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ON THE GENERAL SOLUTION OF A FUNCTIONAL EQUATION CONNECTED TO SUM FORM INFORMATION MEASURES ON OPEN DOMAIN — III

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ABSTRACT. In this series, this paper is devoted to the study of a functional equation connected with the characterization of weighted entropy and weighted entropy of degree β. Here, we find the general solution of the functional equation (2) on an open domain, without using 0-probability and 1-probability.

KEY WORDS AND PHRASES. Functional equation, weighted entropy, weighted entropy of degree β*, open domain, sum form.*
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1. INTRODUCTION.

Let $\Gamma_n^0 = \{P = (p_1,p_2,\ldots,p_n) \mid 0 < p_j < 1, \sum_{k=1}^{n} p_k = 1\}$ and Γ_n be the closure of Γ_n^0. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, where $\mathbb{R}$ is the set of real numbers. Let (Ω,A,μ) be a probability space and let us consider an experiment that is a finite measurable partition $\{A_1,A_2,\ldots,A_n\}$, $(n > 1)$ of Ω. The weighted entropy of such an experiment is defined by Belis and Guiasu [1] as

$$H_n^1(P,U) = -\sum_{k=1}^{n} u_k p_k \log p_k$$

where $p_k = \mu(A_k)$ is the objective probability of the event A_k,
$P = (p_1,p_2,\ldots,p_n) \in \Gamma_n$ and $U = (u_1,u_2,\ldots,u_n) \in \mathbb{R}_+^n$. The weigthed entropy of degree β $(\beta \in \mathbb{R}-\{1\})$ of an experiment is defined by Emptoz [2] as

$$H_n^\beta(P,U) = (1-2^{1-\beta})^{-1} \sum_{k=1}^{n} u_k(p_k-p_k^\beta).$$

The measures $H_n^1(P,U)$ satisfy the following functional equation (see Kannappan [3])

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j,u_iv_j) = \sum_{i=1}^{n} p_iu_j\cdot\sum_{j=1}^{m} f(q_j.v_j) + \sum_{j=1}^{m} q_jv_j\cdot\sum_{i=1}^{n} f(p_i,u_i) \qquad (1.1)$$

for all $P\in\Gamma_n$, $Q \in \Gamma_m$, $u_i,v_j \in \mathbb{R}_+$. A generalization of (1) is the following:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j,u_iv_j) = \sum_{i=1}^{n} p_i^\alpha u_i\cdot\sum_{j=1}^{m} f(q_j,v_j)+\sum_{j=1}^{m} q_j^\beta v_j\cdot\sum_{j=1}^{n} f(p_i,u_i), \qquad (1.2)$$

where $P \in \Gamma_n$, $Q \in \Gamma_m$, $(u_1,u_2,\dots,u_n) \in \mathbb{R}_+^n$, $(v_1,v_2,\dots,v_m) \in \mathbb{R}_+^m$, $\alpha,\beta \in \mathbb{R}-\{0,1\}$. The measurable solution of (1.2) for $\alpha = 1$ was given by Kannappan in [3]. In a recent paper of Kannappan and Sahoo [4], measurable solution of a more general functional equation than (1.2) was given using the result of this paper. In this paper, we determine *the general solution of (1.2)* where $P \in \Gamma_n^0$, $Q \in \Gamma_m^0$, $(u_1,u_2,\dots,u_n) \in \mathbb{R}_+^n$, $(v_1,v_2,\dots,v_m) \in \mathbb{R}_+^m$, $\alpha,\beta \in \mathbb{R}-\{0,1\}$ and m,n (fixed and) ≥ 3, on an open domain.

2. SOLUTION OF (1.2) ON AN OPEN DOMAIN

We need the following result in this sequel.

Result 1 [5]. Let $f,g\colon]0,1[\to \mathbb{R}$ be real valued functions and satisfy

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j) = \sum_{i=1}^{n} p_i^{\alpha}\cdot \sum_{j=1}^{m} g(q_j)+\sum_{j=1}^{m} q_j^{\beta}\cdot \sum_{i=1}^{n} f(p_i) \tag{2.1}$$

for $P \in \Gamma_n^0$, $Q \in \Gamma_m^0$, $\alpha,\beta \in \mathbb{R}-\{0.1\}$ and m,n (≥ 3) are arbitrary but fixed integers. Then the general solutions of (2.1) are given by

$$\left.\begin{aligned} f(p) &= A(p) + ap^{\alpha}+bp^{\beta},\\ g(p) &= A'(p) + a(p^{\alpha}-p^{\beta})+c \end{aligned}\right\} \text{ for } \alpha\neq\beta$$

and

$$\left.\begin{aligned} f(p) &= A(p)+D(p)p^{\alpha}+dp^{\beta},\\ g(p) &= A'(p)+D(p)p^{\alpha}+c \end{aligned}\right\} \text{ for } \alpha=\beta$$

where a,b,c,d are arbitrary constants, A,A' are additive functions on $\mathbb{R}$ with $A(1) = 0$, $A'(1)+mc = 0$ and D is a real valued function satisfying

$$D(pq) = D(p)+D(q), \quad p,q \in]0,1[. \tag{2.2}$$

Now we proceed to determine the general solution of (1.2) on $]0,1[$. Let $f\colon]0,1[\times \mathbb{R}_+ \to \mathbb{R}$ be a real valued function and satisfy the functional equation (1.2) for an arbitrary but fixed pair of positive integers m,n (≥ 3), for $P\in \Gamma_n^0$, $Q \in \Gamma_m^0$, with $\alpha,\beta \in \mathbb{R}-\{0,1\}$. Letting $u_i = u$ for all $i = 1,2,\dots,n$ and $v_j = v$ for $j = 1,2,\dots,m$ in (1.2), we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{f(p_iq_j,uv)}{uv} = \sum_{i=1}^{n} p_i^{\alpha}\cdot \sum_{j=1}^{m} \frac{f(q_j,v)}{v} + \sum_{j=1}^{m} q_j^{\beta}\cdot \sum_{i=1}^{n} \frac{f(p_i,u)}{u}, \tag{2.3}$$

where $u,v \in \mathbb{R}_+$. Putting $v = 1$ in (2.3), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{f(p_iq_j,u)}{u} = \sum_{i=1}^{n} p_i^{\alpha}\cdot \sum_{j=1}^{m} f(q_j,1)+ \sum_{j=1}^{m} q_j^{\beta}\cdot \sum_{i=1}^{n} \frac{f(p_i,u)}{u} \tag{2.4}$$

where $u,v \in \mathbb{R}_+$. Putting $v = 1$ in (2.3), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{f(p_iq_j,u)}{u} = \sum_{i=1}^{n} p_i^{\alpha}\cdot \sum_{j=1}^{m} f(q_j,1)+ \sum_{j=1}^{m} q_j^{\beta}\cdot \sum_{i=1}^{n} \frac{f(p_i,u)}{u}$$

for $u \in \mathbb{R}_+$ and $P \in \Gamma_n^0$, $Q \in \Gamma_m^0$. For fixed $u \in \mathbb{R}_+$, (2.4) is of the form (2.1) and hence its general solutions cna be obtained from Result 1.

First we consider the case $\alpha \neq \beta$. Then from Result 1, we have

$$f(p,u) = A_1(p,u)u+a(u)up^{\alpha}+b(u)up^{\beta} \tag{2.5}$$

where $a,b\colon \mathbb{R}_+ \to \mathbb{R}$ are real valued functions of u and A_1 is additive in the first variable, with $A_1(1,u) = 0$. Letting (2.5) into (2.3), we get

$$\begin{aligned}(a(uv)-a(v))\sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} q_j^{\alpha}+(b(uv)-b(u))\sum_{i=1}^{n} p_i^{\beta}\cdot \sum_{j=1}^{m} q_j^{\beta}\\ -(b(v)+a(u))\sum_{i=1}^{n} p_i^{\alpha}\cdot \sum_{j=1}^{m} q_j^{\beta} = 0\ .\end{aligned} \tag{2.6}$$

Noting $\alpha \neq \beta$, $(\alpha \neq 1,\ \beta \neq 1)$ equating the coefficients of $\sum_{i=1}^{n} p_i^{\alpha}$ and $\sum_{i=1}^{n} p_i^{\beta}$ (then using the same for $\sum_{j=1}^{m} q_j^{\alpha}$ and $\sum_{j=1}^{m} q_j^{\beta}$) in (2.6), we get

$$a(uv) = a(v),\quad b(uv) = b(u)\quad \text{and}\quad b(v) = -a(u).$$

From these it is easy to see that

$$a(u) = -b(v) = a,\quad \text{constant} \tag{2.7}$$

for all $u,v \in \mathbb{R}_+$. Now putting (2.7) into (2.5), we get

$$f(p,u) = A_1(p,u)u+au(p^{\alpha}-p^{\beta}) \tag{2.8}$$

with $A_1(1,u) = 0$. Again letting (2.8) into (1.2), we get

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{m} A_1(p_iq_j,u_iv_j)u_iv_j = \sum_{j=1}^{m} A_1(q_j,v_j)v_j \sum_{i=1}^{n} u_ip_i^{\alpha}\ +\\ +\sum_{i=1}^{n} A_1(p_i,u_i)u_i\cdot \sum_{j=1}^{m} v_jq_j^{\beta}.\end{aligned} \tag{2.9}$$

Since A_1 is additive in the first variable, by putting $u_i = 1$ and $p_i = \frac{1}{n}$ (note that $\alpha \neq 1$), we have

$$\sum_{j=1}^{m} A_1(q_j,v_j)v_j = 0. \tag{2.10}$$

We let $v_1 = v_2,\dots, = v_{m-1} = v$ and $v_m = v'$, where $v,v' \in \mathbb{R}_+$, into (2.10) and obtain

$$\sum_{j=1}^{m-1} A_1(q_j,v)v + A_1(q_m,v')v' = 0.$$

Since A_1 is additive in the first variable, and $A_1(1,v) = 0$, we get

$$A_1(q_m,v)v = A_1(q_m,v')v' \tag{2.11}$$

for all $q_m \in\ ]0,1[$, and $v,v' \in \mathbb{R}_+$. From equation (2.11) it is clear that

$$A_1(x,y)y = A(x) \tag{2.12}$$

where A is an additive function with $A(1) = 0$. Now using (2.12) in (2.8), we obtain

$$f(p,u) = A(p) + au(p^\alpha - p^\beta), \quad p \in]0,1[, \ u \in \mathbb{R}_+ \tag{2.13}$$

where A is an additive function on $\mathbb{R}$ with $A(1) = 0$ and a is an arbitrary constant.

Next we consider the case $\underline{\alpha = \beta}$. Again the general solution of (2.4) from Result 1 can be obtained as

$$f(p,u) = uA_2(p,u) + D_1(p,u)p^\alpha u + d(u)p^\alpha u \tag{2.14}$$

where $d: \mathbb{R}_+ \to \mathbb{R}$ is a real valued function of u and A_2 is an additive function in the first variable with $A_2(1,u) = 0$ and $D_1:]0,1[\times \mathbb{R}_+ \to \mathbb{R}$ satisfies (2.2). Putting (2.14) into (2.4), we get by equating the coefficient of $\sum_{i=1}^{n} p_i^\alpha$ (note $\alpha \neq 1$)

$$\sum_{j=1}^{m} [D_1(q_j,u) - D_1(q_j,1) - d_1]q_j^\alpha = 0. \tag{2.15}$$

Using $u = 1$ in (2.15), gives $d_1 = 0$. Hence (2.15) with $d_1 = 0$, by the use of the Result 1 of [5], yields

$$(D_1(x,u) - D_1(x,1))x^\alpha = A_3(x - \tfrac{1}{m}, u) \tag{2.16}$$

for all $x \in]0,1[$ and A_3 is an additive function in the first variable. Since D_1 satisfies (2.2), we get

$$A_3(x - \tfrac{1}{m}, u)y^\alpha + A_3(y - \tfrac{1}{m}, u)x^\alpha = A_3(xy - \tfrac{1}{m}, u). \tag{2.17}$$

Putting $y = \frac{1}{m}$ and using $A_3(0,u) = 0$ in (2.17), we get

$$A_3(x,u) = c_1 A_3(1,u). \tag{2.18}$$

Since A_3 is additive in the first variable we obtain from (2.18) that $A_3 \equiv 0$ for $x \in]0,1[$, and all $u \in \mathbb{R}_+$. Thus, (2.16) reduces to

$$D_1(x,u) - D_1(x,1) = 0. \tag{2.19}$$

From (2.19), we see that D_1 is independent of u, i.e.

$$D_1(x,y) = D(x), \qquad x \in]0,1[\tag{2.20}$$

and since D_1 satisfies (2.2), D also satisfies (2.2). Using (2.20) in (2.14), we get

$$f(p,u) = uA_2(p,u) + D(p)up^\alpha + d(u)up^\alpha \tag{2.21}$$

where A_2 is additive with $A_2(1,u) = 0$. Letting (2.21) into (2.3), we get

$$(d(uv) - d(u) - d(v)) \sum_{i=1}^{n} \sum_{j=1}^{m} (p_i q_j)^\alpha = 0 \tag{2.22}$$

for all $u,v \in \mathbb{R}_+$. Since $\sum_{i=1}^{n}\sum_{j=1}^{m}(p_i q_j)^\alpha \neq 0$ we obtain

$$d(uv) = d(u) + d(v), \quad u,v \in \mathbb{R}_+. \tag{2.23}$$

Again putting (2.21) into (1.2) and using (2.23) and (2.2), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m} A_2(p_i q_j, u_i v_j)u_i v_j = \sum_{i=1}^{n} u_i p_i^\alpha \cdot \sum_{j=1}^{m} A_2(q_j, v_j)v_j + \sum_{j=1}^{m} v_j q_j^\alpha \cdot \sum_{i=1}^{n} A_2(p_i, u_i)u_i. \tag{2.24}$$

Putting $u_i = 1$ and $p_i = \frac{1}{n}$ in (2.4), we obtain

$$\sum_{j=1}^{m} A_2(q_j, v_j)v_j = 0. \tag{2.25}$$

Note that (2.25) is of the form of (2.10) and hence by a similar argument we get

$$A_2(q,u)u = A(q) \tag{2.26}$$

where A is additive with $A(1) = 0$. Using (2.26) in (2.21), we obtain

$$f(p.u) = A(p) + D(p)up^\alpha + d(u)up^\alpha \tag{2.27}$$

where A is additive on $\mathbb{R}$ with $A(1) = 0$ and $D\colon]0,1[\to \mathbb{R}$, $d\colon \mathbb{R}_+ \to \mathbb{R}$, are functions satisfying (2.2) and (2.23) respectively.

Thus we have proved the following theorem.

Theorem. Let $f\colon]0,1[\times \mathbb{R}_+ \to \mathbb{R}$ be a real valued function satisfying (1.2) for arbitrary but fixed pair of m,n (≥ 3) and $\alpha,\beta \notin \{0,1\}$, $P \in \Gamma_n^0$, $Q \in \Gamma_m^0$. Then f is given by (2.13) when $\alpha \neq \beta$ and by (2.27) when $\alpha = \beta$.

Corollary. If f is measurable in the Theorem then

$$f(p,u) = a(p^\alpha - p^\beta) \qquad \alpha \neq \beta$$

and

$$f(p,u) = bup^\alpha \log p + cp^\alpha u \log u, \quad \alpha = \beta$$

where a,b,c are arbitrary constants.

Remark. Because of the occurrence of the parameters α,β as powers, f is independent of m and n.

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