alpha+\gamma,\beta)$ for $\gamma \geqq 0$.

PROOF. Note that

$$\frac{(\alpha + \gamma)F_1'(z)}{F_1(z)^{1-(\alpha+\gamma)}} = \gamma z^{\gamma-1} f(z)^{\alpha} + \frac{\alpha z^{\gamma} f'(z)}{f(z)^{1-\alpha}} , \tag{3.7}$$

or

$$\frac{(\alpha + \gamma)zF_1'(z)}{F_1(z)^{1-(\alpha+\gamma)} z^{\alpha+\gamma}} = \gamma \left(\frac{f(z)}{z} \right)^{\alpha} + \frac{\alpha z f'(z)}{f(z)^{1-\alpha} z^{\alpha}} . \tag{3.8}$$

Therefore, by using Lemma 6, we have

$$\operatorname{Re}\{zF_1'(z)F_1(z)^{(\alpha+\gamma)-1}/z^{\alpha+\gamma}\} > \beta \tag{3.9}$$

which implies $F_1(z) \in B_1(\alpha+\gamma,\beta)$. Thus we completes the theorem.

Applying Lemma 3, we derive

THEOREM 4. Let the function $f(z)$ be in the class $B_1(\alpha,\beta)$ with $\alpha > 0$ and $0 \leqq \beta < 1$. Then

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\alpha} > \frac{1+2\alpha\beta}{1+2\alpha} \qquad (z \in U). \tag{3.10}$$

PROOF. We define the function $p(z)$ by

$$A\{f(z)/z\}^{\alpha} = p(z) + B, \tag{3.11}$$

where $A = (1+2\alpha)/2\alpha(1-\beta)$ and $B = (1+2\alpha\beta)/2\alpha(1-\beta)$. Then $p(z)$ is analytic in the unit disk U and $p(0) = 1$. Differentiating both sides of (3.11) logarithmically, we obtain

$$\frac{zf'(z)}{f(z)} = \frac{1}{\alpha}\left\{\frac{zp'(z)}{p(z)+B} + \alpha\right\}, \tag{3.12}$$

or

$$zf'(z)f(z)^{\alpha-1}/z^{\alpha} = \{zp'(z) + \alpha(p(z)+B)\}/\alpha A. \tag{3.13}$$

Since $f(z) \in B_1(\alpha,\beta)$, (3.13) gives

$$\operatorname{Re}\{zp'(z) + \alpha(p(z)+B)\} - \alpha\beta A > 0. \tag{3.14}$$

Letting $p(z) = u = u_1 + iu_2$ and $zp'(z) = v = v_1 + iv_2$, we consider the function

$$\phi(u,v) = v + \alpha(u+B) - \alpha\beta A \tag{3.15}$$

which is continuous in $D = C \times C$, and which $(1,0) \in D$ and $\operatorname{Re}\{\phi(1,0)\} = 3/2 > 0$. Then, for all (iu_2, v_1) such that $v_1 \leqq -(1+u_2^2)/2$, we have

$$\begin{aligned}\operatorname{Re}\{\phi(iu_2,v_1)\} &= v_1 + \alpha\beta - \alpha\beta A \\ &\leqq -u_2^2/2 \\ &\leqq 0. \end{aligned} \tag{3.16}$$

Consequently, with the aid of Lemma 3, we conclude that

$$\operatorname{Re}\{p(z)\} > 0 \qquad (z \in U), \tag{3.17}$$

that is, that

$$\operatorname{Re}\left\{ A\left(\frac{f(z)}{z}\right)^{\alpha} \right\} > B . \tag{3.18}$$

This completes the proof of Theorem 4.

Putting $\beta = 0$ in Theorem 4, we have

COROLLARY 3 ([8, Theorem 3]). Let the function f(z) be in the class $B_1(\alpha,0)$ with $\alpha > 0$. Then

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\alpha} > \frac{1}{1+2\alpha} \qquad (z \in U). \tag{3.19}$$

Taking $\alpha = 1$ in Theorem 4, we have

COROLLARY 4. If the function f(z) belonging to A satisfies $\operatorname{Re}\{f'(z)\} > \beta$ with $0 \leqq \beta < 1$, then

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) > \frac{1+2\beta}{3} \qquad (z \in U). \tag{3.20}$$

REMARK 1. Letting $\beta = 0$ in Corollary 4, we have the corresponding result due to Obradović [7, Theorem 2].

Next, we prove

THEOREM 5. Let $\alpha > 1$, $0 \leqq \beta < 1$, and $\gamma(\beta)$ define by (3.3). Let $-1/4 \leqq \alpha - \beta - (\alpha - 1)\gamma(\beta) \leqq 1/4$. If the function $f(z)$ belongs to the class $K(\beta)$, then $f(z) \in B_1(\alpha,\beta')$, where

$$\beta' = 1/[2\{\alpha - \beta - (\alpha - 1)\gamma(\beta)\} + 1].$$

PROOF. Define the function $p(z)$ by

$$Azf'(z)f(z)^{\alpha-1}/z^{\alpha} = p(z) + A - 1 , \tag{3.21}$$

where $A = 1 + 1/2\{\alpha - \beta - (\alpha - 1)\gamma(\beta)\}$. Differentiating both sides of (3.21) logarithmically, we know that

$$1 - \alpha + \frac{zf''(z)}{f'(z)} + (\alpha - 1)\frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z) + A - 1}, \tag{3.22}$$

or

$$1 + \frac{zf''(z)}{f'(z)} - \beta + (\alpha - 1)\left(\frac{zf'(z)}{f'(z)} - \gamma(\beta)\right)$$

$$= \frac{zp'(z)}{p(z) + A - 1} + \alpha - \beta - (\alpha - 1)\gamma(\beta). \tag{3.23}$$

With the help of Lemma 5, (3.23) implies

$$\mathrm{Re}\left\{\frac{zp'(z)}{p(z) + A - 1}\right\} + \alpha - \beta - (\alpha - 1)\gamma(\beta) > 0. \tag{3.24}$$

Let the function $\phi(u,v)$ be defined by

$$\phi(u,v) = \frac{v}{u + A - 1} + \alpha - \beta - (\alpha - 1)\gamma(\beta) \tag{3.25}$$

with $p(z) = u = u_1 + iu_2$ and $zp'(z) = v = v_1 + iv_2$. Then $\phi(u,v)$ is continuous in $D = (C - \{1-A\}) \times C$. Further, $(1,0) \in D$ and

$$\begin{aligned}\mathrm{Re}\{\phi(1,0)\} &= \alpha - \beta - (\alpha - 1)\gamma(\beta)\\ &> (\alpha - 1)\{1 - \gamma(\beta)\}\\ &> 0.\end{aligned} \tag{3.26}$$

Consequently, for all (iu_2, v_1) such that $v_1 \leqq -(1 + u_2^2)/2$, we obtain

$$\mathrm{Re}\{\phi(iu_2, v_1)\} = \frac{(A - 1)v_1}{(A - 1)^2 + u_2^2} + \alpha - \beta - (\alpha - 1)\gamma(\beta)$$

$$\leqq \frac{2\{\alpha - \beta - (\alpha - 1)\gamma(\beta)\}\{(A - 1)^2 + u_2^2\} - (A - 1)(1 + u_2^2)}{2\{(A - 1)^2 + u_2^2\}}$$

$$\leqq 0. \tag{3.27}$$

By virtue of Lemma 3, we have

$$\mathrm{Re}\{p(z)\} > 0 \qquad (z \in U),$$

that is,

$$\mathrm{Re}\{Azf'(z)f(z)^{\alpha-1}/z^{\alpha}\} > A - 1. \tag{3.28}$$

It follows from (3.28) that

$$\mathrm{Re}\left\{\frac{zf'(z)f(z)^{\alpha-1}}{z^{\alpha}}\right\} > \frac{1}{2\{\alpha - \beta - (\alpha - 1)\gamma(\beta)\} + 1} \tag{3.29}$$

which completes the assertion of Theorem 5.

Finally, we prove

THEOREM 6. Let $f(z) \in A$, $\alpha > 0$, $0 \leqq \beta < 1$, and $0 \leqq t \leqq 1$. If

$$g(z) = \int_0^z \left(\frac{f(s)}{s}\right)^{1-\alpha} ds \in S \tag{3.30}$$

and

$$G(z,t) = f((1-t)z) - f(1-t^2)z)$$

$$+ (1 - t^2)\int_0^z \left(\frac{f(s)}{s}\right)^{1-\alpha} ds + zt\beta\left(\frac{f(z)}{z}\right)^{1-\alpha} \prec g(z), \tag{3.31}$$

then $f(z) \in B_1(\alpha,\beta)$.

PROOF. Note that

$$G(z) = \lim_{t \to +0} \frac{G(z,t) - G(z,0)}{zt}$$

$$= \lim_{t \to +0} \frac{\partial G(z,t)/\partial t}{z}$$

$$= \beta\left(\frac{f(z)}{z}\right)^{1-\alpha} - f'(z) \tag{3.32}$$

and $g'(z) = \{f(z)/z\}^{1-\alpha}$. It is clear from (3.32) that $\mathrm{Re}\{G(0)\} = \beta - 1 \neq 0$. Consequently, applying Lemma 4 when $p = 1$, we have

$$\mathrm{Re}\left\{\beta - \frac{zf'(z)f(z)^{\alpha-1}}{z^{\alpha}}\right\} < 0 , \tag{3.33}$$

or

$$\mathrm{Re}\{zf'(z)f(z)^{\alpha-1}/z^{\alpha}\} > \beta \tag{3.34}$$

which shows $f(z) \in B_1(\alpha,\beta)$.

REMARK 2. Letting $\beta = 0$ in Theorem 6, we have the corresponding theorem by Obradović [8, Theorem 1].

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