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ON A GENERALIZATION OF THE CORONA PROBLEM

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ABSTRACT. Let $g, f_1,\dots, f_m \in H^\infty(\Delta)$. We provide conditions on $f_1,\dots,f_m$ in order that $|g(z)|\leq|f_1(z)|+\dots+|f_m(z)|$, for all z in Δ, imply that g, or g^2, belong to the ideal generated by $f_1,\dots,f_m$ in H^∞.

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1. INTRODUCTION.

Let $H(\Delta)=H$ be the space of all holomorphic functions on $\Delta=\{z\in\mathbb{C}: |z|<1\}$, and let $H^\infty(\Delta)=H^\infty$ be the subspace of all bounded functions of $H(\Delta)$. Let $f_1,\dots,f_m$ be functions in H^∞ and let $g\in H^\infty$ satisfy the following condition:

$$|g(z)|\leq|f_1(z)|+\dots+|f_m(z)| \qquad \text{(any } z\in\Delta). \tag{1.1}$$

As a generalization of the corona problem (which was first solved by Carleson [1]) it is natural to ask if (1.1) implies that g belongs to the ideal $I_{H^\infty}(f_1,\dots,f_m)$ generated in H^∞ by $f_1,\dots,f_m$, i.e. if (1.1) implies the existence of $g_1,\dots,g_m$ in H^∞ such that, on Δ,

$$g = f_1g_1+\dots+f_mg_m. \tag{1.2}$$

Rao, [2], has shown that the answer to this question is negative in general. On the other hand Wolff (see [3], th. 2.3) has proved that (1.1) implies that g^3 belongs to $I_{H^\infty}(f_1,\dots,f_m)$. The question whether (1.1) implies the existence of $g_1,\dots g_m$ in H^∞ such that

$$g^2 = f_1g_1+\dots+f_mg_m \tag{1.3}$$

is still open, as Garnett has pointed out ([4], problem 8.20).

In this work we obtain some results on this generalized corona problem, making use of techniques which appear in the theory of A_p spaces, the spaces of entire functions with growth conditions introduced by Hörmander [5].

With the same aim of Berenstein and Taylor [6] in A_p, we introduce in H^∞ the notion of jointly invertible functions (definition 3) and prove that if $f_1,\dots,f_m$ are jointly invertible, condition (1.1) implies that g belongs to $I_{H^\infty}(f_1,\dots f_m)$ (proposition 5). We also prove that if the ideal $I_{H^\infty}(f_1,\dots,f_m)$ contains a weakly invertible function having simple interpolating zeroes (see [3]), then again (1.1) implies that g belongs to $I_{H^\infty}(f_1,\dots,f_m)$ (theorem 6).

Finally, in the same spirit of Kelleher and Taylor [7] we introduce the notion of congeniality for m-tuples of functions in H^∞, and give a partial answer to the problem posed by Garnett ([4]): we prove that if $(f_1,\dots,f_m)\in(H^\infty)^m$ is congenial, then (1.1) implies $g^2\in I_{H^\infty}(f_1,\dots,f_m)$ (theorem 8).

2. WEAK INVERTIBILITY.

We first study some conditions under which (1.1) implies that $g\in I_{H^\infty}(f_1,\dots,f_m)$.

DEFINITION 1. A function f in $H^\infty(\Delta)$ is called weakly invertible if there exists a Blaschke product B such that $f(z)=B(z)\tilde{f}(z)$ (z in Δ) with $\tilde{f}$ invertible in H^∞.

The reason for this definition is the following simple criterion of divisibility for functions in H^∞.

PROPOSITION 2. Let $f\in H^\infty$. Then f is weakly invertible if, and only if, for all $g\in H^\infty$ the fact that $g/f\in H$ implies $g/f\in H^\infty$.

PROOF. Suppose f is weakly invertible: then there exists a Blaschke product B such that $f(z)=B(z)\tilde{f}(z)$, with $\tilde{f}$ invertible in H^∞. Since g/f is holomorphic and since B contains exactly the zeroes of f, it follows that $g/B\in H$; however, since B is a Blaschke product, $g/B\in H$ implies, [8], that $g/B\in H^\infty$. Since $1/\tilde{f}\in H^\infty$ one has $g/f=(g/B)(1/\tilde{f})$, i.e. $g/f\in H^\infty$. Conversely, suppose that for all $g\in H^\infty$ such that $g/f\in H$, it follows $g/f\in H^\infty$. Write $f(z)=B(z)\tilde{f}(z)$, where B is the Blaschke product of all the zeroes of f (see [8]). Then B/f is holomorphic on Δ and therefore $1/\tilde{f}$ must belong to H^∞. ¶

An extension of the notion of weak invertibility to m-tuples of functions in H^∞ is given by the following definition, analogous to the one given by Berenstein and Taylor for the spaces A_p in [6].

DEFINITION 3. The functions $f_1,\dots,f_m\in H^\infty$ are called jointly invertible if the ideal generated by $f_1,\dots,f_m$ in H^∞ coincides with $I_{loc}(f_1,\dots,f_m)=\{g\in H^\infty(\Delta)$: for any $z\in\Delta$, there exists a neighborhood U of z and $\lambda_1,\dots,\lambda_m$ in H(U) such that $g=\lambda_1f_1+\dots+\lambda_mf_m$ on U$\}$.

In view of Cartan's theorem B, it follows immediately that $f_1,\dots,f_m$ are jointly invertible if, and only if, $I_{H^\infty}(f_1,\dots,f_m)=I_H(f_1,\dots,f_m)$, the latter being the ideal generated by $f_1,\dots,f_m$ in $H(\Delta)$. As a consequence of the corona theorem, all m-tuples $f_1,\dots,f_m$ in H^∞ for which there exists $\delta>0$ such that $|f_1(z)|+\dots+|f_m(z)|\geq\delta$ for all z in Δ, are jointly invertible ($I_H=I_{H^\infty}=H^\infty$). More generally one has:

PROPOSITION 4. Let $b\in H^\infty$ be weakly invertible, and let $f_1(z)=b(z)\tilde{f}_1(z),\dots,f_m(z)=b(z)\tilde{f}_m(z)$, for $\tilde{f}_1,\dots,\tilde{f}_m$ in H^∞ such that $|\tilde{f}_1(z)|+\dots+|\tilde{f}_m(z)|\geq\delta>0$ for some δ and all z in Δ. Then $f_1,\dots,f_m$ are jointly invertible.

PROOF. Let $g\in H^\infty$ belong to $I_H(f_1,\dots,f_m)$. There exist $\lambda_1,\dots,\lambda_m$ in $H(\Delta)$ such that

$$g(z) = \lambda_1(z)f_1(z)+\dots\lambda_m(z)f_m(z) \qquad (\text{all } z\in\Delta) \tag{2.1}$$

i.e., for all z in Δ,

$$g(z) = b(z)[\lambda_1(z)\tilde{f}_1(z)+\dots+\lambda_m(z)\tilde{f}_m(z)]. \tag{2.2}$$

Since b is invertible, and $g/b\in H$, it follows that $\tilde{g}=g/b=\lambda_1\tilde{f}_1+\dots+\lambda_m\tilde{f}_m\in H^\infty$. By the corona theorem, then, it follows that there are $h_1,\dots,h_m$ in H^∞ such that

$$\tilde{g}(z) = h_1(z)\tilde{f}_1(z)+\dots+h_m(z)\tilde{f}_m(z), \tag{2.3}$$

therefore

$$g(z) = \tilde{g}(z)b(z) = h_1(z)f_1(z)+\dots+h_m(z)f_m(z) \tag{2.4}$$

and the assertion is proved.

Let now $f_1,\dots,f_m, g\in H^\infty(\Delta)$, and suppose that (1.1) holds. It is well known, [2], that in general (1.1) does not imply that $g\in I_{H^\infty}(f_1,\dots,f_m)$. However, (1.1) certainly implies that $g\in I_{loc}(f_1,\dots,f_m)$ and hence

PROPOSITION 5. Let $f_1,\dots,f_m$ be jointly invertible. Then if g satisfies condition (1.1), it follows that $g\in I_{H^\infty}(f_1,\dots,f_m)$.

A different situation in which (1.1) implies that $g\in I_{H^\infty}(f_1,\dots,f_m)$ occurs when at least one of the f_j's, say f_1, is weakly invertible and has simple zeroes which form an interpolating sequence ([3]); this happens, for example, when f_1 is an interpolating Blaschke product with simple zeroes ([3]). Indeed, following an analogous result proved in [7] for the space of entire functions of exponential type, one has:

THEOREM 6. Let $f_1,\dots,f_m\in H^\infty$, and suppose f_1 is weakly invertible with simple, interpolating zeroes. Then if $g\in H^\infty$ satisfies condition (1.1) it follows that g belongs to $I_{H^\infty}(f_1,\dots,f_m)$.

PROOF. Choose $a_{ij}\in\mathbb{C}$, $i=2,\dots m$, $j\geq 1$, such that for $\{z_j\}=\{z\in\Delta: f_1(z)=0\}$ it is $|a_{ij}|=1$ and $a_{ij}f_i(z_j)\geq 0$. Define now $b_{ij}\in\mathbb{C}$ (i,j as before) by

$$b_{ij}=\begin{cases} 0 \text{ if } f_2(z_j)=\dots=f_m(z_j)=0 \\ a_{ij}g(z_j)/(|f_2(z_j)|+\dots+|f_m(z_j)|) \quad \text{otherwise.}\end{cases}$$

By (1.1) it follows $|b_{ij}|\leq 1$ (all i,j), and since $\{z_j\}$ is interpolating, one finds h_2, $\dots,h_m$ in H^∞ such that $h_i(z_j)=b_{ij}$. Therefore the function $h=g-(h_2f_2+\dots+h_mf_m)$ belongs to H^∞ and vanishes at each z_j. The simplicity of the zeroes of f_1 shows that $f/f_1\in H$, and the invertibility of f_1 implies $h/f_1=h_1\in H^\infty$. The thesis now follows, since $g=f_1h_1+$ $+\dots+f_mh_m$. ¶

It is worthwhile noticing that the hypotheses of Proposition 5 and Theorem 6 are not comparable. Consider, indeed, the following conditions on $f_1,\dots,f_m\in H^\infty$:

(C_1) $f_1,\dots,f_m$ are jointly invertible.

(C_2) there exists j ($1\leq j\leq m$) such that f_j is invertible, with an interpolating sequence of zeroes, all of which are simple.

Then (C_1) does not imply (C_2): take m=1 and f_1 weakly invertible with non-simple zeroes. On the other hand, also (C_2) does not imply (C_1): consider f_1 invertible with simple interpolating zeroes $\{z_n\}$; let $f_2\in H^\infty$ be a function such that $f_2(z_n)=1/n$ (such a function certainly exists since $\{z_n\}$ is an interpolating sequence); now f_1 and f_2 have no common zeroes, and hence $1\in I_{loc}(f_1,f_2)$; however $1\notin I_{H^\infty}(f_1,f_2)$ since if $1=\lambda_1f_1+\lambda_2f_2$, then it is $\lambda_2(z_n)=n$, i.e. $\lambda_2\notin H^\infty$. Therefore the pair (f_1,f_2) satisfies (C_2) but not (C_1).

3. CONGENIALITY.

In this section we describe a class of m-tuples of functions in $H^\infty(\Delta)$, for which condition (1.1) implies that $g^2\in I_{H^\infty}(f_1,\dots,f_m)$.

DEFINITION 7. An m-tuple $(f_1,\dots,f_m)$ of functions in H^∞ is called congenial if, for all $i,j=1,\dots,m$,

$$(f_if_j'-f_jf_i')/\ \|f\|^2\|f'\| \quad \text{belongs to } L^\infty(\Delta),$$

where $\|f(z)\|^2=|f_1(z)|^2+\dots+|f_m(z)|^2$, $\|f'(z)\|^2=|f_1'(z)|^2+\dots+|f_m'(z)|^2$, and $f_i'=\partial f_i/\partial z$.

Notice that the class of congenial m-tuples is not empty. Indeed, one might consider pairs f_1, f_2 in H^∞ which, at their common zeroes, satisfy some simple conditions on their vanishing order easily deducible from Definition 7. For example, one can ask that $f_1(z_0)=f_2(z_0)=0$, $f_2'(z_0)\neq 0$, $f_1'(z_0)=0$. As a partial answer to problem 8.20 in [4], we prove the following

THEOREM 8. Let $f_1,\dots,f_m, g\in H^\infty(\Delta)$, and suppose $(f_1,\dots,f_m)$ be congenial. If g satisfies (1.1), then $g^2 \in I_{H^\infty}(f_1,\dots,f_m)$, i.e. there are $g_1,\dots,g_m$ in H^∞ such that (on Δ)

$$g^2(z) = f_1(z)g_1(z)+\dots+f_m(z)g_m(z) \tag{3.1}$$

PROOF. We mainly follow the proof due to Wolff, [3], of the fact that (1.1) implies that $g^3\in I_{H^\infty}$. We can assume $\|f_j\|_\infty\leq 1$, $\|g\|_\infty\leq 1$, and $f_j, g\in H(\bar\Delta)$ $(j=1,\dots,m)$. Put $\psi_j = g\bar f_j/|f|^2$ (ψ_j is bounded and C^∞ on $\bar\Delta$) and consider the differential equation

$$\partial b_{j,k}/\partial\bar z = \psi_j\partial\psi_k/\partial\bar z = g^2 G_{j,k} \qquad (1\leq j,k\leq m) \tag{3.2}$$

for

$$G_{j,k} = \bar f_j \sum_\ell f_\ell(\overline{f_\ell f_k' - f_k f_\ell'})/|f|^6.$$

If solutions $b_{j,k}\in L^\infty$ exist, then clearly $g_j = g\psi_j + \sum_k (b_{j,k}-b_{k,j})f_k \in H^\infty$ and (3.1) holds (indeed $\bar\partial g_j = 0$ and g_j is bounded on Δ). In order to prove that (3.2) admits a solution in L^∞ it is enough to show that $|g^2G_{j,k}|^2\log(1/|z|)dxdy$ and $\partial(g^2G_{j,k})/\partial z$ are Carleson measures for $1\leq j,k\leq m$.

As far as $|g^2G_{j,k}|^2\log(1/|z|)dxdy$ is concerned, notice that, by the congeniality of $(f_1,\dots,f_m)$, it is

$$|g^2G_{j,k}|^2 \leq |g|^4|\bar f_j|^2|\sum_\ell f_\ell(\overline{f_\ell f_k'-f_kf_\ell'})|^2/|f|^{12} \leq c|f'|^2.$$

On the other hand,

$$\partial(g^2G_{j,k})/\partial z = 2gg'G_{j,k} + g^2\partial G_{j,k}/\partial z;$$

again by the congeniality of $(f_1,\dots,f_m)$, one has

$$|gg'G_{j,k}| \leq |g||g'||\bar f_j||\sum_\ell f_\ell(\overline{f_\ell f_k'-f_kf_\ell'})|/|f|^6 \leq c(|g'|^2+\|f'\|^2)/|f| \leq$$
$$\leq c(|g'|^2/|g| + |f'|^2/|f|),$$

and

$$|g^2\partial G_{j,k}/\partial z| = |g|^2\cdot|f_j||\sum_\ell \bar f_\ell f_\ell'|\cdot|\sum_\ell f_\ell(\overline{f_\ell f_k'-f_kf_\ell'})|/|f|^8 +$$
$$+|g|^2|\bar f_j|/|f|^2\cdot(|\sum_\ell f_\ell'(\overline{f_\ell f_k'-f_kf_\ell'})|/|f|^4 + 2|\sum_\ell f_\ell'\bar f_\ell||\sum_\ell f_\ell(\overline{f_\ell f_k'-f_kf_\ell'})|/|f|^6) \leq$$
$$\leq c\sum_\ell |f_\ell'|^2/|f_\ell|.$$

This concludes the proof. ¶

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