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Research Article

Existence Results of Nondensely Defined Fractional Evolution Differential Inclusions

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This paper deals with the existence results of integral solutions for nondensely defined fractional evolution differential inclusions. Our approach is based on integrated semigroup theory and a fixed point theorem for condensing map due to Martelli. An example is also given to illustrate our results.

1. Introduction

In the past decades, the theory of fractional differential equations and inclusions has become an important area of investigation because of its wide applicability in many branches of physics, economics, and technical sciences [1–10].

Our aim in this paper is to study the existence of the integral solutions for the fractional semilinear differential inclusions, of the form

$$\begin{gathered} D^q x(t) \in Ax(t) + F(t, x(t)), \quad t \in (0, b], \\ x(0) = x_0, \end{gathered} \tag{1.1}$$

where D^q is the Caputo fractional derivative of order $0 < q < 1$, $b > 0$. $A : D(A) \subset X \to X$ is a nondensely closed linear operator on X, X is a real Banach space with the norm $|\cdot|$. $F : [0, \infty) \times X \to \mathcal{P}(X)$ is a nonempty, bounded, closed, and convex multivalued map, and $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

It is well known that one important way to introduce the concepts of mild solutions for fractional evolution equations is based on some probability densities and Laplace transform. This method was initialed by El-Borai [11] and developed by Zhou and Jiao [12]. Since then, many interesting existence results of mild solutions for fractional evolution equations appeared [13–16]. We will point out that the unbounded operators A in their papers were assumed to be densely defined and generate a strongly continuous semigroup.

However, as indicated in [17], we sometimes need to deal with nondensely defined operators and there are extensive work on this subject when equations involve the integral-order derivative, see monograph [18–23] and references therein. Very recently, Wang and Zhou [24] considered problem (1.1) in the case when A is densely defined and generates a strongly continuous semigroup. As far as we know, there are few papers dealing with semilinear fractional differential systems with nondense domain. Motivated by this, we discuss the integral solution to problem (1.1) by using probability densities and integral semigroup. We turn the integral solutions of problem (1.1) to a new formula something like the mild solutions. This new formula of integral solutions is firstly introduced even in fractional evolution equations. Thus, our work can be seen as a supplement to work [24] and a contribution to this emerging field of fractional differential equations with nondense domain.

This paper will be organized as follows. In Section 2, we recall some basic definitions and preliminary facts for integrated semigroup, fractional calculus, and multivalued map which will be used later. Section 3 is devoted to the existence results of integral solutions for problem (1.1). We will present in Section 4 an example which illustrates our main theorem.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary results which are used in the rest of the paper.

We denote by $C([0,b],X)$ the Banach space of all continuous functions from $[0,b]$ into X with the norm

$$\|y\| = \sup\{|y(t)| : t \in [0,b]\}. \tag{2.1}$$

$B(X)$ denotes the Banach space of bounded linear operators from X into X, with the norm

$$\|N\| = \sup\{|N(y)| : |y| = 1\}, \tag{2.2}$$

where $N \in B(X)$ and $y \in X$.

Assume that $J \subset \mathbf{R}$ and $1 \le p \le \infty$. For a measurable function $m : J \to \mathbf{R}$, define the norm

$$\|m\|_{L^p J} = \begin{cases} \left(\displaystyle\int_J |m(t)|^p dt\right)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\inf_{\mu(\bar{J})=0} \left\{ \sup_{t \in J - \bar{J}} |m(t)| \right\}, & p = \infty, \end{cases} \tag{2.3}$$

where $\mu(\overline{J})$ is the Lebesgue measure on $\overline{J}$. Let $L^p(J,\mathbf{R})$ be the Banach space of all Lebesgue measurable functions $m : J \to \mathbf{R}$ with $\|\cdot\|_{L^pJ} < \infty$.

Lemma 2.1 (HöLder inequality). *Assume that $r,p \geq 1$ and $(1/r)+(1/p)=1$. If $l \in L^r(J,\mathbf{R})$, $m \in L^p(J,\mathbf{R})$, then for $1 \leq p \leq \infty$, $lm \in L^1(J,\mathbf{R})$ and*

$$\|lm\|_{L^1J} \leq \|l\|_{L^rJ}\|m\|_{L^pJ}. \tag{2.4}$$

Lemma 2.2 (Bochner theorem). *A measurable function $H : [0,b] \to X$ is Bochner's integrable if $|H|$ is Lebesgue integrable.*

Definition 2.3 (see [25]). Let X be a Banach space; an integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on X with the following properties:

(i) $S(0)=0$;

(ii) $t \to S(t)$ is strongly continuous;

(iii) $S(s)S(t) = \int_0^s (S(t+r)-S(r))dr$ for all $t,s \geq 0$.

Definition 2.4 (see [26]). An operator A is called a generator of an integrated semigroup, if there exists $\omega \in \mathbf{R}$ such that $(\omega,+\infty) \subset \rho(A)$ and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0)=0$ and $(\lambda I - A)^{-1} = \lambda\int_0^\infty e^{-\lambda t}S(t)dt$ for all $\lambda > \omega$.

Proposition 2.5 (see [25]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in X$ and $t \geq 0$,*

$$\int_0^t S(s)xds \in D(A), \qquad S(t)x = A\int_0^t S(s)xds + tx. \tag{2.5}$$

Definition 2.6 (see [26]). We say that linear operator A satisfies the Hille-Yosida condition if there exist $M \geq 0$ and $\omega \in \mathbf{R}$ such that $(\omega,+\infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda-\omega)^n \left\|R(\lambda,A)^n\right\|, n \in \mathbf{N}, \lambda > \omega\right\} \leq M. \tag{2.6}$$

Here and hereafter, we assume that A satisfies the Hille-Yosida condition. Let us introduce the part A_0 of A in $\overline{D(A)}$: $A_0 = A$ on $D(A_0) = \{x \in D(A); Ax \in \overline{D(A)}\}$. Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A. We note that $(S'(t))_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by A_0 and $\|S'(t)\| \leq Me^{\omega t}$, $t \geq 0$, where M and ω are the constants considered in the Hille-Yosida condition ([19, 27]).

Let $B_\lambda = \lambda R(\lambda,A) := \lambda(\lambda I - A)^{-1}$; then for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition it is easy to see that $\lim_{\lambda\to\infty}|B_\lambda x| \leq M|x|$.

For more properties on integral semigroup theory the interested readers may refer to [18, 27].

Definition 2.7 (see [3]). The Riemann-Liouville fractional integral of order $\alpha \in \mathbf{R}^+$ of a function $f : \mathbf{R}^+ \to X$ is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0, \tag{2.7}$$

provided the right-hand side is pointwise defined on $\mathbf{R}^+$, where Γ is the gamma function.

Remark 2.8. According to [3], $I_0^q I_0^\beta = I_0^{q+\beta}$ holds for all $q, \beta \geq 0$.

Definition 2.9 (see [3]). The Caputo fractional derivative of order $0 < \alpha < 1$ of a function $f \in C^1([0,\infty), X)$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \quad t > 0. \tag{2.8}$$

We will remark that integrals which appear in Definitions 2.7 and 2.9 are taken in Bochner's sense.

Lemma 2.10 (see [28]). *Suppose $\beta > 0$, $a(t)$ is a nonnegative, function locally integrable on $0 \leq t < T$ and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T$, $g(t) \leq M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) ds \tag{2.9}$$

on this interval. Then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, \quad 0 \leq t < T. \tag{2.10}$$

Corollary 2.11 (see [28]). *Under the hypothesis of Lemma 2.10, let $a(t)$ be a nondecreasing function on $[0,T)$. Then*

$$u(t) \leq a(t) E_\beta \left(g(t) \Gamma(\beta) t^\beta \right), \tag{2.11}$$

where E_β is the Mittag-Leffler function defined by $E_\beta(z) = \sum_{k=0}^\infty (z^k / \Gamma(k\beta + 1))$.

We also introduce some basic definitions and results of multivalued maps. See [29] for more details.

Let (X, d) be a metric space; $\mathcal{P}(X)$ denotes the family for all nonempty subsets of X. We use the following notations:

$$\begin{aligned} P_{\mathrm{cl}}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ closed}\}, \qquad P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}, \\ P_c(X) &= \{Y \in \mathcal{P}(X) : Y \text{ convex}\}, \qquad P_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}. \end{aligned} \tag{2.12}$$

A multivalued map $F : X \to \mathcal{P}(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$ and F is bounded on bounded sets if $F(B) = U_{x\in B}F(x)$ is bounded in X for all $B \in P_b(X)$, that is, $\sup_{x\in B}\{\sup\{|y| : y \in F(x)\}\} < \infty$. F is called upper semicontinuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $F(\mathcal{V}) \subset \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map F is completely continuous with nonempty compact valued, the F is u.s.c. if and only if F has closed graph, that is, $x_n \to x_*$, $y_n \to y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

Definition 2.12 (see [30]). An upper semicontinuous map $G : X \to X$ is said to be condensing if for any bounded subset $V \subset X$ with $\alpha(V) \neq 0$, one has $\alpha(G(V)) < \alpha(V)$, where α denotes the Kuratowski measure of noncompactness.

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

Theorem 2.13 (see [30]). *Let J be a compact interval and X a Banach space. Let $F : J \times C(J,X) \to P_{b,\mathrm{cl},c}(X)$, $(t,u) \mapsto F(t,u)$ be measurable with respect to t for each $u \in X$, upper semicontinuous with respect to u for each $t \in J$. Moreover, for each fixed $u \in C(J,X)$ the set*

$$N_{F,u} = \left\{ f \in L^1(J,X) : f(t) \in F(t,u) \text{ for a.e. } t \in J \right\} \tag{2.13}$$

is nonempty. Also let $\mathcal{T}$ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$; then the operator

$$\mathcal{T} \circ N_F : C(J,X) \longrightarrow P_{b,\mathrm{cl},c}(C(J,X)), \qquad u \longrightarrow (\mathcal{T} \circ N_F)(u) = \mathcal{T}(N_{F,u}) \tag{2.14}$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Theorem 2.14 (Martelli, [31]). *Let X be a Banach space and $\Phi : X \to P_{b,\mathrm{cl},c}(X)$ a condensing map. If the set*

$$U = \{x \in X : \delta x \in \Phi x \text{ for some } \delta > 1\} \tag{2.15}$$

is bounded, then Φ has a fixed point.

3. Existence of Integral Solutions

In this section we will establish the existence results for problem (1.1). Let us consider the following problem:

$$\begin{gathered} D^q x(t) = Ax(t) + f(t, x(t)), \qquad t \in (0,b], \\ x(0) = x_0, \end{gathered} \tag{3.1}$$

where $f : [0,\infty) \times X \to X$ is a given function and A is the same as that in problem (1.1).

Definition 3.1. One says that a continuous function $x : [0,b] \to X$ is an integral solution of problem (3.1) if

(i) $(1/\Gamma(q)) \int_0^t (t-s)^{q-1} x(s)ds \in D(A)$ for $t \in [0,b]$,

(ii) $x(t) = x_0 + (1/\Gamma(q)) A \int_0^t (t-s)^{q-1} x(s)ds + (1/\Gamma(q)) \int_0^t (t-s)^{q-1} f(s,x(s))ds$, $t \in [0,b]$.

Lemma 3.2. *If x is an integral solution of* (1.1), *then for all $t \in [0,b]$, $x(t) \in \overline{D(A)}$. In particular, $x(0) = x_0 \in \overline{D(A)}$.*

Proof. By Remark 2.8 and $I_0^q x(t) \in D(A)$, for each $t \in (0,b]$, we get that $I_0^1 x(t) = I_0^{1-q} I_0^q x(t) \in D(A)$. From $I_0^1 x(t) = \int_0^t x(s)ds \in D(A)$ we have $(1/h)\int_t^{t+h} x(s)ds \in D(A)$ for $h > 0$, $t+h \in (0,b]$. Hence, we deduce that

$$x(t) = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} x(s)ds \in \overline{D(A)}. \tag{3.2}$$

The proof is completed. □

Lemma 3.3 (see [32]). *Let $\Psi_q(\theta) = (1/\pi)\sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} (\Gamma(nq+1)/n!) \sin(n\pi q)$, $\theta \in \mathbf{R}^+$; then $\Psi_q(\theta)$ is a one-sided stable probability density function and its Laplace transform is given by*

$$\int_0^{\infty} e^{-p\theta} \Psi_q(\theta) d\theta = e^{-p^q}, \quad q \in (0,1),\ p > 0. \tag{3.3}$$

Lemma 3.4. *The integral solution in Definition 3.1 is given by*

$$x(t) = \int_0^{\infty} h_q(\theta) S'(t^q\theta) x_0 d\theta + \lim_{\lambda \to \infty} q \int_0^t \int_0^{\infty} \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q \theta\right) B_\lambda f(s,x(s)) d\theta ds, \tag{3.4}$$

where $h_q(\theta) = (1/q)\theta^{-1-(1/q)} \Psi_q(\theta^{-1/q})$ is the probability density function defined on $\mathbf{R}^+$.

Proof. From the definition, we have

$$x(t) = x_0 + \frac{1}{\Gamma(q)} A \int_0^t (t-s)^{q-1} x(s) ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s)) ds, \quad t \in [0,b]. \tag{3.5}$$

Consider the Laplace transform

$$v(p) = \int_0^{\infty} e^{-pt} B_\lambda x(t) dt, \qquad w(p) = \int_0^{\infty} e^{-pt} B_\lambda f(t,x(t)) dt, \quad p > 0. \tag{3.6}$$

Note that for each $t > 0$, $B_\lambda x(t)$, $B_\lambda f(t, x(t)) \in D(A)$, then we have $v(p)$, $w(p) \in \overline{D(A)}$. Applying (3.6) to (3.5) yields

$$\begin{aligned} v(p) &= \frac{1}{p} B_\lambda x_0 + \frac{1}{p^q} A v(p) + \frac{1}{p^q} w(p) \\ &= p^{q-1} \left(p^q I - A\right)^{-1} B_\lambda x_0 + \left(p^q I - A\right)^{-1} w(p) \\ &= p^{q-1} \int_0^\infty e^{-p^q s} S'(s) B_\lambda x_0 ds + \int_0^\infty e^{-p^q s} S'(s) w(p) ds, \end{aligned} \tag{3.7}$$

where I is the identity operator defined on X.

From (3.3), we get

$$\begin{aligned} p^{q-1} \int_0^\infty e^{-p^q s} S'(s) B_\lambda x_0 ds &= \int_0^\infty p^{q-1} e^{-(pt)^q} S'(t^q) B_\lambda x_0 q t^{q-1} dt \\ &= \int_0^\infty -\frac{1}{p} \frac{d}{dt} \left(e^{-(pt)^q}\right) S'(t^q) B_\lambda x_0 dt \\ &= \iint_0^\infty \left[\theta \Psi_q(\theta) e^{-pt\theta} S'(t^q) B_\lambda x_0\right] d\theta dt \\ &= \iint_0^\infty \left[\Psi_q(\theta) e^{-ps} S'\left(\left(\frac{s}{\theta}\right)^q\right) B_\lambda x_0\right] d\theta ds \\ &= \int_0^\infty e^{-pt} \left[\int_0^\infty \Psi_q(\theta) S'\left(\left(\frac{t}{\theta}\right)^q\right) B_\lambda x_0 d\theta\right] dt, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \int_0^\infty e^{-p^q s} S'(s) w(p) ds &= \iint_0^\infty e^{-p^q s} e^{-pt} S'(s) B_\lambda f(t, x(t)) dt\, ds \\ &= \iint_0^\infty q s^{q-1} e^{-(ps)^q} e^{-pt} S'(s^q) B_\lambda f(t, x(t)) dt\, ds \\ &= \iint_0^\infty \int_0^\infty q \Psi_q(\theta) e^{-ps\theta} e^{-pt} S'(s^q) B_\lambda f(t, x(t)) d\theta dt\, ds \\ &= \iint_0^\infty \int_0^\infty q \Psi_q(\theta) e^{-p(s+t)} \frac{s^{q-1}}{\theta^q} S'\left(\left(\frac{s}{\theta}\right)^q\right) B_\lambda f(t, x(t)) d\theta dt\, ds \\ &= \int_0^\infty e^{-ps} q \int_0^s \int_0^\infty \Psi_q(\theta) \frac{(s-t)^{q-1}}{\theta^q} S'\left(\frac{(s-t)^q}{\theta^q}\right) B_\lambda f(t, x(t)) d\theta dt\, ds \\ &= \int_0^\infty e^{-pt} q \int_0^t \int_0^\infty \Psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} S'\left(\frac{(t-s)^q}{\theta^q}\right) B_\lambda f(s, x(s)) d\theta ds\, dt. \end{aligned} \tag{3.9}$$

According to (3.7), (3.8), and (3.9), we have

$$\begin{aligned} v(p) &= \int_0^\infty e^{-pt}\left[\int_0^\infty \Psi_q(\theta) S'\left(\left(\frac{t}{\theta}\right)^q\right) B_\lambda x_0 d\theta\right] dt \\ &\quad + \int_0^\infty e^{-pt} q \int_0^t \int_0^\infty \Psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} S'\left(\frac{(t-s)^q}{\theta^q}\right) B_\lambda f(s, x(s)) d\theta ds\, dt. \end{aligned} \tag{3.10}$$

Inverting the last Laplace transform, we obtain

$$\begin{aligned} B_\lambda x(t) &= \int_0^\infty \Psi_q(\theta) S'\left(\left(\frac{t}{\theta}\right)^q\right) B_\lambda x_0 d\theta \\ &\quad + q \int_0^t \int_0^\infty \Psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} S'\left(\frac{(t-s)^q}{\theta^q}\right) B_\lambda f(s, x(s)) d\theta ds \\ &= \int_0^\infty h_q(\theta) S'(t^q\theta) B_\lambda x_0 d\theta \\ &\quad + q \int_0^t \int_0^\infty \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q\theta\right) B_\lambda f(s, x(s)) d\theta ds. \end{aligned} \tag{3.11}$$

In view of $\lim_{\lambda\to\infty} B_\lambda x = x$ for $x \in \overline{D(A)}$ and Lemma 3.2, we have

$$x(t) = \int_0^\infty h_q(\theta) S'(t^q\theta) x_0 d\theta + \lim_{\lambda\to\infty} q \int_0^t \int_0^\infty \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q\theta\right) B_\lambda f(s, x(s)) d\theta ds. \tag{3.12}$$

The proof is completed. □

Remark 3.5. According to [32], one can easily check that

$$\int_0^\infty \theta h_q(\theta) d\theta = \int_0^\infty \frac{1}{\theta^q} \Psi_q(\theta) d\theta = \frac{1}{\Gamma(1+q)}. \tag{3.13}$$

Based on the Lemma 3.4, we will define the concept of integral solution of (1.1) as follows.

Definition 3.6. One says that a continuous function $x : [0, b] \to X$ is an integral solution of problem (1.1) if

(i) $(1/\Gamma(q)) \int_0^t (t-s)^{q-1} x(s) ds \in D(A)$ for $t \in [0, b]$,

(ii) $x(0) = x_0$ and there exists $f \in L^1([0, b], X)$ such that $f(t) \in F(t, x(t))$ for a.e. $t \in [0, b]$ and

$$x(t) = \int_0^\infty h_q(\theta) S'(t^q\theta) x_0 d\theta + \lim_{\lambda\to\infty} q \int_0^t \int_0^\infty \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q\theta\right) B_\lambda f(s) d\theta ds,\ t \in [0, b]. \tag{3.14}$$

We are now in a position to state and prove our main results of the existence of solutions for problem (1.1).

Let us list the following hypotheses:

(H1) A satisfies the Hille-Yosida condition;

(H2) the operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$ and satisfies $\sup_{t\in[0,\infty]}\|S'(t)\| = M_0 < \infty$, where M_0 is a constant;

(H3) $F : [0,b] \times X \to P_{b,cl,c}(X)$, for each $x \in X$, $F(\cdot,x)$ is measurable and for each $t \in [0,b]$, $F(t,\cdot)$ is upper semicontinuous; for each fixed $x \in X$, the set $N_{F,x} = \{f \in L^1([0,b],X) : f(t) \in F(t,x),\ \text{for a.e. } t \in [0,b]\}$ is not empty;

(H4) for each $x \in X$, there exist $m \in L^{1/q_1}([0,b],\mathbf{R}^+)$ and $n \in C([0,b],\mathbf{R}^+)$ such that

$$\sup\{|f(t)| : f(t) \in F(t,x)\} \le m(t) + n(t)|x| \quad \text{for a.e. } t \in [0,b], \tag{3.15}$$

where $q_1 \in [0,q)$.

Theorem 3.7. *Assume that hypotheses (H1)–(H4) hold; then problem* (1.1) *has an integral solution* $x \in C([0,b],\overline{D(A)})$.

Proof. Denote $C_0 = C([0,b],\overline{D(A)})$, which is a closed subset of $C([0,b],X)$. Obviously, C_0 with the same norm in $C([0,b],X)$ is also a Banach space. Transform the problem (1.1) into a fixed point problem. Consider the multivalued operator $\Phi : C_0 \to \wp(C_0)$ defined by

$$\begin{aligned}\Phi x = \Bigg\{ h \in C_0 : h(t) &= \int_0^\infty h_q(\theta) S'(t^q\theta)x_0 d\theta \\ &+ \lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f(s)d\theta ds \Bigg\},\end{aligned} \tag{3.16}$$

where $f \in N_{F,x} = \{f \in L^1([0,b],X) : f(t) \in F(t,x(t)),\ \text{for a.e. } t \in [0,b]\}$. Obviously, the fixed points of the operator Φ are integral solutions of problem (1.1). Now we will show that Φ satisfies all conditions of Theorem 2.14. The proof would be divided into the following steps.

Step 1 ($\Phi(x)$ is convex for each $x \in C_0$). Indeed, if h_1 and h_2 belong to Φx, then there exist f_1, $f_2 \in N_{F,x}$ such that for each $t \in [0,b]$, we have

$$h_i(t) = \int_0^\infty h_q(\theta)S'(t^q\theta)x_0 d\theta + \lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f_i(s)d\theta ds, \quad i = 1,2. \tag{3.17}$$

Let $0 \le k \le 1$; then for each $t \in [0, b]$, we have

$$\begin{aligned}(kh_1 + (1-k)h_2)(t) = &\int_0^\infty h_q(\theta)S'(t^q\theta)x_0 d\theta \\ &+ \lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda \\ &\times \big(kf_1(s) + (1-k)f_2(s)\big)d\theta ds.\end{aligned} \tag{3.18}$$

Since $N_{F,x}$ is convex, we have $kh_1 + (1-k)h_2 \in \Phi x$.

Step 2 (Φ maps bounded sets into bounded sets in C_0). Indeed, it is enough to show that there exists a positive constant l such that for each $h \in \Phi x$, $x \in \mathcal{B}_r = \{x \in C_0, \|x\| \le r\}$ one has $\|h\| \le l$.

Let $h \in \Phi x$; then there exists $f \in N_{F,x}$ such that for $t \in [0, b]$, we have

$$h(t) = \int_0^\infty h_q(\theta)S'(t^q\theta)x_0 d\theta + \lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f(s)d\theta ds. \tag{3.19}$$

From (H2) and the fact that $\|B_\lambda\| \le M$, for $t \in [0, b]$ we have

$$\begin{aligned}|h(t)| \le &\left|\int_0^\infty h_q(\theta)S'(t^q\theta)x_0 d\theta\right| \\ &+ \left|\lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f(s)d\theta ds\right| \\ \le &M_0|x_0| + MM_0\int_0^t\int_0^\infty \theta h_q(\theta)\left|(t-s)^{q-1}f(s)\right|d\theta ds \\ \le &M_0|x_0| + \frac{qMM_0}{\Gamma(1+q)}\int_0^t\left|(t-s)^{q-1}f(s)\right|ds.\end{aligned} \tag{3.20}$$

From Lemma 2.1 and (H4), for $t \in [0, b]$ we have

$$\begin{aligned}\int_0^t\left|(t-s)^{q-1}f(s)\right|ds &\le \left(\int_0^t (t-s)^{(q-1)/(1-q_1)}ds\right)^{1-q_1}\|m\|_{L^{1/q_1}[0,t]} + \overline{n}r\int_0^t (t-s)^{q-1}ds \\ &\le \frac{M_1}{(1+a)^{1-q_1}}b^{(1+a)(1-q_1)} + \frac{\overline{n}rb^q}{q},\end{aligned} \tag{3.21}$$

where $a = (q-1)/(1-q_1) \in (-1, 0)$, $M_1 = \|m\|_{L^{1/q_1}[0,b]}$, $\overline{n} = \sup\{n(t), t \in [0, b]\}$.

Then from (3.20) and (3.21), we get that

$$\|h\| \le M_0|x_0| + \frac{MM_0}{\Gamma(1+q)}\left(\frac{qM_1}{(1+a)^{1-q_1}}b^{(1+a)(1-q_1)} + \overline{n}rb^q\right) := l. \tag{3.22}$$

Step 3 (Φ maps bounded sets into equicontinuous sets of C_0). Let $t_1, t_2 \in [0,b]$, $t_1 < t_2$, and $\mathcal{B}_r = \{x \in C_0, \|x\| \le r\}$ be a bounded set of C_0. For each $x \in \mathcal{B}_r$ and $h \in \Phi x$, there exists $f \in N_{F,x}$ such that

$$h(t) = \int_0^\infty h_q(\theta) S'(t^q\theta) x_0 d\theta + \lim_{\lambda\to\infty} q \int_0^t \int_0^\infty \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q\theta\right) B_\lambda f(s) d\theta ds. \tag{3.23}$$

Then,

$$\begin{aligned}
|h(t_2) - h(t_1)| &= \bigg| \int_0^\infty h_q(\theta) S'\left(t_2^q\theta\right) x_0 d\theta - \int_0^\infty h_q(\theta) S'\left(t_1^q\theta\right) x_0 d\theta \\
&\quad + \lim_{\lambda\to\infty} q \int_0^{t_2} \int_0^\infty \theta (t_2-s)^{q-1} h_q(\theta) S'\left((t_2-s)^q\theta\right) B_\lambda f(s) d\theta ds \\
&\quad - \lim_{\lambda\to\infty} q \int_0^{t_1} \int_0^\infty \theta (t_1-s)^{q-1} h_q(\theta) S'\left((t_1-s)^q\theta\right) B_\lambda f(s) d\theta ds \bigg| \\
&\le \int_0^\infty h_q(\theta) \left\| S'\left(t_2^q\theta\right) - S'\left(t_1^q\theta\right) \right\| |x_0| d\theta \\
&\quad + \left| \lim_{\lambda\to\infty} q \int_{t_1}^{t_2} \int_0^\infty \theta (t_2-s)^{q-1} h_q(\theta) S'\left((t_2-s)^q\theta\right) B_\lambda f(s) d\theta ds \right| \\
&\quad + \bigg| \lim_{\lambda\to\infty} q \int_0^{t_1} \int_0^\infty \theta (t_2-s)^{q-1} h_q(\theta) S'\left((t_2-s)^q\theta\right) B_\lambda f(s) d\theta ds \\
&\quad - \lim_{\lambda\to\infty} q \int_0^{t_1} \int_0^\infty \theta (t_1-s)^{q-1} h_q(\theta) S'\left((t_2-s)^q\theta\right) B_\lambda f(s) d\theta ds \bigg| \\
&\quad + \bigg| \lim_{\lambda\to\infty} q \int_0^{t_1} \int_0^\infty \theta (t_1-s)^{q-1} h_q(\theta) S'\left((t_2-s)^q\theta\right) B_\lambda f(s) d\theta ds \\
&\quad - \lim_{\lambda\to\infty} q \int_0^{t_1} \int_0^\infty \theta (t_1-s)^{q-1} h_q(\theta) S'\left((t_1-s)^q\theta\right) B_\lambda f(s) d\theta ds \bigg| \\
&= \int_0^\infty h_q(\theta) \left\| S'\left(t_2^q\theta\right) - S'\left(t_1^q\theta\right) \right\| |x_0| d\theta \\
&\quad + \left| \lim_{\lambda\to\infty} q \int_{t_1}^{t_2} \int_0^\infty \theta (t_2-s)^{q-1} h_q(\theta) S'\left((t_2-s)^q\theta\right) B_\lambda f(s) d\theta ds \right| \\
&\quad + \left| \lim_{\lambda\to\infty} q \int_0^{t_1} \int_0^\infty \theta \left[(t_2-s)^{q-1} - (t_1-s)^q \right] h_q(\theta) S'\left((t_2-s)^q\theta\right) B_\lambda f(s) d\theta ds \right| \\
&\quad + \left| \lim_{\lambda\to\infty} q \int_0^{t_1} \int_0^\infty \theta (t_1-s)^{q-1} h_q(\theta) \left[\left(S'(t_2-s)^q\theta\right) - S'\left((t_1-s)^q\theta\right) \right] B_\lambda f(s) d\theta ds \right| \\
&= I_1 + q(I_2 + I_3 + I_4),
\end{aligned} \tag{3.24}$$

where

$$
\begin{aligned}
I_1 &= \int_0^{\infty} h_q(\theta)\left\| S'\left(t_2^q\theta\right) - S'\left(t_1^q\theta\right)\right\| |x_0| d\theta,\\
I_2 &= \left| \lim_{\lambda\to\infty} \int_{t_1}^{t_2}\int_0^{\infty} \theta(t_2-s)^{q-1} h_q(\theta) S'\left((t_2-s)^q\theta\right) B_\lambda f(s)\, d\theta\, ds\right|,\\
I_3 &= \left| \lim_{\lambda\to\infty} \int_0^{t_1}\int_0^{\infty} \theta\left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right] h_q(\theta) S'\left((t_2-s)^q\theta\right) B_\lambda f(s)\, d\theta\, ds\right|,\\
I_4 &= \left| \lim_{\lambda\to\infty} \int_0^{t_1}\int_0^{\infty} \theta(t_1-s)^{q-1} h_q(\theta) \left[S'\left((t_2-s)^q\theta\right) - S'\left((t_1-s)^q\theta\right)\right] B_\lambda f(s)\, d\theta\, ds\right|.
\end{aligned}
\tag{3.25}
$$

By using analogous argument performed in (3.20) and (3.21), we can conclude that

$$
\begin{aligned}
I_2 &\leq \frac{MM_0}{\Gamma(1+q)}\left(\frac{M_1}{(1+a)^{1-q_1}}(t_2-t_1)^{(1+a)(1-q_1)} + \frac{\overline{n}r(t_2-t_1)^q}{q}\right),\\
I_3 &\leq \frac{MM_0}{\Gamma(1+q)}\left[\left(\int_0^{t_1}\left((t_1-s)^{q-1} - (t_2-s)^{q-1}\right)^{1/(1-q_1)} ds\right)^{1-q_1} \|m\|_{L^{1/q_1}[0,t_1]}\right.\\
&\qquad \left. + \overline{n}r\int_0^{t_1}\left((t_1-s)^{q-1} - (t_2-s)^{q-1}\right) ds\right]\\
&\leq \frac{MM_0}{\Gamma(1+q)}\left[M_1\left(\int_0^{t_1}\left((t_1-s)^a - (t_2-s)^a\right) ds\right)^{1-q_1}\right.\\
&\qquad \left. + \overline{n}r\left(\frac{(t_2-t_1)^q}{q} - \frac{t_2^q}{q} + \frac{t_1^q}{q}\right)\right]\\
&= \frac{MM_0}{\Gamma(1+q)}\left(\frac{M_1}{(1+a)^{1-q_1}}\left(t_1^{1+a} - t_2^{1+a} + (t_2-t_1)^{1+a}\right)^{1-q_1}\right.\\
&\qquad \left. + \frac{\overline{n}r}{q}\left[(t_2-t_1)^q - t_2^q + t_1^q\right]\right)\\
&\leq \frac{MM_0}{\Gamma(1+q)}\left(\frac{M_1}{(1+a)^{1-q_1}}(t_2-t_1)^{(1+a)(1-q_1)} + \frac{\overline{n}r}{q}(t_2-t_1)^q\right).
\end{aligned}
\tag{3.26}
$$

Hence $\lim_{t_2\to t_1} I_2 = 0$ and $\lim_{t_2\to t_1} I_3 = 0$.

On the other hand, (H2) implies that $S'(t)$ for $t > 0$ is continuous in the uniform operator topology; then from the Lebesgue dominated convergence theorem, we get $\lim_{t_2 \to t_1} I_1 = 0$ and

$$\begin{aligned}\lim_{t_2 \to t_1} I_4 &\leq \lim_{t_2 \to t_1} M \int_0^{t_1} \int_0^{\infty} \theta (t_1 - s)^{q-1} h_q(\theta) \left\| S'\left((t_2 - s)^q \theta\right) - S'\left((t_1 - s)^q \theta\right) \right\| \left| f(s) \right| d\theta ds \\ &\leq M \int_0^{t_1} \int_0^{\infty} \theta (t_1 - s)^{q-1} h_q(\theta) \lim_{t_2 \to t_1} \left\| S'\left((t_2 - s)^q \theta\right) - S'\left((t_1 - s)^q \theta\right) \right\| \left| f(s) \right| d\theta ds \\ &= 0.\end{aligned} \tag{3.27}$$

Consequently, $|h(t_2) - h(t_1)| \to 0$ independently of $x \in \mathcal{B}_r$ as $t_2 \to t_1$, which means that $\Phi(\mathcal{B}_r)$ is equicontinuous.

Step 4 (For each $t \in [0, b]$, $V(t) = \{(\Phi x)(t), x \in \mathcal{B}_r\}$ is relatively compact in X). Obviously, $V(0) = \{x_0\}$ is relatively compact in X. Let $0 < t < b$ be fixed. For $x \in \mathcal{B}_r$ and $h \in \Phi x$, there exists $f \in N_{F,x}$ such that

$$h(t) = \int_0^{\infty} h_q(\theta) S'(t^q \theta) x_0 d\theta + \lim_{\lambda \to \infty} q \int_0^t \int_0^{\infty} \theta (t - s)^{q-1} h_q(\theta) S'\left((t - s)^q \theta\right) B_\lambda f(s) d\theta ds. \tag{3.28}$$

For arbitrary $\epsilon \in (0, t)$ and $\delta > 0$, define an operator $F_{\epsilon,\delta}$ on $\mathcal{B}_r$ by

$$\begin{aligned}(F_{\epsilon,\delta} x)(t) &= \int_\delta^{\infty} h_q(\theta) S'(t^q \theta) x_0 d\theta \\ &\quad + \lim_{\lambda \to \infty} q \int_0^{t-\epsilon} \int_\delta^{\infty} \theta (t - s)^{q-1} h_q(\theta) S'\left((t - s)^q \theta\right) B_\lambda f(s) d\theta ds \\ &= \int_\delta^{\infty} h_q(\theta) S'(t^q \delta) S'((t^q \theta) - t^q \delta) x_0 d\theta \\ &\quad + \lim_{\lambda \to \infty} q \int_0^{t-\epsilon} \int_\delta^{\infty} \theta (t - s)^{q-1} h_q(\theta) S'(\epsilon^q \delta) S'\left((t - s)^q \theta - \epsilon^q \delta\right) B_\lambda f(s) d\theta ds \\ &= S'(t^q \delta) \int_\delta^{\infty} h_q(\theta) S'((t^q \theta) - t^q \delta) x_0 d\theta \\ &\quad + S'(\epsilon^q \delta) \lim_{\lambda \to \infty} q \int_0^{t-\epsilon} \int_\delta^{\infty} \theta (t - s)^{q-1} h_q(\theta) S'\left((t - s)^q \theta - \epsilon^q \delta\right) B_\lambda f(s) d\theta ds.\end{aligned} \tag{3.29}$$

Then from the compactness of $S'(t)$, $t > 0$, we get that the set $V_{\epsilon,\delta}(t) = \{(F_{\epsilon,\delta}x)(t), x \in \mathcal{B}_r\}$ is relatively compact in X for each $\epsilon \in (0, t)$ and $\delta > 0$. Moreover, for every $x \in \mathcal{B}_r$, we have

$$\begin{aligned}
|(\Phi x)(t) - (F_{\epsilon,\delta}x)(t)| = &\left| \int_0^\delta h_q(\theta) S'(t^q\theta) x_0 d\theta \right| \\
&+ \left| \lim_{\lambda\to\infty} q \int_0^t \int_0^\delta \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q\theta\right) B_\lambda f(s) d\theta ds \right. \\
&\quad + \lim_{\lambda\to\infty} q \int_0^t \int_\delta^\infty \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q\theta\right) B_\lambda f(s) d\theta ds \\
&\quad \left. - \lim_{\lambda\to\infty} q \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q\theta\right) B_\lambda f(s) d\theta ds \right| \\
\le\, & M_0 |x_0| \int_0^\delta h_q(\theta) d\theta \\
&+ q \left| \lim_{\lambda\to\infty} \int_0^t \int_0^\delta \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q\theta\right) B_\lambda f(s) d\theta ds \right| \\
&+ q \left| \lim_{\lambda\to\infty} \int_{t-\epsilon}^t \int_\delta^\infty \theta (t-s)^{q-1} h_q(\theta) S'\left((t-s)^q\theta\right) B_\lambda f(s) d\theta ds \right| \\
\le\, & M_0 |x_0| \int_0^\delta h_q(\theta) d\theta + qMM_0 \int_0^t (t-s)^{q-1} \left| f(s) \right| ds \int_0^\delta \theta h_q(\theta) d\theta \\
&+ qMM_0 \int_{t-\epsilon}^t (t-s)^{q-1} \left| f(s) \right| ds \int_0^\infty \theta h_q(\theta) d\theta.
\end{aligned} \tag{3.30}$$

In view of (3.21), we have

$$\begin{aligned}
&|(\Phi x)(t) - (F_{\epsilon,\delta}x)(t)| \\
&\quad \le M_0 |x_0| \int_0^\delta h_q(\theta) d\theta + qMM_0 \int_0^\delta \theta h_q(\theta) d\theta \left(\frac{M_1}{(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} + \frac{\overline{n} r b^q}{q} \right) \\
&\qquad + \frac{qMM_0}{\Gamma(1+q)} \left[\left(\int_{t-\epsilon}^t (t-s)^{(q-1)/(1-q_1)} ds \right)^{1-q_1} \|m\|_{L^{1/q_1}[t-\epsilon,t]} + \overline{n} r \int_{t-\epsilon}^t (t-s)^{q-1} ds \right] \\
&\quad \le M_0 |x_0| \int_0^\delta h_q(\theta) d\theta + qMM_0 \int_0^\delta \theta h_q(\theta) d\theta \left(\frac{M_1}{(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} + \frac{\overline{n} r b^q}{q} \right) \\
&\qquad + \frac{qMM_0}{\Gamma(1+q)} \left(\frac{M_1}{(1+a)^{1-q_1}} \epsilon^{(1+a)(1-q_1)} + \frac{\overline{n} r \epsilon^q}{q} \right).
\end{aligned} \tag{3.31}$$

From $\lim_{\delta\to 0}\int_0^{\delta} h_q(\theta)d\theta = 0$ and $\lim_{\delta\to 0}\int_0^{\delta} \theta h_q(\theta)d\theta = 0$, we get that there are relatively compact sets arbitrarily close to the set $V(t)$, $t > 0$. Hence the set $V(t)$, $t > 0$, is also relatively compact in X.

Step 5 (Φ has a closed graph). Let $x_n \to x_*$, $h_n \in \Phi x_n$, and $h_n \to h_*$ as $n \to \infty$; we will prove that $h_* \in \Phi x_*$. $h_n \in \Phi x_n$ means that there exists $f_n \in N_{F,x_n}$ such that

$$h_n(t) = \int_0^{\infty} h_q(\theta)S'(t^q\theta)x_0 d\theta + \lim_{\lambda\to\infty} q\int_0^t\int_0^{\infty} \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f_n(s)d\theta ds. \tag{3.32}$$

We must prove that there exists $f_* \in N_{F,x_*}$ such that

$$h_*(t) = \int_0^{\infty} h_q(\theta)S'(t^q\theta)x_0 d\theta + \lim_{\lambda\to\infty} q\int_0^t\int_0^{\infty} \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f_*(s)d\theta ds. \tag{3.33}$$

Consider the linear continuous operator $\mathfrak{T} : L^1([0,b],X) \to C([0,b],X)$ defined by

$$(\mathfrak{T}f)(t) = \lim_{\lambda\to\infty} q\int_0^t\int_0^{\infty} \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f(s)d\theta ds. \tag{3.34}$$

We can easily see that $\mathfrak{T}$ is continuous. On the other hand,

$$\begin{aligned}&\left|\left(h_n(t) - \int_0^{\infty} h_q(\theta)S'(t^q\theta)x_0 d\theta\right) - \left(h_*(t) - \int_0^{\infty} h_q(\theta)S'(t^q\theta)x_0 d\theta\right)\right| \\ &\quad \le \|h_n - h_*\| \longrightarrow 0, \qquad \text{as } n \longrightarrow 0.\end{aligned} \tag{3.35}$$

From Theorem 2.13, it follows that $\mathfrak{T}\circ N_F$ is a closed graph operator. Moreover, we have that

$$h_n - \int_0^{\infty} h_q(\theta)S'(t^q\theta)x_0 d\theta \in \mathfrak{T}(N_{F,x_n}). \tag{3.36}$$

Since $x_n \to x_*$, it follows from Theorem 2.13 that there exists $f_* \in N_{F,x_*}$ such that

$$h_*(t) - \int_0^{\infty} h_q(\theta)S'(t^q\theta)x_0 d\theta = \lim_{\lambda\to\infty} q\int_0^t\int_0^{\infty} \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f_*(s)d\theta ds. \tag{3.37}$$

Thus,

$$h_*(t) = \int_0^\infty h_q(\theta)S'(t^q\theta)x_0 d\theta + \lim_{\lambda\to\infty} q\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f_*(s)d\theta ds. \tag{3.38}$$

This implies that $h_* \in \Phi x_*$.

Therefore Φ is a completely continuous multivalued map, u.s.c. with convex closed values. In order to prove that Φ has a fixed point, we need one more step.

Step 6 (The set $U = \{x \in C_0 : \delta x \in \Phi x,\ \text{for some } \delta > 1\}$ is bounded). Let $x \in U$; then $\delta x \in \Phi x$ for some $\delta > 1$. Thus there exists $f \in N_{F,x}$ such that for $t \in [0,b]$,

$$x(t) = \frac{1}{\delta}\int_0^\infty h_q(\theta)S'(t^q\theta)x_0 d\theta + \lim_{\lambda\to\infty}\frac{q}{\delta}\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f(s)d\theta ds. \tag{3.39}$$

From (H4), for each $t \in [0,b]$ we have

$$\begin{aligned}
|x(t)| &= \left|\frac{1}{\delta}\int_0^\infty h_q(\theta)S'(t^q\theta)x_0 d\theta + \lim_{\lambda\to\infty}\frac{q}{\delta}\int_0^t\int_0^\infty \theta(t-s)^{q-1}h_q(\theta)S'\big((t-s)^q\theta\big)B_\lambda f(s)d\theta ds\right| \\
&\le M_0|x_0| + \frac{qMM_0}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\big|f(s)\big|ds \\
&\le M_0|x_0| + \frac{qMM_0}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}m(s)ds + \frac{q\overline{n}MM_0}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}|x(s)|ds \\
&\le M_0|x_0| + \frac{qMM_0M_1}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)} + \frac{q\overline{n}MM_0}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}|x(s)|ds \\
&\le \overline{a} + \overline{b}\int_0^t (t-s)^{q-1}|x(s)|ds,
\end{aligned} \tag{3.40}$$

where $\overline{a} = M_0|x_0| + (qMM_0M_1/\Gamma(1+q)(1+a)^{(1-q_1)})b^{(1+a)(1-q_1)}$, $\overline{b} = q\overline{n}MM_0/\Gamma(1+q)$.

Then from Corollary 2.11, we have

$$|x(t)| \le \overline{a}E_q\Big(\overline{b}\Gamma(q)t^q\Big). \tag{3.41}$$

Therefore, we obtain that

$$\|x\| \le \overline{a}E_q\Big(\overline{b}\Gamma(q)b^q\Big). \tag{3.42}$$

This shows that U is bounded.

As a consequence of Theorem 2.14, we conclude that Φ has a fixed point which is the integral solution of problem (1.1). This completes the proof. □

4. An Example

As an application of our results we consider the following fractional differential inclusions of the form

$$\begin{aligned} D^q u(t,z) &\in \frac{\partial^2}{\partial z^2} u(t,z) + G(t,u(t,z)), \quad z \in [0,\pi],\ t \in (0,b], \\ u(t,0) &= u(t,\pi) = 0, \quad t \in [0,b], \\ u(0,z) &= u_0, \quad z \in [0,\pi], \end{aligned} \tag{4.1}$$

where $b > 0$, $G : [0,b] \times X \to \mathcal{P}(X)$ satisfies semi-continuous assumptions (H3) and (H4).

Consider $X = C([0,\pi]; \mathbf{R})$ endowed with the supnorm and the operator $A : D(A) \subset X \to X$ defined by

$$D(A) = \left\{ u \in C^2([0,\pi]; \mathbf{R}) : u(t,0) = u(t,\pi) = 0 \right\}, \qquad Au = \frac{\partial^2}{\partial z^2} u(t,z). \tag{4.2}$$

Now, we have $\overline{D(A)} = \{u \in X : u(t,0) = u(t,\pi) = 0\} \neq X$. As we know from [17] that A satisfies the Hille-Yosida condition with $(0,+\infty) \subseteq \rho(A)$ and $\lambda > 0$, $\|R(\lambda, A)\| \leq 1/\lambda$. Hence, operator A satisfies (H1), (H2), and $M = M_0 = 1$.

Then the system (4.1) can be reformulated as

$$\begin{aligned} D^q x(t) &\in Ax(t) + F(t,x(t)), \quad t \in [0,b], \\ x(0) &= u_0, \end{aligned} \tag{4.3}$$

where $x(t)(z) = u(t,z)$, $F(t,x(t))(z) = G(t,u(t,z))$.

If we assume that F satisfies (H3) and (H4), then all conditions of Theorem 3.7 are satisfied and we deduce (4.1) has at least one integral solution.

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