Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 137507, 7 pages
doi:10.1155/2012/137507

Research Article

New Sharp Bounds for the Bernoulli Numbers and Refinement of Becker-Stark Inequalities

Hua-feng Ge

Department of Mathematics, Taizhou University, Linhai, Zhejiang 317000, China

Correspondence should be addressed to Hua-feng Ge, gehuafeng2011@163.com

Received 25 May 2011; Accepted 25 June 2011

Academic Editor: Kai Diethelm

We obtain new sharp bounds for the Bernoulli numbers: $2(2n)!/(\pi^{2n}(2^{2n}-1)) < |B_{2n}| \le (2(2^{2k}-1)/2^{2k})\zeta(2k)(2n)!/(\pi^{2n}(2^{2n}-1))$, $n = k, k+1, \ldots,\ k \in N^{+}$, and establish sharpening of Papenfuss's inequalities, the refinements of Becker-Stark, and Steckin's inequalities. Finally, we show a new simple proof of Ruehr-Shafer inequality.

1. Introduction

The classical Bernoulli numbers B_n $(n = 1, 2, \ldots)$ can be defined by (see [1])

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi. \tag{1.1}$$

Reference [2] shows a upper bound for $|B_{2n}| = (-1)^{n+1} B_{2n}$

$$|B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{1-2n}}, \quad n = 1, 2, \ldots. \tag{1.2}$$

On the other hand, [3] presents a lower bound for $|B_{2n}|$ as follows:

$$|B_{2n}| > \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{-2n}}, \quad n = 1, 2, \ldots. \tag{1.3}$$

On the basis of (1.2) and (1.3), Alzer [4] obtains the further results.

Theorem A. *For all integers $n \geq 1$ one has*

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{\alpha-2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{\beta-2n}}, \tag{1.4}$$

with the best possible constants $\alpha = 0$ and $\beta = 2 + \ln(1-6/\pi^2)/\ln 2 \approx 0.6491\cdots$.

In this paper, we obtain new bounds for the Bernoulli numbers as follows.

Theorem 1.1. *Let $k \in \mathbf{N}^+$, $n = k, k+1, \ldots$, then*

$$\frac{2(2n)!}{\pi^{2n}(2^{2n}-1)} < |B_{2n}| \leq \frac{2(2^{2k}-1)}{2^{2k}}\zeta(2k)\frac{(2n)!}{\pi^{2n}(2^{2n}-1)}. \tag{1.5}$$

The equality holds in (1.5) if and only if $n = k$. Furthermore, 2 and $(2(2^{2k}-1)/2^{2k})\zeta(2k)$ are the best constants in (1.5).

In the following, we study on some trigonometric inequalities.
Mitrinovic [5] gives us a result which belongs to Steckin.

Theorem B. *If $0 < x < \pi/2$, then*

$$\frac{4}{\pi}\frac{x}{\pi-2x} < \tan x. \tag{1.6}$$

Now, we show a upper bound for $\tan x$ and obtain the following sharp Steckin's inequalities.

Theorem 1.2. *If $0 < x < \pi/2$, then*

$$\frac{4}{\pi}\frac{x}{\pi-2x} < \tan x < \pi\frac{x}{\pi-2x} \tag{1.7}$$

or

$$\frac{4}{\pi}\frac{1}{\pi-2x} < \frac{\tan x}{x} < \pi\frac{1}{\pi-2x}. \tag{1.8}$$

Furthermore, $4/\pi$ and π are the best constants in (1.7) and (1.8).

Kuang [6] gives us the further results described as Becker-Stark inequalities

Theorem C. *Let $0 < t < 1$, then*

$$\frac{4}{\pi}\frac{t}{1-t^2} < tan\frac{\pi}{2}t < \frac{\pi}{2}\frac{t}{1-t^2}. \tag{1.9}$$

Furthermore, $4/\pi$ and $\pi/2$ are the best constants in (1.9).

Let $x = (\pi/2)t$ in (1.9), then Theorem C is equivalent to.

Theorem D. *Let* $0 < x < \pi/2$, *then*

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}. \tag{1.10}$$

Furthermore, 8 and π^2 *are the best constants in* (1.10).

Clearly, Becker-Stark inequalities (1.10) are the generalization of the strengthened Steckin's inequalities (1.8).

On the other hand, Papenfuss [7] proposes an open problem described as the following statement.

Theorem E. *Let* $0 \le x < \pi/2$, *then*

$$x\sec^2 x - \tan x \le \frac{8\pi^2 x^3}{(\pi^2 - 4x^2)^2}. \tag{1.11}$$

Bach [8] prove Theorem E and obtain a further result.

Theorem F. *Let* $0 \le x < \pi/2$, *then*

$$x\sec^2 x - \tan x \le \frac{2\pi^4}{3}\frac{x^3}{(\pi^2 - 4x^2)^2}. \tag{1.12}$$

In this section, we first obtain sharp Papenfuss-Bach inequalities described as Theorem 1.3.

Theorem 1.3. *Let* $0 < x < \pi/2$, *then*

$$\frac{64x^3}{(\pi^2 - 4x^2)^2} < x\sec^2 x - \tan x < \frac{2\pi^4}{3}\frac{x^3}{(\pi^2 - 4x^2)^2}. \tag{1.13}$$

Furthermore, 64 and $2\pi^4/3$ *are the best constants in* (1.13).

The inequalities (1.13) are equivalent to

$$\frac{64x}{(\pi^2 - 4x^2)^2} < \frac{x\sec^2 x - \tan x}{x^2} < \frac{2\pi^4}{3}\frac{x}{(\pi^2 - 4x^2)^2}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{1.14}$$

That is,

$$\frac{64x}{(\pi^2 - 4x^2)^2} < \left(\frac{\tan x}{x}\right)' < \frac{2\pi^4}{3}\frac{x}{(\pi^2 - 4x^2)^2}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{1.15}$$

Then, integrating the three functions in (1.15) from 0 to x, where $x \in (0, \pi/2)$, we obtain the following refinement of Becker-Stark inequalities.

Theorem 1.4 (Refinement of Becker-Stark Inequalities). *Let* $0 < x < \pi/2$, *then*

$$\frac{8}{\pi^2 - 4x^2} + \left(1 - \frac{8}{\pi^2}\right) < \frac{\tan x}{x} < \frac{\pi^4}{12}\frac{1}{\pi^2 - 4x^2} + \left(1 - \frac{\pi^2}{12}\right). \tag{1.16}$$

An application of Theorem 1.4 leads to Theorem 1.5 (the refinement of Steckin's inequalities).

Theorem 1.5 (Refinement of Steckin's Inequalities). *If* $0 < x < \pi/2$, *then*

$$\frac{4}{\pi}\frac{1}{\pi - 2x} + \left(1 - \frac{8}{\pi^2}\right) < \frac{\tan x}{x} < \frac{\pi^3}{12}\frac{1}{\pi - 2x} + \left(1 - \frac{\pi^2}{12}\right). \tag{1.17}$$

Finally, we will show a new proof of Ruehr-Shafer inequality.

Theorem G (Ruehr-Shafer Inequality, see [8]). *Let* $0 \le x < \pi/2$, *then*

$$x\sec^2 x - \tan x \le 2\pi^2 \frac{\tan x - x}{\pi^2 - 4x^2}. \tag{1.18}$$

2. Two Lemmas

Lemma 2.1 (see [9, Lemma 2.1]). *The function* $(1-(1/2^n))\zeta(n)$ $(n = 1, 2, \ldots)$ *is decreasing, where* $\zeta(n)$ *is the Riemann's zeta function.*

Lemma 2.2 (see [10]). *Let* l_n *and* m_n $(n = 1, 2, \ldots)$ *be real numbers, and let the power series* $L(x) = \sum_{n=1}^{\infty} l_n x^n$ *and* $M(x) = \sum_{n=1}^{\infty} m_n x^n$ *be convergent for* $|x| < R$. *If* $m_n > 0$ *for* $n = 1, 2, \ldots$ *and if* l_n/m_n *is strictly decreasing for* $n = 1, 2, \ldots$, *then the function* $L(x)/M(x)$ *is strictly decreasing on* $(0, R)$.

3. Proof of Theorem 1.1

Using the representation

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|, \quad n = 1, 2, \ldots, \tag{3.1}$$

(cf. [11, page 266]), we have

$$G(n) = \frac{|B_{2n}|\pi^{2n}(2^{2n} - 1)}{(2n)!} = 2\left(1 - \frac{1}{2^{2n}}\right)\zeta(2n), \quad n = k, k+1, \ldots;\ k \in \mathbf{N}^+. \tag{3.2}$$

From Lemma 2.1, we know that $G(n)$ is decreasing and $G(k) = (2(2^{2k}-1)/2^{2k})\zeta(2k)$, $G(+\infty) = 2\lim_{n\to\infty}\zeta(2n) = 2$. Then, the proof of Theorem 1.1 is complete.

4. Proofs of Theorem 1.3 and G

4.1. Proof of Theorem 1.3.

The following power series expansion can be found in [12]:

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}(-1)^{n-1}B_{2n}x^{2n-1} = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-1}, \quad |x| < \frac{\pi}{2}. \tag{4.1}$$

Then

$$\begin{aligned} \sec^2 x = (\tan x)' &= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-1)|B_{2n}|x^{2n-2}, \quad |x| < \frac{\pi}{2}, \\ x\sec^2 x - \tan x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-2)|B_{2n}|x^{2n-1}, \quad |x| < \frac{\pi}{2}. \end{aligned} \tag{4.2}$$

Let

$$A(x) = \frac{x\sec^2 x - \tan x}{x^3/(\pi^2-4x^2)^2} = \frac{L(x)}{M(x)}, \tag{4.3}$$

where

$$\begin{aligned} L(x) = x\sec^2 x - \tan x &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-2)|B_{2n}|x^{2n-1}, \quad |x| < \frac{\pi}{2}, \\ M(x) = \frac{x^3}{(\pi^2-4x^2)^2} &= \sum_{n=2}^{\infty} \frac{2n-2}{32}\left(\frac{2}{\pi}\right)^{2n} x^{2n-1}, \quad |x| < \frac{\pi}{2}. \end{aligned} \tag{4.4}$$

Then, $l_n = ((2^{2n}(2^{2n}-1)/(2n)!)\ (2n-2)|B_{2n}|$, $m_n = ((2n-2)/32)\ (2/\pi)^{2n} > 0\ (n \geq 2)$ and $l_n/m_n = 64(1-1/2^{2n})\zeta(2n)$. So, l_n/m_n is decreasing by Lemma 2.1. Therefore, $A(x) = L(x)/M(x)$is decreasing on $(0, \pi/2)$ by Lemma 2.2. At the same time, $\lim_{x\to 0^+} A(x) = 2\pi^4/3$ and $\lim_{x\to(\pi/2)^-} A(x) = 64$, so 64 and $2\pi^4/3$ are the best constants in (1.13).

4.2. Proof of Theorem G.

By (4.1) and (4.2), we have

$$\begin{aligned} (x\sec^2 x - \tan x)(\pi^2-4x^2) &= 2\pi^2 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}(n-1)|B_{2n}|x^{2n-1} \\ &\quad -4\sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-2)|B_{2n}|x^{2n+1}, \\ 2\pi^2(\tan x - x) &= 2\pi^2 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-1}, \end{aligned} \tag{4.5}$$

so, (1.18) is equivalent to

$$2\pi^2\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}(n-2)|B_{2n}|x^{2n-1} \le 4\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-2)|B_{2n}|x^{2n+1}, \tag{4.6}$$

that is,

$$2\pi^2\sum_{n=3}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}(n-2)|B_{2n}|x^{2n-1} \le 4\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-2)|B_{2n}|x^{2n+1}, \tag{4.7}$$

or

$$\pi^2\sum_{n=2}^{\infty}\frac{2^{2n+2}(2^{2n+2}-1)}{(2n+2)!}(n-1)|B_{2n+2}|x^{2n+1} \le 4\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}(n-1)|B_{2n}|x^{2n+1}. \tag{4.8}$$

From Lemma 2.1, we have that $(1-1/2^{2n})\zeta(2n)$ is decreasing or

$$\frac{2^{2n+2}-1}{4}\zeta(2n+2) < \left(2^{2n}-1\right)\zeta(2n) \tag{4.9}$$

holds. By (3.1), we get

$$\frac{\pi^2(2^{2n+2}-1)}{(2n+2)!}|B_{2n+2}| < \frac{(2^{2n}-1)}{(2n)!}|B_{2n}|, \tag{4.10}$$

so, (4.8) holds.

5. Remark

In 2010, Zhu and Hua [9] proved for $x \in (0, \pi/2)$ that

$$\frac{\pi^2 + ((4(8-\pi^2))/\pi^2)x^2}{\pi^2-4x^2} < \frac{\tan x}{x} < \frac{\pi^2 + (\pi^2/3-4)x^2}{\pi^2-4x^2}. \tag{5.1}$$

Now, we can compare the results of (5.1) with (1.16). In fact, we can easy check that

$$\begin{gathered}\frac{8}{\pi^2-4x^2} + \left(1-\frac{8}{\pi^2}\right) = \frac{\pi^2 + ((4(8-\pi^2))/\pi^2)x^2}{\pi^2-4x^2},\\ \frac{\pi^4}{12}\frac{1}{\pi^2-4x^2} + \left(1-\frac{\pi^2}{12}\right) = \frac{\pi^2+(\pi^2/3-4)x^2}{\pi^2-4x^2}.\end{gathered} \tag{5.2}$$

So, (5.1) is equivalent to (1.16).

References

[1] W. Scharlau and H. Opolka, *From Fermat to Minkowski*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 1985.

[2] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover Publications, New York, NY, USA, 1965.

[3] C. D'Aniello, "On some inequalities for the Bernoulli numbers," *Rendiconti del Circolo Matematico di Palermo*, vol. 43, no. 3, pp. 329–332, 1994.

[4] H. Alzer, "Sharp bounds for the Bernoulli numbers," *Archiv der Mathematik*, vol. 74, no. 3, pp. 207–211, 2000.

[5] D. S. Mitrinovic, *Analytic Inequalities*, Springer, New York, NY, USA, 1970.

[6] J.C. Kuang, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan City, Shangdong Province, China, 3rd edition, 2004.

[7] M. C. Papenfuss, "E 2739," *The American Mathematical Monthly*, vol. 85, p. 765, 1978.

[8] G. Bach, "Trigonometric inequality," *The American Mathematical Monthly*, vol. 87, p. 62, 1980.

[9] L. Zhu and J. Hua, "Sharpening the Becker-Stark inequalities," *Journal of Inequalities and Applications*, Article ID 931275, 4 pages, 2010.

[10] S. Ponnusamy and M. Vuorinen, "Asymptotic expansions and inequalities for hypergeometric functions," *Mathematika*, vol. 44, no. 2, pp. 278–301, 1997.

[11] G. Pólya and G. Szegő, *Problems and theorems in analysis. Vol. I*, Springer, New York, NY, USA, 1972.

[12] A. Jeffrey, *Handbook of Mathematical Formulas and Integrals*, Elsevier Academic Press, San Diego, Calif, USA, 3rd edition, 2004.