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Research Article

Closed Int Soft *BCI*-Ideals and Int Soft c-*BCI*-Ideals

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The aim of this paper is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notions of closed intersectional soft *BCI*-ideals and intersectional soft commutative *BCI*-ideals are introduced, and related properties are investigated. Conditions for an intersectional soft *BCI*-ideal to be closed are provided. Characterizations of an intersectional soft commutative *BCI*-ideal are established, and a new intersectional soft c-*BCI*-ideal from an old one is constructed.

1. Introduction

The real world is inherently uncertain, imprecise, and vague. Various problems in system identification involve characteristics which are essentially nonprobabilistic in nature [1]. In response to this situation Zadeh [2] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [3]. To solve complicated problem in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can be considered as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [4]. Maji et al. [5] and Molodtsov [4] suggested that one reason for these difficulties may be the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [4] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [5] described the application of soft set theory to a decision making problem. Maji et al. [6] also studied several operations on the theory of soft sets. Aktaş and Çağman [7] studied the basic concepts of soft set theory and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun and Park [8] studied applications of soft sets in ideal theory of BCK/BCI-algebras. In 2012, Jun et al. [9, 10] introduced the notion of intersectional soft sets, and considered its applications to BCK/BCI-algebras. Independent of Jun et al.'s introduction, Çağman and Çitak [11] also studied soft int-group and its applications to group theory. Also, Jun [12] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [13–26] for further information regarding algebraic structures/properties of soft set theory. Present authors [10] introduced the notion of int soft BCK/BCI-ideals in BCK/BCI-algebras. As a continuation of the paper [10], we introduce the notion of closed int soft BCI-ideals and int soft c-BCI-ideals in BCI-algebras and investigate related properties. We discuss relations between a closed int soft BCI-ideal and an int soft BCI-ideal and provide conditions for an int soft BCI-ideal to be closed. We establish characterizations of an int soft c-BCI-ideal and construct a new intersectional soft c-BCI-ideal from an old one.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z\in X)\,(((x*y)*(x*z))*(z*y)=0)$;

(II) $(\forall x,y\in X)\,((x*(x*y))*y=0)$;

(III) $(\forall x\in X)\,(x*x=0)$;

(IV) $(\forall x,y\in X)\,(x*y=0, y*x=0 \;\Rightarrow\; x=y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x\in X)\;(0*x=0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x\in X)\,(x*0=x)$;

(a2) $(\forall x,y,z\in X)\,(x\le y \;\Rightarrow\; x*z\le y*z, z*y\le z*x)$;

(a3) $(\forall x,y,z\in X)\,((x*y)*z=(x*z)*y)$;

(a4) $(\forall x,y,z\in X)\,((x*z)*(y*z)\le x*y)$, where $x \leq y$ if and only if $x * y = 0$. In a BCI-algebra X, the following hold:

(b1) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$;

(b2) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$.

A BCI-algebra X is said to be *commutative* (see [27]) if

$$(\forall x, y \in X) \quad (x \leq y \Longrightarrow x = y * (y * x)). \tag{2.1}$$

Proposition 2.1. *A BCI-algebra X is commutative if and only if it satisfies*

$$(\forall x, y \in X) \quad (x * (x * y) = y * (y * (x * (x * y)))). \tag{2.2}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCI-algebra X is called a *BCI-ideal* of X if it satisfies

$$0 \in I, \tag{2.3}$$

$$(\forall x \in X)\ (\forall y \in I) \quad (x * y \in I \Longrightarrow x \in I). \tag{2.4}$$

A BCI-ideal I of a BCI-algebra X satisfies

$$(\forall x \in X)\ (\forall y \in I) \quad (x \leq y \Longrightarrow x \in I). \tag{2.5}$$

A BCI-ideal I of a BCI-algebra X is said to be *closed* if it satisfies

$$(\forall x \in X) \quad (x \in I \Longrightarrow 0 * x \in I). \tag{2.6}$$

A subset I of a BCI-algebra X is called a *commutative BCI-ideal* (briefly, *c-BCI-ideal*) of X (see [28]) if it satisfies (2.3) and

$$(x * y) * z \in I, \quad z \in I \Longrightarrow x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I \tag{2.7}$$

for all $x, y, z \in X$.

Proposition 2.2 (see [28]). *A BCI-ideal I of a BCI-algebra X is commutative if and only if $x*y \in I$ implies $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I$.*

Proposition 2.3 (see [28]). *Let I be a closed BCI-ideal of a BCI-algebra X. Then I is commutative if and only if it satisfies*

$$(\forall x, y \in X) \quad (x * y \in I \Longrightarrow x * (y * (y * x)) \in I). \tag{2.8}$$

Observe that every c-BCI-ideal is a BCI-ideal, but the converse is not true (see [28]).

We refer the reader to the books [29, 30] for further information regarding BCK/BCI-algebras.

A soft set theory is introduced by Molodtsov [4], and Çağman and Enginoğlu [31] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe.* Let $\mathcal{P}(U)$ denote the power set of U and $A, B, C, \ldots \subseteq E$.

Definition 2.4 (see [4, 31]). A soft set $\mathcal{F}_A$ over U is defined to be the set of ordered pairs

$$\mathcal{F}_A := \{(x, f_A(x)) : x \in E, f_A(x) \in \mathcal{P}(U)\}, \tag{2.9}$$

where $f_A : E \to \mathcal{P}(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

The function f_A is called the approximate function of the soft set $\mathcal{F}_A$. The subscript A in the notation f_A indicates that f_A is the approximate function of $\mathcal{F}_A$.

In what follows, denote by $S(U)$ the set of all soft sets over U.

Let $\mathcal{F}_A \in S(U)$. For any subset γ of U, the *γ-inclusive set* of $\mathcal{F}_A$, denoted by $\mathcal{F}_A^\gamma$, is defined to be the set

$$\mathcal{F}_A^\gamma := \{x \in A \mid \gamma \subseteq f_A(x)\}. \tag{2.10}$$

3. Closed Int Soft BCI-Ideals and Int Soft c-BCI-Ideals

Definition 3.1 (see [10]). Assume that E has a binary operation $\hookrightarrow$. For any nonempty subset A of E, a soft set $\mathcal{F}_A$ over U is said to be *intersectional* over U if its approximate function f_A satisfies

$$(\forall x, y \in A) \quad (x \hookrightarrow y \in A \implies f_A(x) \cap f_A(y) \subseteq f_A(x \hookrightarrow y)). \tag{3.1}$$

Definition 3.2 (see [12]). Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. Then $\mathcal{F}_A$ is called an *intersectional soft BCI-ideal* (briefly, *int soft BCI-ideal*) over U if the approximate function f_A of $\mathcal{F}_A$ satisfies

$$(\forall x \in A) \quad (f_A(0) \supseteq f_A(x)), \tag{3.2}$$

$$(\forall x, y \in A) \quad (f_A(x) \supseteq f_A(x * y) \cap f_A(y)). \tag{3.3}$$

Definition 3.3. Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. Then $\mathcal{F}_A$ is called an *intersectional soft commutative BCI-ideal* (briefly, *int soft c-BCI-ideal*) over U if the approximate function f_A of $\mathcal{F}_A$ satisfies (3.2) and

$$f_A((x * y) * z) \cap f_A(z) \subseteq f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \tag{3.4}$$

for all $x, y, z \in A$.

Example 3.4. Let $(\mathcal{U}, E) = (\mathcal{U}, X)$ where $X = \{0, a, 1, 2, 3\}$ is a BCI-algebra with the following Cayley table:

$$\begin{array}{c|ccccc} * & 0 & a & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 3 & 2 & 1 \\ a & a & 0 & 3 & 2 & 1 \\ 1 & 1 & 1 & 0 & 3 & 2 \\ 2 & 2 & 2 & 1 & 0 & 3 \\ 3 & 3 & 3 & 2 & 1 & 0 \end{array} \tag{3.5}$$

For subsets γ_1, γ_2, and γ_3 of $\mathcal{U}$ with $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3$, let $\mathcal{F}_E \in S(\mathcal{U})$ in which its approximation function f_E is defined as follows:

$$f_E : E \longrightarrow \mathcal{P}(\mathcal{U}), \qquad x \longmapsto \begin{cases} \gamma_1, & \text{if } x = 0, \\ \gamma_2, & \text{if } x = a, \\ \gamma_3, & \text{if } x \in \{1, 2, 3\}. \end{cases} \tag{3.6}$$

Then $\mathcal{F}_E$ is an int soft c-BCI-ideal over $\mathcal{U}$.

Theorem 3.5. *Let $(\mathcal{U}, E) = (\mathcal{U}, X)$ where X is a BCI-algebra. Then every int soft c-BCI-ideal is an int soft BCI-ideal.*

Proof. Let $\mathcal{F}_A$ be an int soft c-BCI-ideal over $\mathcal{U}$ where A is a subalgebra of E. Taking $y = 0$ in (3.4) and using (a1) and (III) imply that

$$\begin{aligned} f_A(x) &= f_A(x * 0) = f_A(x * ((0 * (0 * x)) * (0 * (0 * (x * 0))))) \\ &\supseteq f_A((x * 0) * z) \cap f_A(z) = f_A(x * z) \cap f_A(z) \end{aligned} \tag{3.7}$$

for all $x, z \in A$. Therefore $\mathcal{F}_A$ is an int soft BCI-ideal over $\mathcal{U}$. □

The following example shows that the converse of Theorem 3.5 is not true.

Example 3.6. Let $(\mathcal{U}, E) = (\mathcal{U}, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCI-algebra with the following Cayley table:

$$\begin{array}{c|ccccc} * & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 4 & 3 & 0 \end{array} \tag{3.8}$$

Let γ_1, γ_2, and γ_3 be subsets of U such that $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3$. Let $\mathcal{F}_E \in S(U)$ in which its approximation function f_E is defined as follows:

$$f_E : E \longrightarrow \mathcal{P}(U), x \longmapsto \begin{cases} \gamma_1, & \text{if } x = 0, \\ \gamma_2, & \text{if } x = 1, \\ \gamma_3, & \text{if } x \in \{2,3,4\}. \end{cases} \tag{3.9}$$

Routine calculations show that $\mathcal{F}_E$ is an int soft BCI-ideal over U. But it is not an int soft c-BCI-ideal over U since

$$f_E(2 * ((3 * (3 * 2)) * (0 * (0 * (2 * 3))))) = \gamma_3 \not\supseteq \gamma_1 = f_E((2 * 3) * 0) \cap f_E(0). \tag{3.10}$$

We provide conditions for an int soft BCI-ideal to be an int soft c-BCI-ideal.

Theorem 3.7. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. Then the following are equivalent:*

(1) *$\mathcal{F}_A$ is an int soft c-BCI-ideal over U;*

(2) *$\mathcal{F}_A$ is an int soft BCI-ideal over U and its approximate function f_A satisfies:*

$$(\forall x, y \in A) \quad (f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \supseteq f_A(x * y)). \tag{3.11}$$

Proof. Assume that $\mathcal{F}_A$ is an int soft c-BCI-ideal over U. Then $\mathcal{F}_A$ is an int soft BCI-ideal over U (see Theorem 3.5). If we take $z = 0$ in (3.4) and use (a1) and (3.2), then we have (3.11).

Conversely, let $\mathcal{F}_A$ be an int soft BCI-ideal over U such that its approximate function f_A satisfies (3.11). Then $f_A(x * y) \supseteq f_A((x * y) * z) \cap f_A(z)$ for all $x, y, z \in A$ by (3.3), which implies from (3.11) that

$$f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \supseteq f_A((x * y) * z) \cap f_A(z) \tag{3.12}$$

for all $x, y, z \in A$. Therefore $\mathcal{F}_A$ is an int soft c-BCI-ideal over U. $\square$

Definition 3.8. Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. An int soft BCI-ideal $\mathcal{F}_A$ over U is said to be *closed* if the approximate function f_A of $\mathcal{F}_A$ satisfies

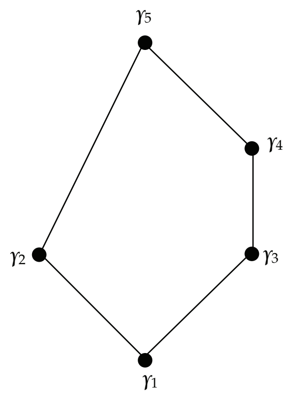
$$(\forall x \in A) \quad (f_A(0 * x) \supseteq f_A(x)). \tag{3.13}$$

Example 3.9. Let $(\mathcal{U}, E) = (\mathcal{U}, X)$ where $X = \{0, 1, 2, a, b\}$ is a BCI-algebra with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

(3.14)

Let $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ be a class of subsets of $\mathcal{U}$ which is a poset with the following Hasse diagram:



(3.15)

Let $\mathcal{F}_E \in S(\mathcal{U})$ in which its approximation function f_E is defined as follows:

$$f_E : E \longrightarrow \mathcal{P}(\mathcal{U}), \qquad x \longmapsto \begin{cases} \gamma_5, & \text{if } x = 0, \\ \gamma_2, & \text{if } x = 1, \\ \gamma_4, & \text{if } x = 2, \\ \gamma_3, & \text{if } x = a, \\ \gamma_1, & \text{if } x = b. \end{cases} \tag{3.16}$$

Then $\mathcal{F}_E$ is a closed int soft BCI-ideal over $\mathcal{U}$.

Example 3.10. Let $(\mathcal{U}, E) = (\mathcal{U}, X)$ where $X = \{2^n \mid n \in \mathbb{Z}\}$ is a BCI-algebra with a binary operation "÷" (usual division). Let $\mathcal{F}_E \in S(\mathcal{U})$ in which its approximation function f_E is defined as follows:

$$f_E : E \longrightarrow \mathcal{P}(\mathcal{U}), \qquad x \longmapsto \begin{cases} \gamma_1, & \text{if } n \geq 0, \\ \gamma_2, & \text{if } n < 0, \end{cases} \tag{3.17}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \supsetneq \gamma_2$. Then $\mathcal{F}_E$ is an int soft BCI-ideal over U which is not closed since

$$f_E\left(1 \div 2^3\right) = f_E\left(2^{-3}\right) = \gamma_2 \not\supseteq \gamma_1 = f_E\left(2^3\right). \tag{3.18}$$

Theorem 3.11. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then an int soft BCI-ideal over U is closed if and only if it is an int soft algebra over U.*

Proof. Let $\mathcal{F}_A$ be an int soft BCI-ideal over U. If $\mathcal{F}_A$ is closed, then $f_A(0 * x) \supseteq f_A(x)$ for all $x \in A$. It follows from (3.3) that

$$f_A(x * y) \supseteq f_A((x * y) * x) \cap f_A(x) = f_A(0 * y) \cap f_A(x) \supseteq f_A(x) \cap f_A(y) \tag{3.19}$$

for all $x, y \in A$. Hence $\mathcal{F}_A$ is an int soft algebra over U.

Conversely, let $\mathcal{F}_A$ be an int soft BCI-ideal over U which is also an int soft algebra over U. Then

$$f_A(0 * x) \supseteq f_A(0) \cap f_A(x) = f_A(x) \tag{3.20}$$

for all $x \in A$. Therefore $\mathcal{F}_A$ is closed. □

Let X be a BCI-algebra and $B(X) := \{x \in X \mid 0 \leq x\}$. For any $x \in X$ and $n \in \mathbb{N}$, we define x^n by

$$x^1 = x, \qquad x^{n+1} = x * (0 * x^n). \tag{3.21}$$

If there is an $n \in \mathbb{N}$ such that $x^n \in B(X)$, then we say that x is of *finite periodic* (see [32]), and we denote its period $|x|$ by

$$|x| = \min\{n \in \mathbb{N} \mid x^n \in B(X)\}. \tag{3.22}$$

Otherwise, x is of infinite period and denoted by $|x| = \infty$.

Theorem 3.12. *Let $(U, E) = (U, X)$ where X is a BCI-algebra in which every element is of finite period. Then every int soft BCI-ideal over U is closed.*

Proof. Let $\mathcal{F}_E$ be an int soft BCI-ideal over U. For any $x \in E$, assume that $|x| = n$. Then $x^n \in B(X)$. Note that

$$\begin{aligned}
\left(0 * x^{n-1}\right) * x &= \left(0 * \left(0 * \left(0 * x^{n-1}\right)\right)\right) * x \\
&= (0 * x) * \left(0 * \left(0 * x^{n-1}\right)\right) = 0 * \left(x * \left(0 * x^{n-1}\right)\right) \\
&= 0 * x^n = 0,
\end{aligned} \tag{3.23}$$

and so $f_E((0*x^{n-1})*x) = f_E(0) \supseteq f_E(x)$ by (3.2). It follows from (3.3) that

$$f_E\left(0*x^{n-1}\right) \supseteq f_E\left(\left(0*x^{n-1}\right)*x\right) \cap f_E(x) \supseteq f_E(x). \tag{3.24}$$

Also, note that

$$\begin{aligned}\left(0*x^{n-2}\right)*x &= \left(0*\left(0*\left(0*x^{n-2}\right)\right)\right)*x \\ &= (0*x)*\left(0*\left(0*x^{n-2}\right)\right) = 0*\left(x*\left(0*x^{n-2}\right)\right) \\ &= 0*x^{n-1},\end{aligned} \tag{3.25}$$

which implies from (3.24) that

$$f_E\left(\left(0*x^{n-2}\right)*x\right) = f_E\left(0*x^{n-1}\right) \supseteq f_E(x). \tag{3.26}$$

Using (3.3), we have

$$f_E\left(0*x^{n-2}\right) \supseteq f_E\left(\left(0*x^{n-2}\right)*x\right) \cap f_E(x) \supseteq f_E(x). \tag{3.27}$$

Continuing this process, we have $f_E(0*x) \supseteq f_E(x)$ for all $x \in E$. Therefore $\mathcal{F}_E$ is closed. □

Lemma 3.13 (see [10]). *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. If $\mathcal{F}_A$ is an int soft BCI-ideal over U, then the approximate function f_A satisfies the following condition:*

$$(\forall x, y, z \in A) \quad (x*y \leq z \Longrightarrow f_A(x) \supseteq f_A(y) \cap f_A(z)). \tag{3.28}$$

Proposition 3.14. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. If the approximate function f_A of $\mathcal{F}_A$ satisfies* (3.2) *and* (3.28)*, then $\mathcal{F}_A$ is an int soft BCI-ideal over U.*

Proof. Note that $x*(x*y) \leq y$ by (II), and thus $f_A(x) \supseteq f_A(x*y) \cap f_A(y)$ by (3.28). Therefore $\mathcal{F}_A$ is an int soft BCI-ideal over U. □

Theorem 3.15. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathcal{F}_A$ be a closed int soft BCI-ideal over U. Then the following are equivalent:*

(1) *$\mathcal{F}_A$ is an int soft c-BCI-ideal over U;*

(2) *the approximate function f_A of $\mathcal{F}_A$ satisfies:*

$$(\forall x, y \in A) \quad (f_A(x*(y*(y*x))) \supseteq f_A(x*y)). \tag{3.29}$$

Proof. Assume that $\mathcal{F}_A$ is an int soft c-BCI-ideal over U. Note that

$$\begin{aligned}(x*(y*(y*x)))*(x*((y*(y*x))*(0*(0*(x*y)))))\\ \leq ((y*(y*x))*(0*(0*(x*y))))*(y*(y*x))\\ = ((y*(y*x))*(y*(y*x)))*(0*(0*(x*y)))\\ = 0*(0*(0*(x*y))) = 0*(x*y)\end{aligned} \tag{3.30}$$

for all $x, y \in A$. Using Lemma 3.13, (3.11), and (3.13), we have

$$\begin{aligned}&f_A(x*(y*(y*x)))\\ &\supseteq f_A(x*((y*(y*x))*(0*(0*(x*y))))) \cap f_A(0*(x*y))\\ &\supseteq f_A(x*y) \cap f_A(0*(x*y)) = f_A(x*y)\end{aligned} \tag{3.31}$$

for all $x, y \in A$. Now suppose that the approximate function f_A of $\mathcal{F}_A$ satisfies (3.29). Since

$$\begin{aligned}(x*((y*(y*x))*(0*(0*(x*y)))))*(x*(y*(y*x)))\\ \leq (y*(y*x))*((y*(y*x))*(0*(0*(x*y))))\\ \leq 0*(0*(x*y)),\end{aligned} \tag{3.32}$$

it follows from Lemma 3.13, (3.13), and (3.29) that

$$\begin{aligned}&f_A(x*((y*(y*x))*(0*(0*(x*y)))))\\ &\supseteq f_A(x*(y*(y*x))) \cap f_A(0*(0*(x*y)))\\ &\supseteq f_A(x*y) \cap f_A(0*(0*(x*y))) = f_A(x*y)\end{aligned} \tag{3.33}$$

for all $x, y \in A$. By Theorem 3.7, $\mathcal{F}_A$ is an int soft c-BCI-ideal over U. □

Theorem 3.16. *Let $(U, E) = (U, X)$ where X is a commutative BCI-algebra. Then every closed int soft BCI-ideal is an int soft c-BCI-ideal.*

Proof. Let $\mathcal{F}_A$ be a closed int soft BCI-ideal over U where A is a subalgebra of E. Using (a3), (b1), (I), (III), and Proposition 2.1, we have

$$\begin{aligned}(x*(y*(y*x)))*(x*y) &= (x*(x*y))*(y*(y*x))\\ &= (y*(y*(x*(x*y))))*(y*(y*x))\\ &= (y*(y*(y*x)))*(y*(x*(x*y)))\\ &= (y*x)*(y*(x*(x*y)))\\ &\leq (x*(x*y))*x = 0*(x*y).\end{aligned} \tag{3.34}$$

It follows from Lemma 3.13 and (3.13) that

$$f_A(x*(y*(y*x))) \supseteq f_A(x*y) \cap f_A(0*(x*y)) = f_A(x*y) \tag{3.35}$$

for all $x, y \in A$. Therefore, by Theorem 3.15, $\mathcal{F}_A$ is an int soft c-BCI-ideal over U. □

Using the notion of γ-inclusive sets, we consider a characterization of an int soft c-BCI-ideal.

Lemma 3.17 (see [25]). *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. Then the following are equivalent:*

(1) *$\mathcal{F}_A$ is an int soft BCI-ideal over U;*

(2) *the nonempty γ-inclusive set of $\mathcal{F}_A$ is a BCI-ideal of A for any $\gamma \subseteq U$.*

Theorem 3.18. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. Then the following are equivalent:*

(1) *$\mathcal{F}_A$ is an int soft c-BCI-ideal over U;*

(2) *the nonempty γ-inclusive set of $\mathcal{F}_A$ is a c-BCI-ideal of A for any $\gamma \subseteq U$.*

Proof. Assume that $\mathcal{F}_A$ is an int soft c-BCI-ideal over U. Then $\mathcal{F}_A$ is an int soft BCI-ideal over U by Theorem 3.5. Hence $\mathcal{F}_A^{\gamma}$ is a BCI-ideal of A for all $\gamma \subseteq U$ by Lemma 3.17. Let $\gamma \subseteq U$ and $x, y \in A$ be such that $x*y \in \mathcal{F}_A^{\gamma}$. Then $f_A(x*y) \supseteq \gamma$, and so

$$f_A(x*((y*(y*x))*(0*(0*(x*y))))) \supseteq f_A(x*y) \supseteq \gamma \tag{3.36}$$

by Theorem 3.7. Thus

$$x*((y*(y*x))*(0*(0*(x*y)))) \in \mathcal{F}_A^{\gamma}. \tag{3.37}$$

It follows from Proposition 2.2 that $\mathcal{F}_A^{\gamma}$ is a c-BCI-ideal of A.

Conversely, suppose that the nonempty γ-inclusive set of $\mathcal{F}_A$ is a c-BCI-ideal of A for any $\gamma \subseteq U$. Then $\mathcal{F}_A^{\gamma}$ is a BCI-ideal of A for all $\gamma \subseteq U$. Hence $\mathcal{F}_A$ is an int soft BCI-ideal over U by Lemma 3.17. Let $x, y \in A$ be such that $f_A(x*y) = \gamma$. Then $x*y \in \mathcal{F}_A^{\gamma}$, and so

$$x*((y*(y*x))*(0*(0*(x*y)))) \in \mathcal{F}_A^{\gamma} \tag{3.38}$$

by Proposition 2.2. Hence

$$f_A(x*((y*(y*x))*(0*(0*(x*y))))) \supseteq \gamma = f_A(x*y). \tag{3.39}$$

It follows from Theorem 3.7 that $\mathcal{F}_A$ is an int soft c-BCI-ideal over U. □

The c-BCI-ideals $\mathcal{F}_A^{\gamma}$ in Theorem 3.18 are called the *inclusive* c-*BCI-ideals* of $\mathcal{F}_A$.

Theorem 3.19. *Let $(\mathcal{U}, E) = (\mathcal{U}, X)$ where X is a BCI-algebra. Let $\mathcal{F}_E, \mathcal{G}_E \in S(\mathcal{U})$ such that*

(i) $(\forall x \in E) \quad (f_E(x) \supseteq g_E(x))$;

(ii) *$\mathcal{F}_E$ and $\mathcal{G}_E$ are int soft BCI-ideals over $\mathcal{U}$.*

If $\mathcal{F}_E$ is closed and $\mathcal{G}_E$ is an int soft c-BCI-ideal over $\mathcal{U}$, then $\mathcal{F}_E$ is also an int soft c-BCI-ideal over $\mathcal{U}$.

Proof. Assume that $\mathcal{F}_E$ is closed and $\mathcal{G}_E$ is an int soft c-BCI-ideal over $\mathcal{U}$. Let γ be a subset of $\mathcal{U}$ such that $\mathcal{F}_E^{\gamma} \neq \emptyset \neq \mathcal{G}_E^{\gamma}$. Then $\mathcal{F}_E^{\gamma}$ and $\mathcal{G}_E^{\gamma}$ are BCI-ideals of E and obviously $\mathcal{F}_E^{\gamma} \supseteq \mathcal{G}_E^{\gamma}$. Let $x \in \mathcal{F}_E^{\gamma}$. Then $f_E(x) \supseteq \gamma$, and so $f_E(0 * x) \supseteq f_E(x) \supseteq \gamma$ since $\mathcal{F}_E$ is closed. Thus $0 * x \in \mathcal{F}_E^{\gamma}$, and thus $\mathcal{F}_E^{\gamma}$ is a closed BCI-ideal of E. Since $\mathcal{G}_E$ is an int soft c-BCI-ideal over $\mathcal{U}$, it follows from Theorem 3.18 that $\mathcal{G}_E^{\gamma}$ is a c-BCI-ideal of E. Let $x, y \in E$ be such that $x * y \in \mathcal{F}_E^{\gamma}$. Then $0 * (x * y) \in \mathcal{F}_E^{\gamma}$. Since $(x * (x * y)) * y = 0 \in \mathcal{G}_E^{\gamma}$, it follows from Proposition 2.2 that

$$\begin{aligned}
&(x * (x * y)) * (y * (y * (x * (x * y)))) \\
&\quad = (x * (x * y)) * ((y * (y * (x * (x * y)))) * (0 * (0 * ((x * (x * y)) * y)))) \\
&\quad \in \mathcal{G}_E^{\gamma} \subseteq \mathcal{F}_E^{\gamma},
\end{aligned} \tag{3.40}$$

and so from (a3) that

$$(x * (y * (y * (x * (x * y))))) * (x * y) \in \mathcal{F}_E^{\gamma}. \tag{3.41}$$

Hence $x * (y * (y * (x * (x * y)))) \in \mathcal{F}_E^{\gamma}$ by (2.4). Note that

$$\begin{aligned}
&(x * (y * (y * x))) * (x * (y * (y * (x * (x * y))))) \\
&\quad \leq (y * (y * (x * (x * y)))) * (y * (y * x)) \\
&\quad \leq (y * x) * (y * (x * (x * y))) \\
&\quad \leq (x * (x * y)) * x = 0 * (x * y) \in \mathcal{F}_E^{\gamma}.
\end{aligned} \tag{3.42}$$

Using (2.5) and (2.4), we have $x * (y * (y * x)) \in \mathcal{F}_E^{\gamma}$. Hence $\mathcal{F}_E^{\gamma}$ is a c-BCI-ideal of E. Therefore $\mathcal{F}_E$ is an int soft c-BCI-ideal over $\mathcal{U}$ by Theorem 3.18. □

Theorem 3.20. *Let $(\mathcal{U}, E) = (\mathcal{U}, X)$ where X is a BCI-algebra. Let $\mathcal{F}_E \in S(\mathcal{U})$ and define a soft set $\mathcal{F}_E^*$ over $\mathcal{U}$ by*

$$f_E^* : E \longrightarrow \mathcal{P}(\mathcal{U}), \qquad x \longmapsto \begin{cases} f_E(x), & \text{if } x \in \mathcal{F}_E^{\gamma}, \\ \delta, & \text{otherwise}, \end{cases} \tag{3.43}$$

where γ and δ are subset of $\mathcal{U}$ with $\delta \subsetneq f_E(x)$. If $\mathcal{F}_E$ is an int soft c-BCI-ideal over $\mathcal{U}$, then so is $\mathcal{F}_E^*$.

Proof. If $\mathcal{F}_E$ is an int soft c-BCI-ideal over U, then $\mathcal{F}_E^{\gamma}$ is a c-BCI-ideal of A for any $\gamma \subseteq U$. Hence $0 \in \mathcal{F}_E^{\gamma}$, and so $f_E^*(0) = f_E(0) \supseteq f_E(x) \supseteq f_E^*(x)$ for all $x \in A$. Let $x, y, z \in A$. If $(x * y) * z \in \mathcal{F}_E^{\gamma}$ and $z \in \mathcal{F}_E^{\gamma}$, then $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in \mathcal{F}_E^{\gamma}$ and so

$$
\begin{aligned}
&f_E^*(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\
&\quad = f_E(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\
&\quad \supseteq f_E((x * y) * z) \cap f_E(z) = f_E^*((x * y) * z) \cap f_E^*(z).
\end{aligned}
\tag{3.44}
$$

If $(x * y) * z \notin \mathcal{F}_E^{\gamma}$ or $z \notin \mathcal{F}_E^{\gamma}$, then $f_E^*((x * y) * z) = \delta$ or $f_E^*(z) = \delta$. Hence

$$
f_E^*(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \supseteq \delta = f_E^*((x * y) * z) \cap f_E^*(z). \tag{3.45}
$$

This shows that $\mathcal{F}_E^*$ is an int soft c-BCI-ideal over U. □

Theorem 3.21. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then any c-BCI-ideal of E can be realized as an inclusive c-BCI-ideal of some int soft c-BCI-ideal over U.*

Proof. Let A be a c-BCI-ideal of E. For any subset $\gamma \subsetneq U$, let $\mathcal{F}_A$ be a soft set over U defined by

$$
f_A : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \gamma, & \text{if } x \in A, \\ \emptyset, & \text{if } x \notin A. \end{cases} \tag{3.46}
$$

Obviously, $f_A(0) \supseteq f_A(x)$ for all $x \in E$. For any $x, y, z \in E$, if $(x * y) * z \in A$ and $z \in A$ then $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in A$. Hence

$$
f_A((x * y) * z) \cap f_A(z) = \gamma = f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))). \tag{3.47}
$$

If $(x * y) * z \notin A$ or $z \notin A$ then $f_A((x * y) * z) = \emptyset$ or $f_A(z) = \emptyset$. It follows that

$$
f_A(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \supseteq \emptyset = f_A((x * y) * z) \cap f_A(z). \tag{3.48}
$$

Therefore $\mathcal{F}_A$ is an int soft c-BCI-ideal over U, and clearly $\mathcal{F}_A^{\gamma} = A$. This completes the proof. □

4. Conclusion

We have introduced the notions of closed int soft BCI-ideals and int soft commutative BCI-ideals, and investigated related properties. We have provided conditions for an int soft BCI-ideal to be closed, and established characterizations of an int soft commutative BCI-ideal. We have constructed a new int soft c-BCI-ideal from old one.

On the basis of these results, we will apply the theory of int soft sets to the another type of ideals, filters, and deductive systems in BCK/BCI-algebras, Hilbert algebras, MV-algebras, MTL-algebras, BL-algebras, and so forth, in future study.

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