

Research Article

The System of Mixed Equilibrium Problems for Quasi-Nonexpansive Mappings in Hilbert Spaces

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We first introduce the iterative procedure to approximate a common element of the fixed-point set of two quasinonexpansive mappings and the solution set of the system of mixed equilibrium problem (SMEP) in a real Hilbert space. Next, we prove the weak convergence for the given iterative scheme under certain assumptions. Finally, we apply our results to approximate a common element of the set of common fixed points of asymptotic nonspreading mapping and asymptotic TJ mapping and the solution set of SMEP in a real Hilbert space.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, let C be a nonempty closed convex subset of H , and let T be a mapping of C into H , then $T : C \rightarrow H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \rightarrow H$ is said to be quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T) := \{x \in C : Tx = x\}$. It is well known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is a closed and convex set [1]. A mapping $T : C \rightarrow C$ is said to be firmly nonexpansive [2] if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \quad (1.1)$$

for all $x, y \in C$, and it is an important example of nonexpansive mappings in a Hilbert space.

Let $\varphi : C \rightarrow \mathbb{R}$ be a real-valued function, and let $F : C \times C \rightarrow \mathbb{R}$ be an equilibrium bifunction, that is, $F(u, u) = 0$ for each $u \in C$. The mixed equilibrium problem is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0 \quad \forall y \in C. \quad (1.2)$$

Denote the set of solution of (1.2) by $\text{MEP}(F, \varphi)$. In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem, which is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \forall y \in C. \quad (1.3)$$

The set of solution of (1.3) is denoted by $\text{EP}(F)$.

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, Min-Max problems, the Nash equilibrium problems in noncooperative games, and others; see, for example, Blum and Oettli [3] and Moudafi [4]. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.3).

Let $F_1, F_2 : C \times C \rightarrow \mathbb{R}$ be two monotone bifunctions and $\mu > 0$ is constant. In 2009, Moudafi [5] introduced an alternating algorithm for approximating a solution of the system of equilibrium problems, finding $(x', y') \in C \times C$ such that

$$(\text{SEP}) \begin{cases} F_1(x', z) + \frac{1}{\mu} \langle y' - x', x' - z \rangle \geq 0, & \forall z \in C, \\ F_2(y', z) + \frac{1}{\mu} \langle x' - y', y' - z \rangle \geq 0, & \forall z \in C. \end{cases} \quad (1.4)$$

For such mappings F_1 and F_2 and two given positive constants $\lambda, \mu > 0$, Plubtieng and Sombut [6] considered the following system of mixed equilibrium problem, finding $(x', y') \in C \times C$ such that

$$(\text{SMEP}) \begin{cases} F_1(x', z) + \varphi(z) - \varphi(x') + \frac{1}{\lambda} \langle y' - x', x' - z \rangle \geq 0, & \forall z \in C, \\ F_2(y', z) + \varphi(z) - \varphi(y') + \frac{1}{\mu} \langle x' - y', y' - z \rangle \geq 0, & \forall z \in C. \end{cases} \quad (1.5)$$

In particular, if $\lambda = \mu$ and $\varphi \equiv 0$, then problem (SMEP) reduces to (SEP). Furthermore, Plubtieng and Sombut [6] introduced the following iterative procedure to approximate a common element of the fixed-point set of a quasi-nonexpansive mapping T and the solution set of (SMEP) in a Hilbert space H . Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be given by

$$\begin{aligned} & x_1 \in C \text{ chosen arbitrary,} \\ & u_n \in C, \quad F_2(u_n, z) + \varphi(z) - \varphi(u_n) + \frac{1}{\mu} \langle z - u_n, u_n - x_n \rangle \geq 0, \quad \forall z \in C, \\ & y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda} \langle z - y_n, y_n - u_n \rangle \geq 0, \quad \forall z \in C, \\ & x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \quad \forall n \in \mathbb{N}, \end{aligned} \quad (1.6)$$

where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and satisfying appropriate conditions. The weak convergence theorems are obtained in a real Hilbert space.

On the other hand, in 1953, Mann [7] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H :

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \quad (1.7)$$

where the initial point x_0 is taken in C arbitrarily, and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

For two nonexpansive mappings T_1, T_2 of C into itself, Moudafi [4] studied weak convergence theorems in the following iterative process:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrary,} \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)(\beta_n T_1 x_n + (1 - \beta_n) T_2 x_n), \end{aligned} \quad (1.8)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0, 1]$ and $F(T_1) \cap F(T_2) \neq \emptyset$. Recently, Iemoto and Takahashi [8] also considered this iterative procedure for T_1 is a nonexpansive mapping and $T_2 : C \rightarrow C$ is a nonspreading mapping. Very recently, Kim [9] studied the weak and strong convergence for the Moudafi's iterative scheme (1.8) of two quasi-nonexpansive mappings.

In this paper, inspired and motivated by Plubtieng and Sombut [6], Moudafi [4], Iemoto and Takahashi [8], and Kim [9], we first introduce the iterative procedure to approximate a common element of the common fixed point set of two quasi-nonexpansive mappings and the solution set of SMEP in a real Hilbert space. Next, we prove the weak convergence theorem for the given iterative scheme under certain assumptions. Finally, we apply our results to approximate a common element of the set of common fixed point of asymptotic nonspreading mapping and asymptotic TJ mapping and the solution set of SMEP in a real Hilbert space.

2. Preliminaries

Throughout this paper, let \mathbb{N} be the set of positive integers, and let \mathbb{R} be the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively, and let C be a closed convex subset of H . We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively.

From [10], for each $x, y \in H$ and $\lambda \in [0, 1]$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \quad (2.1)$$

For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C. \quad (2.2)$$

P_C is called the metric projection of H onto C . It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \quad \forall x, y \in H. \quad (2.3)$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$,

$$\begin{aligned} \langle x - P_C x, P_C y - y \rangle &\geq 0, \\ \|x - y\|^2 &\geq \|x - P_C x\|^2 + \|y - P_C y\|^2 \quad \forall x \in H, y \in C. \end{aligned} \quad (2.4)$$

Further, for all $x \in H$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, y - z \rangle \geq 0$, for all $z \in C$.

Lemma 2.1 (see [11]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let P be the metric projection of H onto C , and let $\{x_n\}_{n \in \mathbb{N}}$ be in H . If*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad (2.5)$$

for all $u \in C$ and $n \in \mathbb{N}$. Then, $\{P_C x_n\}$ converges strongly to an element of C .

Theorem 2.2 (Opial's theorem, [10]). *Let H be a real Hilbert space, and suppose that $x_n \rightharpoonup x$, then*

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|, \quad (2.6)$$

for all $y \in H$ with $x \neq y$.

All Hilbert space and l^p ($1 < p < \infty$) satisfy Opial's condition, while L^p with $1 < p \neq 2 < \infty$ do not.

For solving the mixed equilibrium problem for an equilibrium bifunction $F : C \times C \rightarrow \mathbb{R}$, let us assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$,
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$,
- (A3) for each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous,
- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and semicontinuous.

The following lemma appears implicitly in [3, 12].

Lemma 2.3 (see [3]). *Let C be a nonempty closed convex subset of H , and let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $r > 0$ and $x \in H$, then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \forall y \in C. \quad (2.7)$$

The following lemma was also given in [12].

Lemma 2.4 (see [12]). *Let C be a nonempty closed convex subset of H and let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4), then, for any $r > 0$ and $x \in H$, define a mapping $T_r x : H \rightarrow C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}, \quad (2.8)$$

for all $z \in H, r \in \mathbb{R}$. Then the following hold:

- (i) T_r is single valued,
- (ii) T_r is firmly nonexpansive, that is,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H; \quad (2.9)$$

- (iii) $F(T_r) = \text{EP}(F)$,
- (iv) $\text{EP}(F)$ is closed and convex.

We note that Lemma 2.4 is equivalent to the following lemma.

Lemma 2.5 (see [6]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $F : C \times C \rightarrow \mathbb{R}$ be an equilibrium bifunction satisfying (A1)–(A4) and let $\varphi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex functional. For each $r > 0$ and $x \in H$, define a mapping*

$$S_r(x) = \left\{ y \in C : F(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r} \langle y - x, z - y \rangle \geq 0, \forall z \in C \right\}, \quad \forall x \in H. \quad (2.10)$$

Then, the following results hold:

- (i) for each $x \in C$, $S_r(x) \neq \emptyset$,
- (ii) S_r is single valued,
- (iii) S_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y \rangle, \quad (2.11)$$

- (iv) $F(S_r) = \text{MEP}(F, \varphi)$,
- (v) $\text{MEP}(F, \varphi)$ is closed and convex.

3. Main Results

In this section, we prove the weak convergence for approximating a common element of the common fixed point set of two quasi-nonexpansive mappings and the solution set of the system of mixed equilibrium problems in a Hilbert space.

To begin with, let us state and prove the following characterizations of the solution set of GMEP.

Lemma 3.1. *Let C be a closed convex subset of a real Hilbert space H . Let F_1 and F_2 be two mappings from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4), and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as in Lemma 2.5 associated to F_1 and F_2 , respectively. For given $x', y' \in C$, (x', y') is a solution of problem (1.5) if and only if x' is a fixed point of the mapping $G : C \rightarrow C$ defined by*

$$G(x) = S_{1,\lambda}(S_{2,\mu}x), \quad \forall x \in C, \quad (3.1)$$

where $y' = S_{2,\mu}x'$.

Next, we prove that

$$\lim_{n \rightarrow \infty} \|x_n - x^*\| \text{ exists.} \quad (3.6)$$

Since T_1 and T_2 are quasi-nonexpansive, we obtain that

$$\begin{aligned} \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|^2 &= \|b_n (T_1 y_n - x^*) + (1 - b_n) (T_2 y_n - x^*)\|^2 \\ &= b_n \|T_1 y_n - x^*\|^2 + (1 - b_n) \|T_2 y_n - x^*\|^2 \\ &\quad - b_n (1 - b_n) \|T_1 y_n - T_2 y_n\|^2 \\ &\leq b_n \|T_1 y_n - x^*\|^2 + (1 - b_n) \|T_2 y_n - x^*\|^2 \\ &\leq b_n \|y_n - x^*\|^2 + (1 - b_n) \|y_n - x^*\|^2 \\ &= \|y_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2, \end{aligned} \quad (3.7)$$

which gives that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|a_n x_n + (1 - a_n) (b_n T_1 y_n + (1 - b_n) T_2 y_n) - x^*\|^2 \\ &= \|a_n (x_n - x^*) + (1 - a_n) (b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*)\|^2 \\ &= a_n \|x_n - x^*\|^2 + (1 - a_n) \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|^2 \\ &\quad - a_n (1 - a_n) \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\|^2 \\ &\leq a_n \|x_n - x^*\|^2 + (1 - a_n) \|x_n - x^*\|^2 \\ &\quad - a_n (1 - a_n) \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\|^2 \\ &= \|x_n - x^*\|^2 - a_n (1 - a_n) \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \quad (3.8)$$

Hence, $\{\|x_n - x^*\|\}$ is a nonincreasing sequence, and hence, $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists. This implies that $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{T_1 y_n\}$, and $\{T_2 y_n\}$ are bounded. From (3.8), we have

$$a_n (1 - a_n) \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \quad (3.9)$$

Since $\liminf_{n \rightarrow \infty} a_n (1 - a_n) > 0$, this implies that

$$\lim_{n \rightarrow \infty} \|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\| = 0. \quad (3.10)$$

Furthermore, since $0 < a \leq a_n \leq b < 1$, we have

$$\begin{aligned}
 \|x_{n+1} - x_n\| &= \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n) - x_n\| \\
 &= \|(1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n) - (1 - a_n) x_n\| \\
 &= (1 - a_n) \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x_n\| \\
 &\leq (1 - a) \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x_n\|.
 \end{aligned} \tag{3.11}$$

From (3.10), we conclude that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

From (3.7), we have

$$\begin{aligned}
 b_n(1 - b_n) \|T_1 y_n - T_2 y_n\|^2 &= b_n \|T_1 y_n - x^*\|^2 + (1 - b_n) \|T_2 y_n - x^*\|^2 \\
 &\quad - \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|^2 \\
 &\leq b_n \|y_n - x^*\|^2 + (1 - b_n) \|y_n - x^*\|^2 \\
 &\quad - \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|^2 \\
 &= \|y_n - x^*\|^2 - \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|^2 \\
 &\leq \|x_n - x^*\|^2 - \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|^2 \\
 &= (\|x_n - x^*\| - \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|) \\
 &\quad \times (\|x_n - x^*\| + \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|) \\
 &\leq M (\|x_n - x^*\| - \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|) \\
 &\leq M (\|x_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\|),
 \end{aligned} \tag{3.13}$$

where M is a constant satisfying $M \geq \sup_{n \geq 1} [\|x_n - x^*\| + \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|]$. Again from (3.10), we conclude that

$$\lim_{n \rightarrow \infty} b_n(1 - b_n) \|T_1 y_n - T_2 y_n\| = 0. \tag{3.14}$$

Using $\liminf_{n \rightarrow \infty} b_n(1 - b_n) > 0$, we have

$$\lim_{n \rightarrow \infty} \|T_1 y_n - T_2 y_n\| = 0. \tag{3.15}$$

Now, we prove that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_1 y_n\| = 0, \quad \lim_{n \rightarrow \infty} \|x_{n+1} - T_2 y_n\| = 0. \tag{3.16}$$

We observe that

$$\begin{aligned}
\|x_{n+1} - T_1 y_n\| &= \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n) - T_1 y_n\| \\
&= \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n) - (a_n + (1 - a_n)) T_1 y_n\| \\
&= \|a_n(x_n - T_1 y_n) + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n - T_1 y_n)\| \\
&= \|a_n(x_n - T_1 y_n) + (1 - a_n)(b_n T_1 y_n + T_2 y_n - b_n T_2 y_n - T_1 y_n)\| \\
&= \|a_n(x_n - T_1 y_n) + (1 - a_n)((1 - b_n) T_2 y_n - (1 - b_n) T_1 y_n)\| \tag{3.17} \\
&= \|a_n(x_n - T_1 y_n) + (1 - a_n)(1 - b_n)(T_2 y_n - T_1 y_n)\| \\
&\leq a_n \|x_n - T_1 y_n\| + (1 - a_n)(1 - b_n) \|T_2 y_n - T_1 y_n\| \\
&= a_n \|x_n - x_{n+1} + x_{n+1} - T_1 y_n\| + (1 - a_n)(1 - b_n) \|T_2 y_n - T_1 y_n\| \\
&\leq a_n \|x_n - x_{n+1}\| + a_n \|x_{n+1} - T_1 y_n\| + (1 - a_n)(1 - b_n) \|T_2 y_n - T_1 y_n\|,
\end{aligned}$$

which gives that

$$(1 - a_n) \|x_{n+1} - T_1 y_n\| \leq a_n \|x_n - x_{n+1}\| + (1 - a_n)(1 - b_n) \|T_2 y_n - T_1 y_n\|. \tag{3.18}$$

Since $0 < a \leq a_n \leq b < 1$, we have

$$\begin{aligned}
a_n(1 - a_n) \|x_{n+1} - T_1 y_n\| &\leq (1 - a_n) \|x_{n+1} - T_1 y_n\| \\
&\leq a_n \|x_n - x_{n+1}\| + (1 - a_n)(1 - b_n) \|T_2 y_n - T_1 y_n\|.
\end{aligned} \tag{3.19}$$

Using (3.12) and (3.15), we conclude that

$$\lim_{n \rightarrow \infty} a_n(1 - a_n) \|x_{n+1} - T_1 y_n\| = 0, \tag{3.20}$$

which gives that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_1 y_n\| = 0, \tag{3.21}$$

since $\liminf_{n \rightarrow \infty} a_n(1 - a_n) > 0$. Similarly, we have

$$\begin{aligned}
\|x_{n+1} - T_2 y_n\| &= \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n) - T_2 y_n\| \\
&= \|a_n(x_n - T_2 y_n) + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n - T_2 y_n)\| \\
&= \|a_n(x_n - T_2 y_n) + (1 - a_n)b_n(T_1 y_n - T_2 y_n)\| \\
&\leq a_n \|x_n - T_2 y_n\| + (1 - a_n)b_n \|T_1 y_n - T_2 y_n\| \\
&= a_n \|x_n - x_{n+1} + x_{n+1} - T_2 y_n\| + (1 - a_n)b_n \|T_2 y_n - T_1 y_n\| \\
&\leq a_n \|x_n - x_{n+1}\| + a_n \|x_{n+1} - T_2 y_n\| + (1 - a_n)b_n \|T_2 y_n - T_1 y_n\|,
\end{aligned} \tag{3.22}$$

which implies that

$$(1 - a_n)\|x_{n+1} - T_2 y_n\| \leq a_n\|x_n - x_{n+1}\| + (1 - a_n)b_n\|T_2 y_n - T_1 y_n\|. \quad (3.23)$$

Thus, we have

$$\begin{aligned} a_n(1 - a_n)\|x_{n+1} - T_2 y_n\| &\leq (1 - a_n)\|x_{n+1} - T_2 y_n\| \\ &\leq a_n\|x_n - x_{n+1}\| + (1 - a_n)b_n\|T_2 y_n - T_1 y_n\| \\ &\leq b\|x_n - x_{n+1}\| + (1 - a_n)\|T_2 y_n - T_1 y_n\| \\ &\leq b\|x_n - x_{n+1}\| + (1 - a)\|T_2 y_n - T_1 y_n\|. \end{aligned} \quad (3.24)$$

Hence, $\lim_{n \rightarrow \infty} a_n(1 - a_n)\|x_{n+1} - T_2 y_n\| = 0$. Since $\liminf_{n \rightarrow \infty} a_n(1 - a_n) > 0$, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_2 y_n\| = 0. \quad (3.25)$$

Next, we prove that

$$\lim_{n \rightarrow \infty} \|y_n - T_1 y_n\| = 0, \quad \lim_{n \rightarrow \infty} \|y_n - T_2 y_n\| = 0. \quad (3.26)$$

Since $S_{1,r}$ and $S_{2,\lambda}$ are firmly nonexpansive, it follows that

$$\|u_n - y^*\|^2 = \|S_{2,\lambda} x_n - S_{2,\lambda} x^*\|^2 \leq \langle S_{2,\lambda} x_n - S_{2,\lambda} x^*, x_n - x^* \rangle = \langle u_n - y^*, x_n - x^* \rangle, \quad (3.27)$$

which gives that

$$\begin{aligned} \|(u_n - y^*) - (x_n - x^*)\|^2 &= \|u_n - y^*\|^2 - 2\langle u_n - y^*, x_n - x^* \rangle + \|x_n - x^*\|^2 \\ &\leq \|u_n - y^*\|^2 - 2\|u_n - y^*\|^2 + \|x_n - x^*\|^2 \\ &= -\|u_n - y^*\|^2 + \|x_n - x^*\|^2. \end{aligned} \quad (3.28)$$

This implies that

$$\|u_n - y^*\|^2 \leq \|x_n - x^*\|^2 - \|(u_n - y^*) - (x_n - x^*)\|^2. \quad (3.29)$$

By the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n) - x^*\|^2 \\ &= \|a_n(x_n - x^*) + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*)\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq a_n \|x_n - x^*\|^2 + (1 - a_n) \|b_n T_1 y_n + (1 - b_n) T_2 y_n - x^*\|^2 \\
&\leq a_n \|x_n - x^*\|^2 + (1 - a_n) \|y_n - x^*\|^2 \\
&\leq a_n \|x_n - x^*\|^2 + (1 - a_n) \|u_n - y^*\|^2 \\
&\leq a_n \|x_n - x^*\|^2 + (1 - a_n) \left(\|x_n - x^*\|^2 - \|(u_n - y^*) - (x_n - x^*)\|^2 \right) \\
&= \|x_n - x^*\|^2 - (1 - a_n) \|(u_n - y^*) - (x_n - x^*)\|^2.
\end{aligned} \tag{3.30}$$

Thus,

$$(1 - a_n) \|(u_n - y^*) - (x_n - x^*)\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.31}$$

Since $0 < a \leq a_n \leq b < 1$, we have

$$(1 - b) \|(u_n - y^*) - (x_n - x^*)\|^2 \leq (1 - a_n) \|(u_n - y^*) - (x_n - x^*)\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.32}$$

Since $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists, we have

$$\lim_{n \rightarrow \infty} \|u_n - x_n + x^* - y^*\| = 0. \tag{3.33}$$

Similarly, we have

$$\|y_n - x^*\|^2 = \|S_{1,r_n} u_n - S_{1,r_n} y^*\|^2 \leq \langle S_{1,r_n} u_n - S_{1,r_n} y^*, u_n - y^* \rangle = \langle y_n - x^*, u_n - x^* \rangle, \tag{3.34}$$

which gives that

$$\begin{aligned}
\|(y_n - x^*) - (u_n - y^*)\|^2 &= \|y_n - x^*\|^2 - 2\langle y_n - x^*, u_n - x^* \rangle + \|u_n - y^*\|^2 \\
&\leq \|y_n - x^*\|^2 - 2\|y_n - x^*\|^2 + \|u_n - y^*\|^2 \\
&= -\|y_n - x^*\|^2 + \|u_n - y^*\|^2.
\end{aligned} \tag{3.35}$$

This implies that

$$\|y_n - x^*\|^2 \leq \|u_n - y^*\|^2 - \|y_n - u_n - x^* + y^*\|^2. \tag{3.36}$$

By the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n) - x^*\|^2 \\
&\leq a_n \|x_n - x^*\|^2 + (1 - a_n) \|y_n - x^*\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq a_n \|x_n - x^*\|^2 + (1 - a_n) \left(\|u_n - y^*\|^2 - \|y_n - u_n - x^* + y^*\|^2 \right) \\
&\leq a_n \|x_n - x^*\|^2 + (1 - a_n) \|x_n - x^*\|^2 - (1 - a_n) \|y_n - u_n - x^* + y^*\|^2 \\
&= \|x_n - x^*\|^2 - (1 - a_n) \|(y_n - x^*) - (u_n - y^*)\|^2.
\end{aligned} \tag{3.37}$$

Thus,

$$(1 - a_n) \|y_n - u_n - x^* + y^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.38}$$

Since $0 < a \leq a_n \leq b < 1$, we have

$$(1 - b) \|y_n - u_n - x^* + y^*\|^2 \leq (1 - a_n) \|y_n - u_n - x^* + y^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.39}$$

Since $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists, we have

$$\lim_{n \rightarrow \infty} \|y_n - u_n - x^* + y^*\| = 0. \tag{3.40}$$

Hence,

$$\begin{aligned}
&\|(b_n T_1 y_n + (1 - b_n) T_2 y_n) - y_n\| \\
&= \|(b_n T_1 y_n + (1 - b_n) T_2 y_n) - x_n + x_n - u_n + u_n - y^* + y^* - x^* + x^* - y_n\| \\
&\leq \|(b_n T_1 y_n + (1 - b_n) T_2 y_n) - x_n\| + \|x_n - u_n - x^* + y^*\| + \|u_n - y_n + x^* - y^*\|.
\end{aligned} \tag{3.41}$$

It follow from (3.10), (3.33), and (3.40) that

$$\lim_{n \rightarrow \infty} \|(b_n T_1 y_n + (1 - b_n) T_2 y_n) - y_n\| = 0, \tag{3.42}$$

from which it follows that

$$\begin{aligned}
\|y_n - x_n\| &= \|y_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n) + (b_n T_1 y_n + (1 - b_n) T_2 y_n) - x_n\| \\
&\leq \|y_n - (b_n T_1 y_n + (1 - b_n) T_2 y_n)\| \\
&\quad + \|(b_n T_1 y_n + (1 - b_n) T_2 y_n) - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{3.43}$$

that is,

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{3.44}$$

Thus,

$$\|y_n - T_1 y_n\| \leq \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - T_1 y_n\| \longrightarrow 0. \tag{3.45}$$

Similarly, we have $\|y_n - T_2 y_n\| \rightarrow 0$. Since $\{y_n\}$ is bounded sequence, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightarrow \bar{x}$ as $i \rightarrow \infty$. Since T_1 and T_2 are demiclosed at 0, we conclude that $\bar{x} \in F(T_1) \cap F(T_2)$. Let G be a mapping which is defined as in Lemma 3.1. Thus, we have

$$\|y_n - G(y_n)\| = \|S_{1,r} S_{2,\lambda} x_n - G(y_n)\| = \|G(x_n) - G(y_n)\| \leq \|x_n - y_n\|, \quad (3.46)$$

and hence,

$$\begin{aligned} \|x_n - G(x_n)\| &= \|x_n - y_n + y_n - G(y_n) + G(y_n) - G(x_n)\| \\ &\leq \|x_n - y_n\| + \|y_n - G(y_n)\| + \|G(y_n) - G(x_n)\| \\ &\leq \|x_n - y_n\| + \|x_n - y_n\| + \|y_n - x_n\| \\ &= 3\|x_n - y_n\| \rightarrow 0. \end{aligned} \quad (3.47)$$

This together with $x_{n_i} \rightarrow \bar{x}$ implies that $\bar{x} \in F(G) := \Omega$, if $\{y_{n_j}\}$ is another subsequence of $\{y_n\}$ such that $y_{n_j} \rightarrow \hat{x}$ as $j \rightarrow \infty$. Since T_1 and T_2 are demiclosed at 0, we conclude that $\hat{x} \in F(T_1) \cap F(T_2) \cap \Omega$. From $x_{n_i} \rightarrow \bar{x}$ and $x_{n_j} \rightarrow \hat{x}$, we will show that $\bar{x} = \hat{x}$. Assume that $\bar{x} \neq \hat{x}$. Since $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists for all $x^* \in F(T_1) \cap F(T_2) \cap \Omega$, by Opial's Theorem 2.2, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - \bar{x}\| &= \liminf_{i \rightarrow \infty} \|x_{n_i} - \bar{x}\| \\ &< \liminf_{i \rightarrow \infty} \|x_{n_i} - \hat{x}\| \\ &= \lim_{n \rightarrow \infty} \|x_n - \hat{x}\| \\ &= \liminf_{j \rightarrow \infty} \|x_{n_j} - \hat{x}\| \\ &< \liminf_{j \rightarrow \infty} \|x_{n_j} - \bar{x}\| \\ &= \lim_{n \rightarrow \infty} \|x_n - \bar{x}\|. \end{aligned} \quad (3.48)$$

This is a contradiction. Thus, we have $\bar{x} = \hat{x}$. This implies that $y_n \rightarrow \bar{x} \in F(T_1) \cap F(T_2) \cap \Omega$. Since $\|x_n - y_n\| \rightarrow 0$, we have $x_n \rightarrow \bar{x}$. Put $z_n = P_{F(T_1) \cap F(T_2) \cap \Omega} x_n$. Finally, we show that $\bar{x} = \lim_{n \rightarrow \infty} z_n$. Now from (2.4) and $\bar{x} \in F(T_1) \cap F(T_2) \cap \Omega$, we have

$$\langle \bar{x} - z_n, z_n - x_n \rangle \geq 0. \quad (3.49)$$

Since $\{\|x_n - x^*\|\}$ is nonnegative and nonincreasing for all $x^* \in F(T_1) \cap F(T_2) \cap \Omega$, it follows by Lemma 2.1 that $\{z_n\}$ converges strongly to some $\hat{x} \in F(T_1) \cap F(T_2) \cap \Omega$. By (3.49), we have

$$\langle \bar{x} - \hat{x}, \hat{x} - \bar{x} \rangle \geq 0. \quad (3.50)$$

Therefore, $\bar{x} = \hat{x}$. □

Setting $T := T_1 = T_2$ in Theorem 3.2, we have the following result.

Corollary 3.3 (see [6]). *Let C be a closed convex subset of a real Hilbert space H . Let F_1 and F_2 be two bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4). Let $\lambda, \mu > 0$, and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as in Lemma 2.5 associated to F_1 and F_2 , respectively. Let $T : C \rightarrow C$ be a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero and $F(T) \cap \Omega \neq \emptyset$. Suppose that $x_0 = x \in C$ and $\{x_n\}, \{y_n\}$, and $\{z_n\}$ are given by*

$$\begin{aligned} z_n \in C, \quad F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\mu} \langle z - z_n, z_n - x_n \rangle &\geq 0, \quad \forall z \in C, \\ y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda} \langle z - y_n, y_n - z_n \rangle &\geq 0, \quad \forall z \in C, \\ x_{n+1} &= a_n x_n + (1 - a_n) T y_n, \end{aligned} \quad (3.51)$$

for all $n \in \mathbb{N}$, where $\{a_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, and satisfy $\liminf_{n \rightarrow \infty} a_n(1 - a_n) > 0$, then $\{x_n\}$ converges weakly to $\bar{x} = \lim_{n \rightarrow \infty} P_{F(T) \cap \Omega} x_n$ and (\bar{x}, \bar{y}) is a solution of problem (1.5), where $\bar{y} = S_{2,\mu} \bar{x}$.

Setting $F_1 = F_2 \equiv 0, \varphi \equiv 0$ in Theorem 3.2, we have the following result.

Corollary 3.4 (see [9]). *Let H be a Hilbert space, let C be a nonempty, closed, and convex subset of H , and let T_1, T_2 be two quasi-nonexpansive mappings of C into itself such that $I - T_1, I - T_2$ are demiclosed at zero with $F(T_1) \cap F(T_2) \neq \emptyset$. For any x_1 in C , let $\{x_n\}$ be defined by*

$$x_{n+1} = (1 - a_n)x_n + a_n(b_n T_1 x_n + (1 - b_n) T_2 x_n), \quad (3.52)$$

where $\{a_n\}$ and $\{b_n\}$ are chosen so that

$$\liminf_{n \rightarrow \infty} a_n(1 - a_n) > 0 \quad \liminf_{n \rightarrow \infty} b_n(1 - b_n) > 0, \quad (3.53)$$

then $x_n \rightarrow p \in F(T_1) \cap F(T_2)$.

4. Applications

In this section, we apply our results to approximate a common element of the set of common fixed points of an asymptotic nonspreading mapping and an asymptotic TJ mapping and the solution set of SMEP in a real Hilbert space. We recall the following definitions. A mapping $T : C \rightarrow C$ is called nonspreading [13] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \quad \forall x, y \in C. \quad (4.1)$$

Furthermore, Takahashi and Yao [14] also introduced two nonlinear mappings in Hilbert spaces. A mapping $T : C \rightarrow C$ is called a $TJ - 1$ mapping [14] if

$$2\|Tx - Ty\|^2 = \|x - y\|^2 + \|Tx - y\|^2, \quad (4.2)$$

for all $x, y \in C$. A mapping $T : C \rightarrow C$ is called a $TJ - 2$ [14] mapping if

$$3\|Tx - Ty\|^2 = 2\|Tx - y\|^2 + \|Ty - x\|^2, \quad (4.3)$$

for all $x, y \in C$. For these two nonlinear mappings, $TJ - 1$ and $TJ - 2$ mappings, Takahashi and Yao [14] studied the existence results of fixed points in Hilbert spaces. Very recently, Lin et al. [15] introduced the following definitions of new mappings.

Definition 4.1. Let C be a nonempty closed convex subset of a Hilbert space H . We say that $T : C \rightarrow C$ is an asymptotic nonspreading mapping if there exist two functions $\alpha : C \rightarrow [0, 2)$ and $\beta : C \rightarrow [0, k], k < 2$, such that

$$(A1) \quad 2\|Tx - Ty\|^2 \leq \alpha(x)\|Tx - y\|^2 + \beta(x)\|Ty - x\|^2 \text{ for all } x, y \in C,$$

$$(A2) \quad 0 < \alpha(x) + \beta(x) \leq 2 \text{ for all } x \in C.$$

Remark 4.2. The class of asymptotic nonspreading mappings contains the class of nonspreading mappings and the class of $TJ - 2$ mappings in a Hilbert space. Indeed, in Definition 4.1, we know that

(i) if $\alpha(x) = \beta(x) = 1$ for all $x \in C$, then T is a nonspreading mapping,

(ii) if $\alpha(x) = 4/3$ and $\beta(x) = 2/3$ for all $x \in C$, then T is a $TJ - 2$ mapping.

Definition 4.3. Let C be a nonempty closed convex subset of a Hilbert space H . We say $T : C \rightarrow C$ is an asymptotic TJ mapping if there exists two functions $\alpha : C \rightarrow [0, 2]$ and $\beta : C \rightarrow [0, k], k < 2$, such that

$$(B1) \quad 2\|Tx - Ty\|^2 \leq \alpha(x)\|x - y\|^2 + \beta(x)\|Tx - y\|^2 \text{ for all } x, y \in C,$$

$$(B2) \quad \alpha(x) + \beta(x) \leq 2 \text{ for all } x \in C.$$

Remark 4.4. The class of asymptotic TJ mappings contains the class of $TJ - 1$ mappings and the class of nonexpansive mappings in a Hilbert space. Indeed, in Definition 4.3, we know that

(i) if $\alpha(x) = 2$ and $\beta(x) = 0$ for each $x \in C$, then T is a nonexpansive mapping,

(ii) if $\alpha(x) = \beta(x) = 1$ for each $x \in C$, then T is a $TJ - 1$ mapping.

It is well known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is a closed and convex set [1]. Hence, if $T : C \rightarrow C$ is an asymptotic nonspreading mapping (resp., asymptotic TJ mapping) with $F(T) \neq \emptyset$, then T is a quasi-nonexpansive mapping, and this implies that $F(T)$ is a nonempty closed convex subset of C .

Theorem 4.5 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $T : C \rightarrow C$ be an asymptotic nonspreading mapping, then $I - T$ is demiclosed at 0.*

Theorem 4.6 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $T : C \rightarrow C$ be an asymptotic TJ mapping, then $I - T$ is demiclosed at 0.*

Applying the above results, we have the following theorem.

Theorem 4.7. Let C be a nonempty closed convex subset of a real Hilbert space H . Let F_1 and F_2 be two bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4). Let $r, \lambda > 0$ and $S_{1,r}$ and $S_{2,\lambda}$ be defined as in Lemma 2.5 associated to F_1 and F_2 , respectively. Let $T_i : C \rightarrow C$, $i = 1, 2$, be any one of asymptotic nonspreading mapping and asymptotic TJ mapping such that $F(T_1) \cap F(T_2) \cap \Omega \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be given by

$$\begin{aligned} x_1 &\in C \text{ chosen arbitrary,} \\ u_n &\in C, \quad F_2(u_n, z) + \varphi(z) - \varphi(u_n) + \frac{1}{\lambda} \langle z - u_n, u_n - x_n \rangle \geq 0, \quad \forall z \in C, \\ y_n &\in C, \quad 2F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{r} \langle z - y_n, y_n - u_n \rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} &= a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{4.4}$$

where $\{a_n\}, \{b_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and satisfying

$$\liminf_{n \rightarrow \infty} a_n(1 - a_n) > 0, \quad \liminf_{n \rightarrow \infty} b_n(1 - b_n) > 0, \tag{4.5}$$

then $x_n \rightarrow \bar{x} := \lim_{n \rightarrow \infty} P_{F(T_1) \cap F(T_2) \cap \Omega} x_n$, and (\bar{x}, \bar{y}) is a solution of problem (1.5), where $\bar{y} = S_{2,\lambda} \bar{x}$.

Setting $F_1 = F_2 := F$ and $\varphi \equiv 0$ in the above theorem, we have the following result.

Corollary 4.8. Let C be a nonempty closed convex subset of a real Hilbert space H . Let F be the bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4). Let $T_i : C \rightarrow C$, $i = 1, 2$, be any one of asymptotic nonspreading mapping and asymptotic TJ mapping such that $\mathcal{F} := F(T_1) \cap F(T_2) \cap \text{EP}(F) \neq \emptyset$. For given $u \in C$ and $r > 0$, let the sequences $\{x_n\}$ and $\{u_n\}$ be defined by

$$\begin{aligned} x_1 &\in C \text{ chosen arbitrary,} \\ u_n &\in C, \quad F(u_n, y) + \frac{1}{r} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in C, \\ x_{n+1} &= a_n x_n + (1 - a_n)(b_n T_1 u_n + (1 - b_n) T_2 u_n) \quad \forall n \in \mathbb{N}, \end{aligned} \tag{4.6}$$

where $\{a_n\}, \{b_n\}$ are two sequences in $(0, 1)$ satisfying

$$\liminf_{n \rightarrow \infty} a_n(1 - a_n) > 0, \quad \liminf_{n \rightarrow \infty} b_n(1 - b_n) > 0, \tag{4.7}$$

then $x_n \rightarrow w$ for some $w \in \mathcal{F}$.

Setting $F_1 = F_2 \equiv 0$ and $\varphi \equiv 0$ in Theorem 4.7, we have the following result.

Corollary 4.9 (see [15]). Let C be a nonempty closed convex subset of a real Hilbert space H , and let $T_i : C \rightarrow C$, $i = 1, 2$, be any one of asymptotic nonspreading mapping and asymptotic TJ mapping. Let $\mathcal{F} := F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $(0, 1)$. Let $\{x_n\}$ be defined by

$$\begin{aligned} x_1 &\in C \text{ chosen arbitrary,} \\ x_{n+1} &= a_n x_n + (1 - a_n)(b_n T_1 x_n + (1 - b_n) T_2 x_n). \end{aligned} \tag{4.8}$$

Assume that $\liminf_{n \rightarrow \infty} a_n(1 - a_n) > 0$ and $\liminf_{n \rightarrow \infty} b_n(1 - b_n) > 0$, then $x_n \rightarrow w$ for some $w \in \mathcal{F}$.

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