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CHARACTERIZATIONS OF SOME NEAR-CONTINUOUS FUNCTIONS AND NEAR-OPEN FUNCTIONS

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ABSTRACT. A subset N of a topological space is defined to be a θ-neighborhood of x if there exists an open set U such that $x \in U \subseteq \text{Cl } U \subseteq N$. This concept is used to characterize the following types of functions: weakly continuous, θ-continuous, strongly θ-continuous, almost strongly θ-cbntinuous, weakly δ-continuous, weakly open and almost open functions. Additional characterizations are given for weakly δ-continuous functions. The concept of θ-neighborhood is also used to define the following types of open maps: θ-open, strongly θ-open, almost strongly θ-open, and weakly δ-open functions.

KEY WORDS AND PHRASES. θ-neighborhood, weakly continuous function, θ-continuous function, strongly θ-continuous function, almost strongly θ-continuous function, weakly δ-continuous function, weakly open function, almost open function, θ-open function, strongly θ-open function, almost strongly θ-open function, weakly δ-open function.
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1. INTRODUCTION.

Near-continuity has been investigated by many authors including Levine [1], Long and Herrington [2], Noiri [3], and Rose [4]. Near-openness has been developed by Rose [5] and Singal and Singal [6]. The purpose of this note is to characterize several types of near-continuity and near-openness in terms of the concept of θ-neighborhood. These characterizations clarify both the nature of these functions and the relationships among them. Additional characterizations of weak δ-continuity are given. The concept of θ-neighborhood also leads to the definition of several new types of near-open functions.

2. DEFINITIONS AND NOTATION.

The symbols X and Y denote topological spaces with no separation axioms assumed unless explicitly stated. Let U be a subset of a space X. The closure of U and the interior of U are denoted by Cl U and Int U respectively. The set U is said to be regular open (regular closed) if U = Int Cl U (U = Cl Int U). The θ-closure (δ-closure) (Velicko [7]) of U is the set of all x in X such that every closed neighborhood (the interior of every closed neighborhood) of x intersects

U. The θ-closure and the δ-closure of U are denoted by $Cl_\theta U$ and $Cl_\delta U$ respectively. The set U is called θ-closed (δ-closed) if $U = Cl_\theta U$ ($U = Cl_\delta U$). A set is said to be θ-open (δ-open) if its complement is θ-closed (δ-closed). For a given space X the collection of all θ-open sets and the collection of all δ-open sets both form topologies. The space X with the θ-open (δ-open) topology will be signified by X_θ (X_s).

DEFINITION 1. A function $f: X \to Y$ is said to be weakly continuous (Levine [1]) (θ-continuous (Fomin [8]), strongly θ-continuous (Long and Herrington [2]), almost strongly θ-continuous (Noiri and Kang [9]), weakly δ-continuous (Baker [10])) if for each $x \in X$ and each open neighborhood V of f(x), there exists an open neighborhood U of x such that $f(U) \subseteq Cl\ V$ ($f(Cl\ U) \subseteq Cl\ V$, $f(Cl\ U) \subseteq V$, $f(Cl\ U) \subseteq Int\ Cl\ V$, $f(Int\ Cl\ U) \subseteq Cl\ V$).

DEFINITION 2. A function $f: X \to Y$ is said to be weakly open (Rose [5]) (almost open (Rose [5])) provided that for each open subset U of X, $f(U) \subseteq Int\ f(Cl\ U)$ ($f(U) \subseteq Int\ Cl\ f(U)$).

DEFINITION 3. A subset N of a space X is said to be a θ-neighborhood (δ-neighborhood) of a point x in X if there exists an open set U such that $x \in U \subseteq Cl\ U \subseteq N$ ($x \in U \subseteq Int\ Cl\ U \subseteq N$).

Note that a θ-neighborhood is not necessarily a neighborhood in the θ-topology, but a δ-neighborhood is a neighborhood in the δ-topology.

3. NEAR-CONTINUOUS FUNCTIONS.

The main results can be paraphrased as follows: weak continuity corresponds to "f^{-1} (θ-neighborhood) = neighborhood"; θ-continuity corresponds to "f^{-1} (θ-neighborhood) = θ-neighborhood"; strong θ-continuity corresponds to "f^{-1} (neighborhood) = θ-neighborhood"; almost strong θ-continuity corresponds to "f^{-1} (δ-neighborhood) = θ-neighborhood", and weak δ-continuity corresponds to "f^{-1} (θ-neighborhood) = δ-neighborhood".

THEOREM 1. A function $f: X \to Y$ is weakly continuous if and only if for each x in X and each θ-neighborhood N of f(x), $f^{-1}(N)$ is a neighborhood of x.

PROOF. Assume f is weakly continuous. Let $x \in X$ and let N be a θ-neighborhood of f(x). Then there exists an open set V such that $f(x) \in V \subseteq Cl\ V \subseteq N$. Since f is weakly continuous, there exists an open neighborhood U of x such that $f(U) \subseteq Cl\ V \subseteq N$. Thus $x \in U \subseteq f^{-1}(N)$ and hence $f^{-1}(N)$ is a neighborhood of x.

Assume for each $x \in X$ and each θ-neighborhood N of x that $f^{-1}(N)$ is a neighborhood of x. Let $x \in X$ and let V be an open neighborhood of f(x). Since Cl V is a θ-neighborhood of f(x), $f^{-1}(Cl\ V)$ is a neighborhood of x. Thus there is an open set U for which $x \in U \subseteq f^{-1}(Cl\ V)$ and $f(U) \subseteq Cl\ V$ which proves f is weakly continuous.

THEOREM 2. A function $f: X \to Y$ is θ-continuous if and only if for each x in X and each θ-neighborhood N of f(x), $f^{-1}(N)$ is a θ-neighborhood of x.

PROOF. Assume $f: X \to Y$ is θ-continuous. Let $x \in X$ and let N be a θ-neighborhood of $f(x)$. Then there exists an open set V for which $f(x) \in V \subseteq \mathrm{Cl}\, V \subseteq N$. By the θ-continuity of f, there exists an open neighborhood U of x such th.t $f(\mathrm{Cl}\ U) \subseteq \mathrm{Cl}\ V \subseteq N$. Thus $x \in U \subseteq \mathrm{Cl}\ U \subseteq f^{-1}(N)$ and hence $f^{-1}(N)$ is a θ-neighborhood of x.

Assume for each x in X and for each θ-neighborhood N of $f(x)$ that $f^{-1}(N)$ is a θ-neighborhood of x. Let $x \in X$ and let V be an open neighborhood of $f(x)$. Since $\mathrm{Cl}\ V$ is a θ-neighborhood of $f(x)$, $f^{-1}(\mathrm{Cl}\ V)$ is a θ-neighborhood of x. Hence there exists an open set U for which $x \in U \subseteq \mathrm{Cl}\ U \subseteq f^{-1}(\mathrm{Cl}\ V)$. That is, $f(\mathrm{Cl}\ U) \subseteq \mathrm{Cl}\ V$ and thus f is θ-continuous.

The proof of the following theorem is similar to that of Theorem 2 and is omitted.

THEOREM 3. A function $f: X \to Y$ is strongly θ-continuous if and only if for each x in X and each neighborhood N of $f(x)$, $f^{-1}(N)$ is a θ-neighborhood of x.

THEOREM 4. A funciton $f: X \to Y$ is almost strongly θ-continuous if and only if for each x in X and each δ-neighborhood N of $f(x)$, $f^{-1}(N)$ is a θ-neighborhood of x.

PROOF. Assume $f: X \to Y$ is almost strongly θ-continuous. Let $x \in X$ and let N be a δ-neighborhood of $f(x)$. Then there exists an open set V such that $f(x) \in V \subseteq \mathrm{Int}\ \mathrm{Cl}\ V \subseteq N$. Since f is almost strongly θ-continuous, there exists an open neighborhood U of x for which $f(\mathrm{Cl}\ U) \subseteq \mathrm{Int}\ \mathrm{Cl}\ V \subseteq N$. Then $x \in U \subseteq \mathrm{Cl}\ U \subseteq f^{-1}(N)$ which proves that $f^{-1}(N)$ is a θ-neighborhood of x.

Assume for each $x \in X$ and each δ-neighborhood N of $f(x)$ that $f^{-1}(N)$ is a θ-neighborhood of x. Let $x \in X$ and let V be an open neighborhood of $f(x)$. Since $\mathrm{Int}\ \mathrm{Cl}\ V$ is a δ-neighborhood of $f(x)$, $f^{-1}(\mathrm{Int}\ \mathrm{Cl}\ V)$ is a θ-neighborhood of x. Hence there is an open set U such that $x \in U \subseteq \mathrm{Cl}\ U \subseteq f^{-1}(\mathrm{Int}\ \mathrm{Cl}\ V)$. That is, $f(\mathrm{Cl}\ U) \subseteq \mathrm{Int}\ \mathrm{Cl}\ V$ and hence f is almost strongly θ-continuous.

THEOREM 5. A function $f: X \to Y$ is weakly δ-continuous if and only if for each $x \in X$ and each θ-neighborhood N of $f(x)$, $f^{-1}(N)$ is a δ-neighborhood of x.

The proof of this theorem is similar to that of Theorem 4. The following theorem gives additional characterizations of weak δ-continuity. These results are analogous to those obtained by Noiri and Kang in [9] for almost strongly θ-continuous functions.

LEMMA. Let X be a space and $H \subseteq X$. Then

(a) $\mathrm{Cl}_\theta\ H = \{x \in X:$ every θ-neighborhood of x intersects $H\}$ and

(b) $\mathrm{Cl}_\delta\ H = \{x \in X:$ every δ-neighobrhood of x intersects $H\}$.

The proof follows easily from the definitions.

THEOREM 6. For $f: X \to Y$ the following statements are equivalent:

(a) $f: X \to Y$ is weakly δ-continuous.

(b) For each $H \subseteq X$, $f(\mathrm{Cl}_\delta\ H) \subseteq \mathrm{Cl}_\theta\ f(H)$.

(c) For each $K \subseteq Y$, $\mathrm{Cl}_\delta\ f^{-1}(K) \subseteq f^{-1}(\mathrm{Cl}_\theta\ K)$.

(d) $f: X_s \to Y$ is weakly continuous.

PROOF. (a) => (b). Let $H \subseteq X$ and let $y \in f(\mathrm{Cl}_\delta H)$. Then there exists an x in $\mathrm{Cl}_\delta H$ such that $y = f(x)$. Let N be a θ-neighborhood of $f(x)$. By Theorem 5 $f^{-1}(N)$ is a δ-neighborhood of x. Since $x \in \mathrm{Cl}_\delta H$, $f^{-1}(N) \cap H \neq \phi$. That is, $N \cap f(H) \neq \phi$. Hence $y \in \mathrm{Cl}_\theta f(H)$. Thus $f(\mathrm{Cl}_\delta H) \subseteq \mathrm{Cl}_\theta f(H)$.

(b) => (c). Let $K \subseteq Y$. By (b) $f(\mathrm{Cl}_\delta f^{-1}(K)) \subseteq \mathrm{Cl}_\theta f(f^{-1}(K)) \subseteq \mathrm{Cl}_\theta K$. Thus $\mathrm{Cl}_\delta f^{-1}(K) \subseteq f^{-1}(\mathrm{Cl}_\theta K)$.

(c) => (d). Let $x \in X$ and let V be an open neighborhood of $f(x)$. Since Cl V is a θ-neighborhood of $f(x)$, $f(x) \notin \mathrm{Cl}_\theta$ $(Y - \mathrm{Cl}\, V)$. Hence $x \notin f^{-1}$ $(\mathrm{Cl}_\theta$ $(Y - \mathrm{Cl}\, V)$.

By (c) $x \notin \mathrm{Cl}_\delta f^{-1}$ $(Y - \mathrm{Cl}\, V)$. Thus there is a neighborhood U of x such that $(\mathrm{Int\,Cl}\, U) \cap f^{-1}$ $(Y - \mathrm{Cl}\, V) = \phi$. Then $f(\mathrm{Int\,Cl}\, U) \subseteq \mathrm{Cl}\, V$. Since Int Cl U is a regular open, $f: X_s \to Y$ is weakly continuous.

(d) => (a). Let $x \in X$ and let V be an open neighborhood of $f(x)$. Since $f: X_s \to Y$ is weakly continuous, there exists a δ-open set W containing x such that $f(W) \subseteq \mathrm{Cl}\, V$. Then there is a regular open set U for which $x \in U \subseteq W$. Then $f(\mathrm{Int\,Cl}\, U) = f(U) \subseteq f(W) \subseteq \mathrm{Cl}\, V$ and hence f is weakly δ-continuous.

4. NEAR-OPEN FUNCTIONS.

In this section weak openness and almost openness are characterized in terms of the concept of θ-neighborhood.

THEOREM 7. A function $f: X \to Y$ is weakly open if and only if for each $x \in X$ and each θ-neighborhood N of x, $f(N)$ is a neighborhood of $f(x)$.

PROOF. Assume f is weakly open. Let $x \in X$ and let N be a θ-neighborhood of x. Then there is an open set U such that $x \in U \subseteq \mathrm{Cl}\, U \subseteq N$. Since f is weakly open $f(x) \in f(U) \subseteq \mathrm{Int}\, f(\mathrm{Cl}\, U) \subseteq \mathrm{Int}\, f(N)$. Hence $f(N)$ is a neighborhood of $f(x)$.

Assume for each x in X and each θ-neighborhood N of x that $f(N)$ is a neighborhood of $f(x)$. Let U be an open set in X. Suppose $x \in U$. Since Cl U is a θ-neighborhood of x, $f(\mathrm{Cl}\, U)$ is a neighborhood of $f(x)$. Hence $f(x) \in \mathrm{Int}\, f(\mathrm{Cl}\, U)$. Thus $f(U) \subseteq \mathrm{Int}\, f(\mathrm{Cl}\, U)$ and f is weakly open.

The proof of the following theorem is similar and is omitted.

THEOREM 8. A function $f: X \to Y$ is almost open if and only if for each $x \in X$ and each neighborhood N of x, Cl $f(N)$ is a θ-neighborhood of $f(x)$.

Theorem 7 and the characterizations of near-continuous functions in Section 3 suggest the following definitions of near-open functions.

DEFINITION 4. A function $f: X \to Y$ is said to be θ-open (strongly θ-open, almost strongly θ-open, weakly δ-open) if for each $x \in X$ and each θ-neighborhood (neighborhood, δ-neighborhood, θ-neighborhood) N of x, $f(N)$ is a θ-neighborhood (θ-neighborhood, θ-neighboehood, δ-neighborhood) of $f(x)$.

The following theorems characterize these near-open functions in terms of the closure and interior operators. Since the proofs are all similar, only the first theorem is proved.

THEOREM 9. A function $f: X \to Y$ is θ-open if and only if for each $x \in X$ and each open neighborhood U of x, there exists an open neighborhood V of $f(x)$ such that $\mathrm{Cl}\, V \subseteq f(\mathrm{Cl}\, U)$.

PROOF. Assume $f: X \to Y$ is θ-open. Let $x \in X$ and let U be an open neighborhood of x. Since f(Cl U) is a θ-neighborhood of f(x), there exists an open set V such that $f(x) \in V \subseteq Cl\ V \subseteq f(Cl\ U)$.

Assume that for each $x \in X$ and each open neighborhood U of x there exists an open neighborhood V of f(x) for which $Cl\ V \subseteq f(Cl\ U)$. Let $x \in X$ and let N be a θ-neighborhood of x. Then there is an open set U for which $x \in U \subseteq Cl\ U \subseteq N$. There exists an open set V such that $f(x) \in V \subseteq Cl\ V \subseteq f(Cl\ U) \subseteq f(N)$. Hence f(N) is a θ-neighborhood of f(x) and f is θ-open.

THEOREM 10. A function $f: X \to Y$ is strongly θ-open if and only if for each $x \in X$ and each open neighborhood U of x, there exists an open neighborhood V of f(x) such that $Cl\ V \subseteq f(U)$.

THEOREM 11. A function $f: X \to Y$ is almost strongly θ-open if and only if for each $x \in X$ and each open neighborhood U of x there exists an open neighborhood V of f(x) such that $Cl\ V \subseteq f(Int\ Cl\ U)$.

THEOREM 12. A function $f: X \to Y$ is weakly δ-open if and only if for each $x \in X$ and each open neighborhood U of x, there exists an open neighborhood V of f(x) such that $Int\ Cl\ V \subseteq f(Cl\ U)$.

We have the following implications: almost open <= st. θ-open => almost st. θ-open => θ-open => weak δ-open => weak open. The following examples show that these implications are not reversible.

EXAMPLE 1. Let X = {a, b}, T_1 = {X, ϕ, {a}}, Y = {a, b, c}, and T_2 ={Y, ϕ, {a} {a, b}}. The inclusion mapping: $(X, T_1) \to Y, T_2)$ is weak open but not weak δ-open.

In the next example the space (Y, T_2) is from Example 2.2 in Noiri and Kang [9].

EXAMPLE 2. Let (X, T_1) be as in Example 1. Let Y = {a, b, c, d}. and T_2 = {Y, ϕ {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}}. The inclusion mapping: $(X, T_1) \to (Y, T_2)$ is weak δ-open, but not θ-open.

EXAMPLE 3. Let (Y, T_2) be as in Example 2. The identity mapping: $(Y, T_2) \to (Y, T_2)$ is θ-open but not almost strongly θ-open.

EXAMPLE 4. Let X = {a, b, c}, T_1 = {X, ϕ, {a}, {a, c}} and T_2 = {X, ϕ, {a}, {c}, {a, c}, {a, b}}. The identity mapping: $(X, T_1) \to (X, T_2)$ is almost strongly θ-open and almost open, but not strongly θ-open.

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