

*Research Article*

# Nonoscillatory Solutions of Second-Order Differential Equations without Monotonicity Assumptions

**Lianwen Wang and Rhonda McKee**

*Department of Mathematics and Computer Science, University of Central Missouri, Warrensburg, MO 64093, USA*

Correspondence should be addressed to Lianwen Wang, [lwang@ucmo.edu](mailto:lwang@ucmo.edu)

Received 25 March 2012; Accepted 7 June 2012

Academic Editor: Chong Lin

Copyright © 2012 L. Wang and R. McKee. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The continuability, boundedness, monotonicity, and asymptotic properties of nonoscillatory solutions for a class of second-order nonlinear differential equations  $[p(t)h(x(t))f(x'(t))]' = q(t)g(x(t))$  are discussed without monotonicity assumption for function  $g$ . It is proved that all solutions can be extended to infinity, are eventually monotonic, and can be classified into disjoint classes that are fully characterized in terms of several integral conditions. Moreover, necessary and sufficient conditions for the existence of solutions in each class and for the boundedness of all solutions are established.

## 1. Introduction

This paper studies the continuability, boundedness, monotonicity, and asymptotic properties of nonoscillatory solutions for a class of second-order nonlinear differential equations

$$[p(t)h(x(t))f(x'(t))] = q(t)g(x(t)), \quad t \geq a. \quad (1.1)$$

Some special cases of (1.1) such as half-linear equation

$$[p(t)\Phi_p(x'(t))] = q(t)\Phi_p(x(t)), \quad (1.2)$$

where  $\Phi_p(r) = |r|^{p-2}r$ ,  $p > 1$ , is the so-called  $p$ -Laplacian operator, Emden-Fowler equation

$$[p(t)\Phi_p(x'(t))] = q(t)\Phi_p(x(t)), \quad (1.3)$$

and differential equation

$$[p(t)h(x(t))x'(t)] = q(t)g(x(t)) \quad (1.4)$$

have been extensively discussed in the literature; see, for example, [1–15] and references cited therein. Equation (1.1) with general nonlinear function  $f(r)$  is investigated in [16–18]. It is worth to point out that  $g(r)$  is assumed to be monotonic in most cited papers, but [2, 6] explain that this assumption does not hold in some applications. The aim of this paper is to investigate the continuability, boundedness, monotonicity, and asymptotic properties of nonoscillatory solutions of (1.1) without monotonic assumption for  $g$ . Some techniques and ideas have been used by the authors in [17].

By solution of (1.1), we mean a differentiable function  $x$  such that  $p(t)h(x(t))f(x'(t))$  is differentiable and satisfies (1.1) on the maximum existence interval  $[a, \alpha_x)$ ,  $\alpha_x \leq \infty$ . A solution  $x$  of (1.1) is said to be eventually monotonic if there exists a  $t_x \geq a$  such that  $x$  is monotonic on  $[t_x, \alpha_x)$ . In this paper, we consider only solutions that are not eventually identically equal to zero.

Throughout the paper, we always assume that

- (H)  $p(t), q(t) : [a, \infty) \rightarrow \mathbb{R}$  are continuous and  $p(t) > 0$  and  $q(t) > 0$ ;  
 $h(r) : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $h(r) > 0$ ;  
 $g(r) : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $rg(r) > 0$  for  $r \neq 0$ ;  
 $f(r) : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, increasing, and  $rf(r) > 0$  for  $r \neq 0$ .

- (H1) There exists a constant  $M_1 > 0$  such that

$$|f^{-1}(uv)| \leq M_1 |f^{-1}(u)| |f^{-1}(v)|, \quad \forall u, v \in \mathbb{R}. \quad (1.5)$$

*Remark 1.1.* (H1) holds for  $p$ -Laplacian operator; indeed,

$$f^{-1}(uv) = f^{-1}(u)f^{-1}(v). \quad (1.6)$$

However, there are nonlinear functions  $f$  that satisfy (H1) but not (1.6); see [17].

The paper is organized as follows: Section 1 briefly addresses the background and the motivation of the paper. Continuability, classification, and boundedness of solutions are discussed in Section 2. Sections 3 and 4 deal with the existence of class A and class B solutions, respectively. Finally, several remarks are provided in Section 5 to compare our results with existing ones.

## 2. Continuability, Classification, and Boundedness of Solutions

In this section we discuss continuability, classification, and boundedness of solutions of (1.1). First of all, we cite a result from [17] that will be used later on.

**Lemma 2.1.** *If  $x$  is a solution of (1.1) with maximal existence interval  $[a, \alpha_x)$ ,  $\alpha_x \leq \infty$ , then  $x$  is eventually monotonic. Moreover, if  $x$  is bounded on all finite subinterval of  $[a, \alpha_x)$ , then  $\alpha_x = \infty$ .*

*Remark 2.2.* From Lemma 2.1 all solutions of (1.1) except eventually trivial solutions can be classified into two classes

$$\begin{aligned} A &= \{x \text{ is defined on } [a, \alpha_x) : x(t)x'(t) > 0 \text{ in a left neighborhood of } \alpha_x\}, \\ B &= \{x \text{ is defined on } [a, \infty) : x(t)x'(t) < 0 \text{ for } t \geq a\}. \end{aligned} \quad (2.1)$$

Next theorem establishes the continuability for all solutions of (1.1), in other words, all solutions can be extended to  $[a, \infty)$ .

**Theorem 2.3.** *Assume the following assumptions hold.*

(H2) *There exists a real number  $m > 0$  and a continuous function  $G(r) : \mathbb{R} \rightarrow \mathbb{R}$  such that  $G(r)$  is increasing and  $|g(r)| \leq |G(r)|$  for  $|r| \geq m$ , and  $rG(r) > 0$  for  $r \neq 0$ ;*

(H3) *There exists a real number  $r_0 > 0$  such that*

$$\int_{r_0}^{\infty} \frac{dr}{f^{-1}(z(r))} = \infty, \quad \int_{-\infty}^{-r_0} \frac{dr}{f^{-1}(z(r))} = -\infty, \quad (2.2)$$

where  $z(r) = G(r)/h(r)$ .

Then all solutions of (1.1) can be extended to  $[a, \infty)$ .

*Proof.* The proof is similar to that of Theorem 2.3 [17]. We point out that as in the proof of Theorem 2.3 [17], for a class A solution  $x$ , we have

$$\begin{aligned} f(x'(t)) &\leq \frac{G(x(t))}{p(t)h(x(t))} \left( \frac{p(d)h(x(d))f(x'(d))}{G(x(d))} + \int_d^t q(s)ds \right), \\ \int_{x(t_1)}^{x(t)} \frac{dr}{f^{-1}(z(r))} &\leq M_1^2 f^{-1}(k) \int_{t_1}^t f^{-1} \left( \frac{1}{p(s)} \int_d^s q(\sigma)d\sigma \right) ds. \end{aligned} \quad (2.3)$$

□

*Remark 2.4.* The function  $g(r) = r + \sin r$  is not monotonic. Clearly,  $|g(r)| \leq 2|r|$ , so  $g$  is bounded by an increasing function  $G(r) = 2r$ . Therefore, the existing results which require the monotonic condition for  $g$  would not apply, but Theorem 2.3 does.

From Remark 2.2 and Theorem 2.3, all solutions of (1.1) can be classified further into four disjoint classes

$$\begin{aligned}
 A_b &= \left\{ x \in A : \lim_{t \rightarrow \infty} |x(t)| = \ell < \infty \right\}, \\
 A_\infty &= \left\{ x \in A : \lim_{t \rightarrow \infty} |x(t)| = \infty \right\}, \\
 B_b &= \left\{ x \in B : \lim_{t \rightarrow \infty} x(t) = \ell \neq 0 \right\}, \\
 B_0 &= \left\{ x \in B : \lim_{t \rightarrow \infty} x(t) = 0 \right\}.
 \end{aligned} \tag{2.4}$$

We will show that the existence of solutions in each class and the boundedness of all solutions are fully characterized by means of convergence or divergence of the following integrals:

$$\begin{aligned}
 J_1 &= \int_a^\infty f^{-1} \left( \frac{1}{p(t)} \int_a^t q(s) ds \right) dt, \\
 J_2 &= \int_a^\infty f^{-1} \left( -\frac{1}{p(t)} \int_a^t q(s) ds \right) dt, \\
 J_3 &= \int_a^\infty f^{-1} \left( \frac{1}{p(t)} \int_t^\infty q(s) ds \right) dt, \\
 J_4 &= \int_a^\infty f^{-1} \left( -\frac{1}{p(t)} \int_t^\infty q(s) ds \right) dt, \\
 J_5 &= \int_a^\infty f^{-1} \left( \frac{1}{p(t)} \right) dt.
 \end{aligned} \tag{2.5}$$

**Theorem 2.5.** *Let (H2) and (H3) hold. Then all positive (negative) solutions of (1.1) are bounded if and only if  $J_1 < \infty$  ( $J_2 > -\infty$ ).*

*Proof.* We consider positive solutions only since the case of negative solutions can be handled similarly.

*Necessity.* Let  $x$  be a positive bounded class A solution. Then  $x(t) > 0$  and  $x'(t) > 0$  for  $t \geq b > a$  and  $\lim_{t \rightarrow \infty} x(t) = l \in (0, \infty)$ . By the Extreme Value Theorem, we have  $L_1 := \min_{x(b) \leq r \leq l} g(r) > 0$ . Hence

$$p(t)h(x(t))f(x'(t)) = p(b)h(x(b))f(x'(b)) + \int_b^t q(s)g(x(s))ds \geq L_1 \int_b^t q(s)ds. \tag{2.6}$$

Since  $x$  is continuous and bounded and  $h(r)$  is continuous, then  $h(x(t))$  is bounded. Let  $h(x(t)) \leq K$  for  $t \in [a, \infty)$ . Then

$$\begin{aligned} Kp(t)f(x'(t)) &\geq p(t)h(x(t))f(x'(t)) \geq L_1 \int_b^t q(s)ds, \\ \frac{K}{L_1}f(x'(t)) &\geq \frac{1}{p(t)} \int_b^t q(s)ds. \end{aligned} \quad (2.7)$$

By (H1), we have

$$f^{-1}\left(\frac{1}{p(t)} \int_b^t q(s)ds\right) \leq f^{-1}\left(\frac{K}{L_1}f(x'(t))\right) \leq M_1 f^{-1}\left(\frac{K}{L_1}\right)x'(t). \quad (2.8)$$

Integrating from  $b$  to  $t$  and letting  $t \rightarrow \infty$ , we have

$$J_1 = \int_b^\infty f^{-1}\left(\frac{1}{p(t)} \int_b^t q(s)ds\right)dt \leq M_1 f^{-1}\left(\frac{K}{L_1}\right)(l - x(b)) < \infty. \quad (2.9)$$

*Sufficiency.* We will prove by contradiction. Let  $x$  be a unbounded class A solution. Then  $x(t) > 0$  and  $x'(t) > 0$  on  $[b, \infty)$ , and there exists a real number  $d \geq b$  such that  $x(t) \geq m$  for  $d \leq t < \infty$ . Similar to the proof of Theorem 2.3, we have

$$\int_{x(t_1)}^{x(t)} \frac{dr}{f^{-1}(z(r))} \leq M_1^2 f^{-1}(k) \int_{t_1}^t f^{-1}\left(\frac{1}{p(s)} \int_d^s q(\sigma)d\sigma\right)ds. \quad (2.10)$$

Letting  $t \rightarrow \infty$  and noting that  $x(\infty) = \infty$ , we have

$$\int_{x(t_1)}^\infty \frac{dr}{f^{-1}(z(r))} \leq M_1^2 f^{-1}(k) \int_{t_1}^\infty f^{-1}\left(\frac{1}{p(s)} \int_b^s q(\sigma)d\sigma\right)ds \leq M_1^2 f^{-1}(k) J_1 < \infty, \quad (2.11)$$

a contradiction to (H3). Therefore,  $x$  is bounded.  $\square$

**Corollary 2.6.** *Let (H2) and (H3) hold. If (1.1) has a positive (negative) bounded class A solution, then all positive (negative) solutions are bounded. On the other hand, if (1.1) has an unbounded positive (negative) class A solution, then all positive (negative) solutions are unbounded.*

### 3. Class A Solutions

In this section, we consider the existence of class  $A_b$  and class  $A_\infty$  solutions of (1.1). The necessary and sufficient conditions for the existence of class  $A_b$  solutions and the sufficient conditions for the existence of class  $A_\infty$  solutions are provided.

**Theorem 3.1.** Equation (1.1) has both positive and negative class A solutions.

*Proof.* Similar to the proof of Theorem 3.1 in [17].  $\square$

**Theorem 3.2.** Equation (1.1) has a positive (negative)  $A_b$  solution if and only if  $J_1 < \infty$  ( $J_2 > -\infty$ ).

*Proof. Necessity.* Without loss of generality, we assume that  $x$  is a positive  $A_b$  solution. In this case, there exists a  $b \geq a$  such that  $x(t) > 0$  and  $x'(t) > 0$  for  $t \geq b$ . Note that  $x(\infty) := \lim_{t \rightarrow \infty} x(t) < \infty$ , we have

$$\begin{aligned} m_1 &:= \min_{x(b) \leq r \leq x(\infty)} g(r) > 0, \\ c_1 &:= \max_{b \leq t < \infty} h(x(t)) \leq \max_{x(b) \leq r \leq x(\infty)} h(r) < \infty. \end{aligned} \quad (3.1)$$

Then

$$p(t)h(x(t))f(x'(t)) = p(b)h(x(b))f(x'(b)) + \int_b^t q(s)g(x(s))ds \geq m_1 \int_b^t q(s)ds, \quad (3.2)$$

and hence

$$\frac{1}{p(t)} \int_b^t q(s)ds \leq \frac{1}{m_1} h(x(t))f(x'(t)). \quad (3.3)$$

Taking  $f^{-1}$  on both sides and applying (H1) imply that

$$f^{-1} \left( \frac{1}{p(t)} \int_b^t q(s)ds \right) \leq f^{-1} \left( \frac{1}{m_1} h(x(t))f(x'(t)) \right) \leq M_1 f^{-1} \left( \frac{c_1}{m_1} \right) x'(t). \quad (3.4)$$

Therefore

$$J_1 = \int_b^\infty f^{-1} \left( \frac{1}{p(t)} \int_b^t q(s)ds \right) dt \leq M_1 f^{-1} \left( \frac{c_1}{m_1} \right) (x(\infty)) - (x(b)) < \infty. \quad (3.5)$$

*Sufficiency.* Define

$$m_2 = \max_{1 \leq r \leq 2} g(r) > 0, \quad c_2 = \min_{1 \leq r \leq 2} h(r) > 0. \quad (3.6)$$

Since  $J_1 < \infty$ , we may select a  $d \geq a$  such that

$$\int_d^\infty f^{-1} \left( \frac{1}{p(t)} \int_d^t q(s)ds \right) dt \leq \frac{1}{M_1 f^{-1}(m_2/c_2)}. \quad (3.7)$$

Let  $CB[d, \infty)$  be the Banach space of all bounded and continuous functions defined on  $[d, \infty)$  endowed with the supremum norm, and let  $X = \{x \in CB[d, \infty) : 1 \leq x(t) \leq 2, t \geq d\}$ . Clearly,  $X$  is a bounded convex subset of  $CB[d, \infty)$ . Define a mapping  $F_1 : X \rightarrow CB[d, \infty)$  by

$$(F_1x)(t) = 1 + \int_d^t f^{-1} \left( \frac{1}{p(s)h(x(s))} \int_d^s q(\sigma)g(x(\sigma))d\sigma \right) ds. \quad (3.8)$$

In order to apply Schauder's fixed-point theorem to show that  $F_1$  has a fixed point in  $X$ , we need to prove that  $F_1$  maps into  $X$  and is continuous, and  $F_1(X)$  is precompact in  $CB[d, \infty)$ .

Let  $x \in X$ . Considering (3.7), we have

$$\begin{aligned} 1 \leq (F_1x)(t) &\leq 1 + M_1 f^{-1} \left( \frac{m_2}{c_2} \right) \int_d^t f^{-1} \left( \frac{1}{p(s)} \int_d^s q(\sigma)d\sigma \right) ds \\ &\leq 1 + M_1 f^{-1} \left( \frac{m_2}{c_2} \right) \int_d^\infty f^{-1} \left( \frac{1}{p(t)} \int_d^t q(s)ds \right) dt \leq 2. \end{aligned} \quad (3.9)$$

Hence,  $F_1$  maps  $X$  into  $X$ .

Now, we show that if  $x_n, x^* \in X$  and  $\|x_n - x^*\| \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\|F_1x_n - F_1x^*\| \rightarrow 0$ . Indeed, for any fixed  $s \in [d, \infty)$ , since  $x_n(s) \rightarrow x^*(s)$  as  $n \rightarrow \infty$ , we have

$$\begin{aligned} &\left| f^{-1} \left( \frac{1}{p(s)h(x_n(s))} \int_d^s q(\sigma)g(x_n(\sigma))d\sigma \right) \right. \\ &\quad \left. - f^{-1} \left( \frac{1}{p(s)h(x^*(s))} \int_d^s q(\sigma)g(x^*(\sigma))d\sigma \right) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.10)$$

Note that

$$\begin{aligned} &\left| f^{-1} \left( \frac{1}{p(s)h(x_n(s))} \int_d^s q(\sigma)g(x_n(\sigma))d\sigma \right) - f^{-1} \left( \frac{1}{p(s)h(x^*(s))} \int_d^s q(\sigma)g(x^*(\sigma))d\sigma \right) \right| \\ &\leq \left| f^{-1} \left( \frac{1}{p(s)h(x_n(s))} \int_d^s q(\sigma)g(x_n(\sigma))d\sigma \right) \right| \\ &\quad + \left| f^{-1} \left( \frac{1}{p(s)h(x^*(s))} \int_d^s q(\sigma)g(x^*(\sigma))d\sigma \right) \right| \\ &\leq 2M_1 f^{-1} \left( \frac{m_2}{c_2} \right) f^{-1} \left( \frac{1}{p(s)} \int_d^s q(\sigma)d\sigma \right) := F(s), \end{aligned} \quad (3.11)$$

and that

$$\int_d^\infty F(s)ds = \int_d^\infty 2M_1 f^{-1} \left( \frac{m_2}{c_2} \right) f^{-1} \left( \frac{1}{p(s)} \int_d^s q(\sigma)d\sigma \right) ds = 2M_1 f^{-1} \left( \frac{m_2}{c_2} \right) J_1. \quad (3.12)$$

By Lebesgue's dominated convergence theorem and considering (3.11) and (3.12) we have

$$\begin{aligned}
& \|F_1x_n - F_1x^*\| \\
& \leq \sup_{b \leq t < \infty} \int_d^t \left| f^{-1} \left( \frac{1}{p(s)h(x_n(s))} \int_d^s q(\sigma)g(x_n(\sigma))d\sigma \right) \right. \\
& \quad \left. - f^{-1} \left( \frac{1}{p(s)h(x^*(s))} \int_d^s q(\sigma)g(x^*(\sigma))d\sigma \right) \right| ds \\
& \leq \int_d^\infty \left| f^{-1} \left( \frac{1}{p(s)h(x_n(s))} \int_d^s q(\sigma)g(x_n(\sigma))d\sigma \right) \right. \\
& \quad \left. - f^{-1} \left( \frac{1}{p(s)h(x^*(s))} \int_d^s q(\sigma)g(x^*(\sigma))d\sigma \right) \right| ds \rightarrow 0
\end{aligned} \tag{3.13}$$

as  $n \rightarrow \infty$ . Therefore,  $F_1$  is continuous in  $X$ .

Finally, we show the precompactness of  $F_1(X)$  in  $CB[d, \infty)$ , which means that for any sequence  $x_n \in X$ ,  $F_1x_n$  has a convergent subsequence in  $CB[d, \infty)$ . This can be proved by showing that  $F_1x_n$  has a convergent subsequence in  $C[b_1, b_2]$  for any compact subinterval  $[b_1, b_2]$  of  $[b, \infty)$  as well as the diagonal rule. In fact,  $F_1x_n$  is uniformly bounded on  $[b_1, b_2]$ . Since

$$(F_1x_n)'(t) = f^{-1} \left( \frac{1}{p(t)h(x_n(t))} \int_d^t q(s)g(x_n(s))ds \right) \leq M_1 f^{-1} \left( \frac{m_2}{c_2} \right) f^{-1} \left( \frac{1}{p(t)} \int_d^t q(t)ds \right). \tag{3.14}$$

By the Mean Value Theorem, we have

$$\begin{aligned}
& |(F_1x_n)(t_1) - (F_1x_n)(t_2)| \\
& = |(F_1x_n)'(\xi)(t_1 - t_2)| \leq M_1 f^{-1} \left( \frac{m_2}{c_2} \right) \max_{b_1 \leq t \leq b_2} f^{-1} \left( \frac{1}{p(t)} \int_d^t q(s)ds \right) |t_1 - t_2|.
\end{aligned} \tag{3.15}$$

Then  $F_1x_n$  is uniformly bounded and equicontinuous in  $C[b_1, b_2]$ . So  $F_1x_n$  has a convergent subsequence in  $C[b_1, b_2]$  by Arzelà-Ascoli Theorem.

Now all conditions of Schauder's fixed-point theorem are satisfied, so  $F_1$  has a fixed point  $\bar{x}$  in  $X$ , that is,

$$\bar{x}(t) = 1 + \int_d^t f^{-1} \left( \frac{1}{p(s)h(\bar{x}(s))} \int_d^s q(\sigma)g(\bar{x}(\sigma))d\sigma \right) ds. \tag{3.16}$$

It is easy to verify that  $[p(t)h(\bar{x}(t))f(\bar{x}'(t))] = q(t)g(\bar{x}(t))$ . Hence,  $\bar{x}$  is a positive  $A_b$  solution of (1.1). The proof is complete.  $\square$



**Theorem 3.3.** *Let (H2) and (H3) hold. Then*

- (a)  $A_\infty = \emptyset$  if and only if  $J_1 < \infty$  and  $J_2 > -\infty$ .
- (b) Equation (1.1) has a positive (negative)  $A_\infty$  solution if  $J_1 = \infty$  ( $J_2 = -\infty$ ).

*Proof.* By Theorem 2.5 all solutions of (1.1) are bounded if and only if  $J_1 < \infty$  and  $J_2 > -\infty$ , so part (a) follows.

If  $J_1 = \infty$ , there is no positive  $A_b$  solution of (1.1) from Theorem 3.2. Therefore, Theorem 3.1 guarantees the existence of a positive  $A_\infty$  solution of (1.1). Similarly, there exists a negative  $A_\infty$  solution of (1.1) if  $J_2 = -\infty$ .  $\square$

## 4. Class B Solutions

In this section the existence of class  $B$ ,  $B_b$ , and  $B_0$  solutions are discussed. We assume that (1.1) has a unique solution for any initial conditions  $x(a) = x_0 \neq 0$  and  $x'(a) = x_1$ .

**Theorem 4.1.** *Assume the following assumptions hold.*

- (H2a) *There exists a continuous function  $G(r) : \mathbb{R} \rightarrow \mathbb{R}$  such that  $G$  is increasing,  $rG(r) > 0$  for  $r \neq 0$  and  $|g(r)| \leq |G(r)|$ ;*
- (H4) *There exists  $r_0 > 0$  such that*

$$\int_0^{\pm r_0} \frac{dr}{f^{-1}(z(r))} = \infty. \quad (4.1)$$

Then (1.1) has

- (a) *both positive and negative solutions in class  $B$ ;*
- (b) *no solution which is eventually identically equal to zero.*

*Proof.* (a) We prove that class  $B$  has a positive solution, the case of having a negative solution is similar. Assume  $x_0 > 0$ . The solution of (1.1) with initial conditions  $x(a) = x_0$  and  $x'(a) = c$ , denoted by  $x(t) := x(t, c)$ , has the form

$$x(t) = x_0 + \int_a^t f^{-1} \left( \frac{p(a)h(x_0)f(c)}{p(s)h(x(s))} + \frac{1}{p(s)h(x(s))} \int_a^s q(\sigma)g(x(\sigma))d\sigma \right) ds. \quad (4.2)$$

Define two sets  $U$  and  $L$  as

$$\begin{aligned} U &= \left\{ c \in \mathbb{R} : \text{there exists some } \bar{t} \geq a \text{ such that } x'(\bar{t}, c) > 0 \right\}, \\ L &= \left\{ c \in \mathbb{R} : \text{there exists some } \bar{t} \geq a \text{ such that } x(\bar{t}, c) < 0 \right\}. \end{aligned} \quad (4.3)$$

Then  $U \cap L = \emptyset$ . Clearly,  $U \neq \emptyset$ . We claim that  $U$  is open. Indeed, if  $c_0 \in U$ , there exists  $\bar{t} > a$  such that  $x'(\bar{t}, c_0) > 0$ . For any  $c \in \mathbb{R}$ , we have

$$\begin{aligned} & p(\bar{t})h(x(\bar{t}, c_0))f(x'(\bar{t}, c_0)) - p(\bar{t})h(x(\bar{t}, c))f(x'(\bar{t}, c)) \\ &= p(a)h(x_0)f(c_0) - p(a)h(x_0)f(c) + \int_a^{\bar{t}} q(s)(g(x(s, c_0)) - g(x(s, c)))ds. \end{aligned} \quad (4.4)$$

Since (1.1) has a unique solution for any initial conditions  $x(a) \neq 0$ ,  $x'(a)$ , this solution is continuously dependent on initial data. If  $c \rightarrow c_0$ , we have  $g(x(s, c)) - g(x(s, c_0)) \rightarrow 0$  uniformly for  $s$  on  $[a, \bar{t}]$ . Hence,  $x'(\bar{t}, c) > 0$  for all  $c$  that are close to  $c_0$ , this proves the openness of  $U$ .

Next we show that  $L \neq \emptyset$ . Define

$$M_2 := \min_{0 \leq r \leq x_0} h(r) > 0, \quad M_3 := \min_{a \leq t \leq a+1} p(t) > 0. \quad (4.5)$$

Let

$$c < f^{-1} \left( \frac{M_2 M_3 f^{-1}(-x_0) - G(x_0) \int_a^{a+1} q(s) ds}{p(a)h(x_0)} \right) < 0. \quad (4.6)$$

If there exists  $b \in (a, a+1]$  such that  $x(b, c) < 0$ , then  $c \in L$  and  $L \neq \emptyset$ . Otherwise,  $x(t, c) \geq 0$  on  $[a, a+1]$ . In this case, we claim  $x'(t, c) < 0$  on  $[a, a+1]$ . If this is not true, since  $x'(a, c) = c < 0$ , there exists  $t_1 \in (a, a+1]$  such that  $x'(t_1, c) = 0$  and  $x'(t, c) < 0$  for  $t \in [a, t_1]$ . Taking into account (4.6) we have

$$\begin{aligned} 0 &= p(t)h(x(t_1, c))f(x'(t_1, c)) \\ &= p(a)h(x_0)f(c) + \int_a^{t_1} q(s)g(x(s, c))ds \\ &\leq p(a)h(x_0)f(c) + G(x_0) \int_a^{a+1} q(s)ds < 0. \end{aligned} \quad (4.7)$$

This is a contradiction and hence  $x'(t, c) < 0$  on  $[a, a+1]$ . Notice that

$$\begin{aligned} x(a+1, c) &= x_0 + \int_a^{a+1} f^{-1} \left( \frac{p(a)h(x_0)f(c)}{p(t)h(x(t))} + \frac{1}{p(t)h(x(t))} \int_a^t q(s)g(x(s))ds \right) dt \\ &\leq x_0 + \int_a^{a+1} f^{-1} \left( \frac{p(a)h(x_0)f(c) + G(x_0) \int_a^{a+1} q(s)ds}{M_2 M_3} \right) dt < 0, \end{aligned} \quad (4.8)$$

we know  $c \in L$ . Clearly,  $L$  is open, then  $\mathbb{R} - (U \cup L) \neq \emptyset$ . Take  $c \in \mathbb{R} - (U \cup L)$ ,  $x(t, c)$  is a nonincreasing nonnegative solution on  $[a, \infty)$ . We will show that  $x(t, c) > 0$  on  $[a, \infty)$ . If not, there exists  $t_0 > a$  such that  $x(t_0) = 0$  and  $x(t) = 0$  for  $t \geq t_0$  and  $x'(t_0) = 0$ . Note that

for  $t \in [a, t_0]$  we have

$$\begin{aligned} x'(t) &= f^{-1}\left(-\frac{1}{p(t)h(x(t))} \int_t^{t_0} q(s)g(x(s))ds\right) \\ &\geq f^{-1}\left(-\frac{G(x(t))}{p(t)h(x(t))} \int_t^{t_0} q(s)ds\right) \\ &\geq M_1 f^{-1}(z(x(t))) f^{-1}\left(-\frac{1}{p(t)} \int_t^{t_0} q(s)ds\right). \end{aligned} \quad (4.9)$$

Dividing both sides by  $f^{-1}(z(x(t)))$  and integrating from  $a$  to  $t_0$ , we have

$$\int_a^{t_0} \frac{x'(t)}{f^{-1}(z(x(t)))} dt \geq M_1 \int_a^{t_0} f^{-1}\left(-\frac{1}{p(t)} \int_t^{t_0} q(s)ds\right) dt. \quad (4.10)$$

That is

$$\int_0^{x_0} \frac{1}{f^{-1}(z(r))} dr \leq -M_1 \int_a^{t_0} f^{-1}\left(-\frac{1}{p(t)} \int_t^{t_0} q(s)ds\right) dt < \infty, \quad (4.11)$$

a contradiction to (H4). Therefore,  $x(t) > 0$  for  $t \geq a$  and  $x \in B$ .

The proof of part (b) follows from the end part of the proof of part (a).  $\square$

**Theorem 4.2.** Equation (1.1) has a positive (negative)  $B_b$  solution if and only if  $J_4 > -\infty$  ( $J_3 < \infty$ ).

*Proof. Necessity.* We assume that  $x$  is a positive  $B_b$  solution. The case of negative  $B_b$  solution is similar. In this case, we have  $x(t) > 0$  and  $x'(t) < 0$  for  $t \geq a$ . Let

$$m_1 = \min_{x(\infty) \leq r \leq x(a)} g(r) > 0, \quad c_1 = \max_{x(\infty) \leq r \leq x(a)} h(r) > 0 \quad (4.12)$$

and note that  $p(t)h(x(t))f(x'(t)) < 0$ ,  $(p(t)h(x(t))f(x'(t)))' > 0$ . Then

$$\lim_{t \rightarrow \infty} p(t)h(x(t))f(x'(t)) = B \leq 0. \quad (4.13)$$

Integrating both sides of (1.1) from  $t$  to  $\infty$  implies that

$$\begin{aligned} m_1 \int_t^\infty q(s)ds &\leq \int_t^\infty q(s)g(x(s))ds = B - (p(t)h(x(t))f(x'(t))) \\ &\leq -p(t)h(x(t))f(x'(t)). \end{aligned} \quad (4.14)$$

Hence,

$$f^{-1}\left(-\frac{1}{p(t)}\int_t^\infty q(s)ds\right) \geq M_1 f^{-1}\left(\frac{c_1}{m_1}\right)x'(t). \quad (4.15)$$

Again, integrating both sides of the above inequality we have

$$J_4 = \int_a^\infty f^{-1}\left(-\frac{1}{p(t)}\int_t^\infty q(s)ds\right)dt \geq M_1 f^{-1}\left(\frac{c_1}{m_1}\right)(x(\infty) - x(a)) > -\infty. \quad (4.16)$$

*Sufficiency.* Let

$$m_2 = \max_{1 \leq r \leq 2} g(r) > 0, \quad c_2 = \min_{1 \leq r \leq 2} h(r) > 0. \quad (4.17)$$

Since  $J_4 > -\infty$  we choose  $d > a$  such that

$$\int_d^\infty f^{-1}\left(-\frac{1}{p(t)}\int_t^\infty q(s)ds\right)dt \geq -\frac{1}{M_1 f^{-1}(m_2/c_2)}. \quad (4.18)$$

Let  $X$  and  $CB[d, \infty)$  as defined in Theorem 3.2. Define  $F_2 : X \rightarrow CB[d, \infty)$  by

$$(F_2x)(t) = 1 - \int_t^\infty f^{-1}\left(-\frac{1}{p(s)h(x(s))}\int_s^\infty q(\sigma)g(x(\sigma))d\sigma\right)ds. \quad (4.19)$$

For any  $x \in X$ , we have

$$1 \leq (F_2x)(t) \leq 1 - \int_d^\infty M_1 f^{-1}\left(\frac{m_2}{c_2}\right) f^{-1}\left(-\frac{1}{p(s)}\int_s^\infty q(\sigma)d\sigma\right)ds \leq 2. \quad (4.20)$$

This proves that  $F_2$  maps  $X$  into  $X$ . Similar to the proof of Theorem 3.2, we are able to show that  $F_2$  is continuous in  $X$ , and  $F_2(X)$  is precompact in  $CB[d, \infty)$ . Then  $F_2$  has a fixed-point  $\bar{x}(t)$  in  $X$  by Schauder's fixed-point theorem, that is,

$$\bar{x}(t) = 1 - \int_t^\infty f^{-1}\left(-\frac{1}{p(s)h(\bar{x}(s))}\int_s^\infty q(\sigma)g(\bar{x}(\sigma))d\sigma\right)ds. \quad (4.21)$$

It is easy to verify that  $\bar{x}(t)$  is a positive  $B_b$  solution of (1.1). The proof is complete.  $\square$

**Theorem 4.3.** *Let (H2a) and (H4) hold and let  $J_5 = \infty$ . Then (1.1) has a positive (negative)  $B_0$  solution if and only if  $J_4 = -\infty$  ( $J_4 = \infty$ ).*

*Proof.* We prove the assertion for positive solutions without loss of generality.

*Necessity.* Assume  $x(t)$  is a positive  $B_0$  solution. Then  $x(t) > 0$  and  $x'(t) < 0$  for  $t \geq a$ ,  $x(\infty) = 0$ , and  $\lim_{t \rightarrow \infty} p(t)h(x(t))f(x'(t)) = L \in (-\infty, 0]$ . We claim  $L = 0$ . In fact, if  $L < 0$ , since  $p(t)h(x(t))f(x'(t))$  is negative and increasing on  $[a, \infty)$ , then  $p(t)h(x(t))f(x'(t)) \leq L$  and

$$x'(t) \leq f^{-1}\left(\frac{L}{c_1 p(t)}\right) \leq M_1 f^{-1}\left(\frac{L}{c_1}\right) f^{-1}\left(\frac{1}{p(t)}\right), \quad (4.22)$$

where  $c_1 = \max_{0 \leq r \leq x(a)} h(r) > 0$ .

Integrating both sides from  $a$  to  $\infty$  and noting that  $x(\infty) = 0$ , we have

$$x(a) \geq -M_1 f^{-1}\left(\frac{L}{c_1}\right) \int_a^\infty f^{-1}\left(\frac{1}{p(t)}\right) dt, \quad (4.23)$$

a contradiction to  $J_5 = \infty$  and hence  $L = 0$ .

Integrating both sides of (1.1) from  $t$  to  $\infty$  we have

$$p(t)h(x(t))f(x'(t)) = - \int_t^\infty q(s)g(x(s))ds. \quad (4.24)$$

Then

$$\begin{aligned} x'(t) &= f^{-1}\left(-\frac{1}{p(t)h(x(t))} \int_t^\infty q(s)g(x(s))ds\right) \\ &\geq M_1 f^{-1}(z(x(t))) f^{-1}\left(-\frac{1}{p(t)} \int_t^\infty q(s)ds\right). \end{aligned} \quad (4.25)$$

Hence,

$$\frac{x'(t)}{f^{-1}(z(x(t)))} \geq M_1 f^{-1}\left(-\frac{1}{p(t)} \int_t^\infty q(s)ds\right). \quad (4.26)$$

Integrating both sides of the above inequality from  $a$  to  $\infty$  implies that

$$\int_0^{x(a)} \frac{dr}{f^{-1}(z(r))} \leq -M_1 \int_a^\infty f^{-1}\left(-\frac{1}{p(t)} \int_t^\infty q(s)ds\right) dt. \quad (4.27)$$

Therefore,  $J_4 = -\infty$  from (H4).

*Sufficiency.* By Theorem 4.1 (1.1) has a positive class B solution  $x$ , either  $x \in B_b$  or  $x \in B_0$ . Note that  $J_4 = -\infty$  implies that  $x \notin B_b$  from Theorem 4.2. So  $x \in B_0$ . The proof is complete.  $\square$

## 5. Remarks

In this section, we present several remarks about comparison of our results with the existing ones in the literature.

Theorems 2.3 and 2.5 improve [10, Theorem 1] since (H3) reduces to (iii) of [10] if  $f(r) = r$  and the differentiability of  $p(\cdot)$  and  $h(\cdot)$  is not required. Theorems 2.3, 2.5, and 4.2 complement and generalize [2, Theorem 8]. Moreover, under (H2), Theorems 2.3, 2.5, and 4.2 improve [2, Theorem 8] since (H3) improves (22) of [2]; see the discussion in [16]. Theorem 2.5 generalizes [13, Theorem 3.9]. Theorems 2.3, 3.1, and 4.2 generalize [16, Theorem 1]. Theorem 3.2 generalizes [2, Theorem 3], [16, Theorem 3], and [18, Theorem 2.1]. Theorem 3.3 generalizes [18, Theorem 2.2]. Theorem 4.1 generalizes [13, Theorem 2.1] and improves [3, Theorem 6] under (H2a) since (hp) in [3] is replaced by a weaker condition (H4). Theorem 4.2 generalizes [2, Theorem 1], [16, Theorem 5], and [18, Theorem 3.1]. Theorem 4.3 generalizes [16, Theorem 6] and [18, Theorem 3.2].

## References

- [1] M. Cecchi, Z. Došlá, and M. Marini, "On the dynamics of the generalized Emden-Fowler equation," *Georgian Mathematical Journal*, vol. 7, no. 2, pp. 269–282, 2000.
- [2] M. Cecchi, Z. Došlá, and M. Marini, "On nonoscillatory solutions of differential equations with  $p$ -Laplacian," *Advances in Mathematical Sciences and Applications*, vol. 11, no. 1, pp. 419–436, 2001.
- [3] M. Cecchi, M. Marini, and G. Villari, "On the monotonicity property for a certain class of second order differential equations," *Journal of Differential Equations*, vol. 82, no. 1, pp. 15–27, 1989.
- [4] M. Cecchi, M. Marini, and G. Villari, "On some classes of continuable solutions of a nonlinear differential equation," *Journal of Differential Equations*, vol. 118, no. 2, pp. 403–419, 1995.
- [5] S. Z. Chen, Q. G. Huang, and L. H. Erbe, "Bounded and zero-convergent solutions of a class of Stieltjes integro-differential equations," *Proceedings of the American Mathematical Society*, vol. 113, no. 4, pp. 999–1008, 1991.
- [6] J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries. Vol. I*, vol. 106 of *Research Notes in Mathematics*, Pitman, Boston, Mass, USA, 1985.
- [7] O. Došlý and P. Rehák, *Half-linear Differential Equations*, vol. 202 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, The Netherlands, 2005.
- [8] K. I. Kamo and H. Usami, "Asymptotic forms of positive solutions of second-order quasilinear ordinary differential equations," *Advances in Mathematical Sciences and Applications*, vol. 10, no. 2, pp. 673–688, 2000.
- [9] K. I. Kamo and H. Usami, "Asymptotic forms of positive solutions of second-order quasilinear ordinary differential equations with sub-homogeneity," *Hiroshima Mathematical Journal*, vol. 31, no. 1, pp. 35–49, 2001.
- [10] M. Marini, "Monotone solutions of a class of second order nonlinear differential equations," *Nonlinear Analysis*, vol. 8, no. 3, pp. 261–271, 1984.
- [11] M. Marini, "On nonoscillatory solutions of a second-order nonlinear differential equation," *Bollettino della Unione Matematica Italiana C*, vol. 3, no. 1, pp. 189–202, 1984.
- [12] J. D. Mirzov, *Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations*, vol. 14 of *Folia Facultatis Scientiarum Naturalium Universitatis Masarykianae Brunensis. Mathematica*, Masaryk University, Brno, Czech Republic, 2004.
- [13] M. Mizukami, M. Naito, and H. Usami, "Asymptotic behavior of solutions of a class of second order quasilinear ordinary differential equations," *Hiroshima Mathematical Journal*, vol. 32, no. 1, pp. 51–78, 2002.
- [14] Y. Nagabuchi and M. Yamamoto, "On the monotonicity of solutions for a class of nonlinear differential equations of second order," *Mathematica Japonica*, vol. 37, no. 4, pp. 665–673, 1992.
- [15] T. Tanigawa, "Existence and asymptotic behavior of positive solutions of second order quasilinear differential equations," *Advances in Mathematical Sciences and Applications*, vol. 9, no. 2, pp. 907–938, 1999.
- [16] L. Wang, "On monotonic solutions of systems of nonlinear second order differential equations," *Nonlinear Analysis*, vol. 70, no. 7, pp. 2563–2574, 2009.
- [17] L. Wang, R. McKee, and L. Usyk, "Continuability and boundedness of solutions to nonlinear second-order differential equations," *Electronic Journal of Differential Equations*, vol. 2010, no. 165, pp. 1–12, 2010.
- [18] L. Wang and J. Ballinger, "Monotonicity and asymptotic behavior of solutions for second-order differential equations," *Journal of Nonlinear Systems and Applications*, pp. 53–57, 2012.



# Hindawi

Submit your manuscripts at  
<http://www.hindawi.com>

