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Research Article

Derivatives of Multivariate *Bernstein* Operators and Smoothness with *Jacobi* Weights

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Using the modulus of smoothness, directional derivatives of multivariate *Bernstein* operators with weights are characterized. The obtained results partly generalize the corresponding ones for multivariate *Bernstein* operators without weights.

1. Introduction

For the simplex $S = S_d$ in $R^d (d = 1, 2, \ldots)$,

$$S = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_d);\ x_i \geq 0, i = 1, 2, \ldots, d, |\mathbf{x}| = \sum_{i=0}^{d} x_i \leq 1 \right\}, \tag{1.1}$$

we denote $C(S)$ the space of continuous functions on S equipped with the norm

$$\|f\| = \sup_{\mathbf{x}\in S} |f(\mathbf{x})|. \tag{1.2}$$

Let $f \in C(S)$, for each $n \in N_0$ $(N_0 = N \cup \{0\},\ N_0^d = N_0 \times N_0 \times \cdots \times N_0 \in R^d)$, the multivariate *Bernstein* polynomial of f is defined by

$$B_{n,d}(f; \mathbf{x}) = \sum_{|\mathbf{k}|\leq n} P_{n,\mathbf{k}}(\mathbf{x}) f\left(\frac{\mathbf{k}}{n}\right), \quad \mathbf{x} \in S, \tag{1.3}$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_d)$ with nonnegative integers k_i $(i = 0, 1, 2, \ldots, n)$, and

$$P_{n,\mathbf{k}}(x) = \frac{n!}{(n-|\mathbf{k}|)!}\mathbf{x}^{\mathbf{k}}(1-|\mathbf{x}|)^{n-|\mathbf{k}|}, \quad |\mathbf{x}| = \sum_{i=0}^{d} x_i, \ |\mathbf{k}| = \sum_{i=0}^{d} k_i , \tag{1.4}$$

with the convention

$$k! = k_1!k_2!\cdots k_d! , \qquad \mathbf{x}^{\mathbf{k}} = x_1^{k_1}x_2^{k_2}\cdots x_d^{k_d}. \tag{1.5}$$

Obviously, the multivariate *Bernstein* operators given in (1.3) can be reduced as the classical *Bernstein* polynomials in case $d = 1$, that is,

$$B_n(f,x) := B_{n,1}(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)P_{n,k}(x), \quad x \in [0,1]. \tag{1.6}$$

Here introduce the crucial notations of our investigation. First, with the simplex S, we denote V_S the set of unit vectors in the directions of the edges of S where e_i and $-e_i$ are considered to be the same vectors. That is, $e_i = (0, 0, \ldots, \overset{i\text{th}}{1}, 0, \ldots, 0)$ $(1 \le i \le d)$ and $e_{ij} = e_i - e_j$ $(1 \le i < j \le d)$. With a direction $\xi \in V_S$ and a point $\mathbf{x} \in S$, we define the step-weight function

$$\varphi_\xi^2(\mathbf{x}) = \inf_{\mathbf{x}+\lambda\xi \notin S,\, \lambda>0} d(\mathbf{x}, \mathbf{x}+\lambda\xi) \inf_{\mathbf{x}-\lambda\xi \notin S,\, \lambda>0} d(\mathbf{x}, \mathbf{x}-\lambda\xi), \tag{1.7}$$

where $d(\mathbf{x}, \mathbf{y})$ is the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$ in R^d. Obviously, as $\mathbf{x} \in S$, the $\varphi_\xi^2(x)$ can further be expressed as:

$$\varphi_\xi^2(\mathbf{x}) = \begin{cases} x_i(1-|x|), & \text{if } \xi = e_i,\ 1 \le i \le d, \\ 2x_ix_j & \text{if } \xi = \dfrac{e_i - e_j}{\sqrt{2}},\ 1 \le i < j \le d. \end{cases} \tag{1.8}$$

It is clear that $\varphi_\xi^2(x)$ can be reduced as the classical *Bernstein* polynomials' step-weight function $\varphi^2(x) = \varphi_\xi(x)^2 = x(1-x)$ $(x \in [0,1])$ in case $d = 1$.

The multivariate *Jacobi* weight function in this paper is denoted as follows:

$$\omega(\mathbf{x}) = \mathbf{x}^{\alpha}(1-|\mathbf{x}|)^{\beta}, \quad \mathbf{x} \in S, \tag{1.9}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in R^d, 0 < \alpha_i, \beta < 1$, $i = 1, 2, \ldots, d$, $\mathbf{x}^{\alpha} = x_1^{\alpha_1}x_2^{\alpha_2}, \ldots, x_d^{\alpha_d}$.

The rth symmetric difference of function f with the direction e is given by

$$\Delta_{he}^{r} f(x) = \begin{cases} \sum_{i=0}^{r} C_r^i (-1)^i f\left(x + \left(\frac{r}{2} - i\right)he\right), & \text{if } x \pm \frac{rhe}{2} \in S, \\ 0, & \text{otherwise.} \end{cases} \tag{1.10}$$

Using the above notation, the weighted Sobolev space in S is then defined by

$$W_{\phi}^{r,\infty}(S) = \left\{ f \in C(S) : \omega f \in C(S),\ f \in C^r\left(\mathring{S}\right),\ \omega\varphi_{ij}^{r} D_{ij}^{r} f \in C(S),\ 1 \le i \le j \le d, r \in N \right\}, \tag{1.11}$$

where $\mathring{S}$ is the inner of S.

Furthermore, the weighted K-functional is defined by

$$K_{\varphi}^{r}(f, t^r)_{\omega} = \inf_{g \in W_{\phi}^{r,p}} \left\{ \|\omega(f-g)\| + t^r \sum_{1 \le i \le j \le d} \left\|\omega\varphi_{ij}^{r} D_{ij}^{r} g\right\| \right\}, \tag{1.12}$$

and the weighted modulus is

$$\Omega_{\varphi}^{r}(f,t)_{\omega} = \sup_{0<h\le t} \sum_{1 \le i \le j \le d} \left\|\omega \Delta_{h\varphi_{ij}e_{ij}}^{r} f\right\|, \tag{1.13}$$

where $\|\omega f\| = \max_{x\in S}|\omega(x)f(x)|$ is the weighted form. From [1], there exists a positive constant C,

$$C^{-1} K_{\varphi}^{r}(f,t^r)_{\omega} \le \Omega_{\varphi}^{r}(f,t)_{\omega} \le C K_{\varphi}^{r}(f,t^r)_{\omega}. \tag{1.14}$$

Throughout the paper, the letter C, appearing in various formulas, denotes a positive constant independent of n, x, and f. Its value may be different at different occurrences, even within the same formula.

The close connection between the derivatives of *Bernstein*-type operators and the smoothness of functions has been well investigated by Ditzian, Totik, Ivanov and some other mathematicians (see [2–6], etc.) In [2], Ditzian has studied the relations between the derivatives of classical *Bernstein* operators $B_{n,1}(f,x)$ and the smoothness of the function f. In [7], we have presented the relation between the derivatives of classical *Bernstein* operators and the smoothness of function f with *Jacobi* weights. Zhou has considered the approximation problems of higher-dimensional *Bernstein* operators with *Jacobi* weights, and has pointed out the unboundedness of *Bernstein* operators with *Jacobi* weights in the usual norm [8]. Because of the unboundedness of $B_{n,d}(f,\mathbf{x})$ operators with weights in $C(S)$, he used the method of space reduction, that is,

$$C_0(S) = \left\{ f \in C(S) : f(\mathbf{x})|_{\mathbf{x}\in\partial S} = 0 \right\} \tag{1.15}$$

has been taken instead of $C(S)(\partial S$ is the boundary of $S)$. He then has shown the characteristic of the two dimensional *Bernstein* operators with *Jacobi* weights. In [1], Cao has yielded the order of approximation of d-dimensional *Bernstein* Operators with *Jacobi* weights by using the equivalence relation (1.14). In [6], Cao has evaluated extensively derivatives of the multivariate *Bernstein* operators on a simplex, and he proved the following.

Theorem 1.1. *Let* $f \in C(S)$, $0 < \alpha \leq r$, $0 \leq \lambda \leq 1$, $r \in N$, *and* $\xi \in V_S$, *and suppose* $\Omega_r^{\xi}(f,t) = O(t^{\alpha})$, *then*

$$\left\| \varphi_{\xi}^{r\lambda} \left(\frac{\partial}{\partial \xi} \right)^{r} B_{n,d}(f,x) \right\| = O\left\{ \min \left(n^{2-\lambda}, \frac{n}{\varphi_{\xi}^{2(1-\lambda)}} \right)^{(r-\alpha)/2} \right\}. \tag{1.16}$$

In this paper, we study the characterization of derivatives of multivariate *Bernstein* polynomials with *Jacobi* weights by using the measure of smoothness in the space $C_0(S)$. The main result is expressed as follows.

Theorem 1.2. *Let* $f \in C_0(S)$, $0 < \alpha \leq r$, $r \in N$, *and* $\xi \in V_S$, *and suppose* $\Omega_r^{\xi}(f,t)_{\omega} = O(t^{\alpha})$, *one has*

$$\left\| \omega \varphi_{\xi}^{2r} \left(\frac{\partial}{\partial \xi} \right)^{2r} B_{n,d}(f,x) \right\| = O(n^{r-\alpha}). \tag{1.17}$$

Remark 1.3. Theorem 1.2 shows that the characterization of derivatives for multivariate bernstein operators with *jacobi* weight by using the measure of smoothness $\Omega_r^{\xi}(f,t)_{\omega}$. conversely, we conjecture that the inverse theorem is also correct, that is,

$$\left\| \omega \varphi_{\xi}^{2r} \left(\frac{\partial}{\partial \xi} \right)^{2r} M_{n,d}(f,x) \right\| = O(n^{r-\alpha}) \Longleftrightarrow \Omega_r^{\xi}(f,t)_{\omega} = O(t^{\alpha}). \tag{1.18}$$

The above equivalent relation without *Jacobi* weight has been proved in [6] when $\lambda = 1$. In fact, the proof of Theorem 1.2 shows that the direct part holds true, we leave the inverse part as an open problem.

2. Lemmas

To prove Theorem 1.2, some lemmas will be shown in this section.

Lemma 2.1. *Consider the following;*

$$\sum_{|k| \leq n} P_{n,k}(x) \omega^{-1}\left(\frac{k}{n} \right) \leq C \omega^{-1}(x). \tag{2.1}$$

Proof. When $d = 1$, one has

$$\sum_{k=0}^{n} P_{n,k}(x)\left\{\frac{n}{k+1}\right\}^{\alpha}\left\{\frac{n}{n-k+1}\right\}^{\beta} \leq \left[\sum_{k=0}^{n} P_{n,k}(x)\left\{\frac{n}{k+1}\right\}^{2\alpha}\right]^{1/2}\left[\sum_{k=0}^{n} P_{n,k}(x)\left\{\frac{n}{n-k+1}\right\}^{2\beta}\right]^{1/2} =: I^{1/2}J^{1/2}. \tag{2.2}$$

Consider different conditions,

(1) if $0 < 2\alpha < 1$,

$$I \leq \left\{\sum_{k=0}^{n} P_{n,k}(x)\frac{n}{k+1}\right\}^{2\alpha}\left\{\sum_{k=0}^{n} P_{n,k}(x)\right\}^{1-2\beta} \leq Cx^{-2\alpha}, \tag{2.3}$$

(2) if $1 < 2\alpha < 2$, let $2\alpha = 1 + r, 0 \leq r < 1$,

$$\begin{aligned} I &= \sum_{k=0}^{n} P_{n,k}(x)\left\{\frac{n}{k+1}\right\}\left\{\frac{n}{k+1}\right\}^{r} \\ &\leq \frac{2}{x}\sum_{k=0}^{n} P_{n+1,k+1}(x)\left\{\frac{n}{k+1}\right\}^{r} \\ &\leq Cx^{-(1+r)} = Cx^{-2\alpha}. \end{aligned} \tag{2.4}$$

By the same methods $J \leq C(1-x)^{-2\beta}$ can also be given.

Suppose the lemma is correct when $d-1$. We prove the lemma is also correct when d. Through a simple computation, the following results can be easily obtained

$$P_{n,k}(x) = P_{n,k_1}(x_1)P_{n-k_1,\overline{k}}\left(\frac{\overline{x}}{1-x_1}\right), \tag{2.5}$$

where $\overline{k} = (k_2, k_3, \ldots, k_d)$ $\overline{x} = (x_2, x_3, \ldots, x_d)$,

$$\begin{aligned} &\omega(x)\sum_{|k|\leq n} P_{n,k}(x)\omega^{-1}\left(\frac{k}{n}\right) \\ &= \omega(x)\sum_{|k|\leq n} P_{n,k_1}(x_1)P_{n-k_1,\overline{k}}\left(\frac{\overline{x}}{1-x_1}\right)\left(\frac{k_1}{n}\right)^{-\alpha_1}\left(\frac{k_2}{n}\right)^{-\alpha_2}\cdots\left(\frac{k_d}{n}\right)^{-\alpha_d}\left(1-\frac{|k|}{n}\right)^{-\beta} \\ &= x_1^{\alpha_1}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)x_2^{\alpha_2}x_3^{\alpha_3}\cdots x_d^{\alpha_d}(1-|x|)^{\beta}\left(\frac{k_1}{n}\right)^{-\alpha_1}\left(\frac{n-k_1}{n}\right)^{-|\overline{\alpha}|-\beta} \\ &\quad\times \sum_{|\overline{k}|\leq n-k_1} P_{n-k_1,\overline{k}}\left(\frac{\overline{x}}{1-x_1}\right)\cdot\left(\frac{k_2}{n-k_1}\right)^{-\alpha_2}\cdots\left(\frac{k_d}{n-k_1}\right)^{-\alpha_d}\left(1-\frac{|\overline{k}|}{n-k_1}\right)^{-\beta} \end{aligned}$$

$$
\begin{aligned}
&= x_1^{\alpha_1}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)x_2^{\alpha_2}x_3^{\alpha_3}\cdots x_d^{\alpha_d}(1-|x|)^{\beta}\left(\frac{k_1}{n}\right)^{-\alpha_1}\left(\frac{n-k_1}{n}\right)^{-|\overline{\alpha}|-\beta} \\
&\quad\times \sum_{|\overline{k}|\le n-k_1} P_{n-k_1,\overline{k}}\left(\frac{\overline{x}}{1-x_1}\right)\omega^{-1}\left(\frac{\overline{k}}{n-k_1}\right) \\
&\le Cx_1^{\alpha_1}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)x_2^{\alpha_2}x_3^{\alpha_3}\cdots x_d^{\alpha_d}(1-|x|)^{\beta}\left(\frac{k_1}{n}\right)^{-\alpha_1}\left(\frac{n-k_1}{n}\right)^{-|\overline{\alpha}|-\beta}\omega^{-1}\left(\frac{\overline{x}}{1-x_1}\right) \\
&\le Cx_1^{\alpha_1}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(1-x_1)^{|\overline{\alpha}|+\beta}\left(\frac{k_1}{n}\right)^{-\alpha_1}\left(\frac{n-k_1}{n}\right)^{-|\overline{\alpha}|-\beta} \\
&\le C.
\end{aligned}
\tag{2.6}
$$

□

Lemma 2.2. *Let $f \in C_0(S)$, $r \in N$, and $\xi \in V_S$, then*

$$
\left\|\omega\varphi_{\xi}(\mathbf{x})^{2r}\left(\frac{\partial}{\partial\xi}\right)^{2r}B_{n,d}(f,x)\right\| \le Cn^r\|\omega f\| \quad f \in C_0(S). \tag{2.7}
$$

Proof. First, we recall the discussion of theorem 4.1 of [9] that will allow us to consider lemma 1 with $\xi = e_2$. it is clear that if $\xi = e_i, i = 1,3,4,\ldots,d$, we may just rename the coordinates. the following transformation will help us to complete the other case of ξ. the transformation $T : S \to S$ is defined by [9]

$$
T : \begin{cases} T(x_1,x_2,\ldots,x_d) = (u_1,u_2,\ldots,u_d), \\ T^2 = I, \end{cases} \tag{2.8}
$$

where $u_i = x_i\ (i \ne j)$; $u_j = 1-|x|$ and I is the identity operator.

Obviously,

$$
\frac{\partial}{\partial u_i} = \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}, \quad i \ne j, \qquad \frac{\partial}{\partial u_j} = -\frac{\partial}{\partial x_j}, \tag{2.9}
$$

$$
B_{n,d}(f;\mathbf{x}) = B_{n,d}(f_T;T\mathbf{x}), \qquad B_{n,d}(f;T\mathbf{x}) = B_{n,d}(f_T;\mathbf{x}), \tag{2.10}
$$

where $f_T(\mathbf{u}) = f(\mathbf{x})$ and $\mathbf{u} = T\mathbf{x}$. So, for $\xi = e_{ij}/\sqrt{2}$, $1 \le i < j \le d$, we have

$$
\begin{aligned}
\left\|\omega\varphi_{\xi}^{2r}\left(\frac{\partial}{\partial\xi}\right)^{2r}B_{n,d}(f)\right\| &= \left\|\omega_T\varphi_{e_i}^{2r}\left(\frac{\partial}{\partial u_i}\right)^{r}B_{n,d}(f_T)\right\| \\
&\le Cn^r\|\omega_T f_T\| \\
&\le Cn^r\|\omega f\|.
\end{aligned}
\tag{2.11}
$$

Secondly, we prove

$$\left\| \omega\varphi_{e_2}^{2r}\left(\frac{\partial}{\partial x_2}\right)^{2r} B_{n,d}(f)\right\| \le Cn^r \|\omega f\|. \tag{2.12}$$

In The following we use mathematical induction on the dimension number d to prove (2.12). When $d = 1$, Lemma 3.2 in [10] proved the above inequality for $r = 1$, for $r > 1$, from the expression of derivatives of *Bernstein* operator in [4] (page125,(9.4.3)), we can easily prove it. Next, suppose that (2.12) is valid for $d - 1$ ($d > 1$); we prove (2.12) is also true for d. Assume

$$S' = \{\overline{\mathbf{x}} : (x_1, \overline{\mathbf{x}}) \in S_d\}, \quad \overline{\mathbf{x}} = (x_2, x_3, \ldots, x_d), \quad \overline{\mathbf{k}} = (k_2, k_3 \ldots, k_d), \quad \mathbf{k} = \left(k_1, \overline{\mathbf{k}}\right). \tag{2.13}$$

Let $\mathbf{z} = \overline{\mathbf{x}}/(1 - x_1) = (x_2/(1 - x_1), x_3/(1 - x_1), \ldots, x_d/(1 - x_1))$. $\omega(\mathbf{x})$ can therefore be rewritten as

$$\omega(\mathbf{x}) = x_1^{\alpha_1}(1 - x_1)^{|\overline{\alpha}|+\beta}\omega(\mathbf{z}), \tag{2.14}$$

and $B_{n,d}(f, \mathbf{x})$ can be decomposed as

$$B_{n,d}(f, \mathbf{x}) = \sum_{k_1=0}^{n} p_{n,k_1}(x_1) B_{n-k_1,d-1}(H, \mathbf{z}), \tag{2.15}$$

where $H(\mathbf{u}) = f(k_1/n, (1 - k_1/n)u)$. Using the inductive assumption, we have

$$\begin{aligned}
&\left| \omega\varphi_{e_2}^{2r}\left(\frac{\partial}{\partial x_2}\right)^{2r} B_{n,d}(f)\right| \\
&\quad = x_1^{\alpha_1}(1 - x_1)^{|\overline{\alpha}|+\beta} \sum_{k_1=0}^{n} p_{n,k_1}(x_1) z_1^{\alpha_2} \cdots z_{d-1}^{\alpha_d}(1 - |z|)^{\beta}\varphi_{e_1}^{2r}(z)\left(\frac{\partial}{\partial z_1}\right)^{2r} B_{n-k_1,d-1}(H, \mathbf{z}) \\
&\quad = x_1^{\alpha_1}(1 - x_1)^{|\overline{\alpha}|+\beta} \sum_{k_1=0}^{n} p_{n,k_1}(x_1)\omega(\mathbf{z})\varphi_{e_1}^{2r}(z)\left(\frac{\partial}{\partial z_1}\right)^{2r} B_{n-k_1,d-1}(H, \mathbf{z}) \\
&\quad \le x_1^{\alpha_1}(1 - x_1)^{|\overline{\alpha}|+\beta} \sum_{k_1=0}^{n} p_{n,k_1}(x_1) C(n - k_1)^r \max_{Z\in S_{d-1}} |\omega(z)H(z)| \\
&\quad \le Cn^r \|\omega f\| x_1^{\alpha_1}(1 - x_1)^{|\overline{\alpha}|+\beta} \sum_{k_1=0}^{n} p_{n,k_1}(x_1)\left(\frac{k_1}{n}\right)^{-\alpha_1}\left(\frac{n - k_1}{n}\right)^{-|\overline{\alpha}|-\beta} \\
&\quad \le Cn^r \|\omega f\|.
\end{aligned} \tag{2.16}$$

Here, the equality

$$\omega(z)H(z) = \left(\frac{k_1}{n}\right)^{-\alpha_1}\left(\frac{n - k_1}{n}\right)^{-|\overline{\alpha}|-\beta}(\omega f)\left(\frac{k_1}{n}, \left(1 - \frac{k_1}{n}\right)z\right), \tag{2.17}$$

and the inequality

$$x_1^{\alpha_1}(1-x_1)^{|\overline{\alpha}|+\beta}\sum_{k_1=0}^{n} p_{n,k_1}(x_1)\left(\frac{k_1}{n}\right)^{-\alpha_1}\left(\frac{n-k_1}{n}\right)^{-|\overline{\alpha}|-\beta} \le C \tag{2.18}$$

have been used in the proof of (2.16). The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let $r \in N$ and $\xi \in V_S$, then*

$$\left\|\omega\varphi_\xi(\mathbf{x})^{2r}\left(\frac{\partial}{\partial\xi}\right)^{2r} B_{n,d}(f,x)\right\| \le C\left\|\omega\varphi_\xi^{2r}\left(\frac{\partial}{\partial\xi}\right)^{2r} f\right\| \quad f \in C_0^r(S). \tag{2.19}$$

Proof. By (2.10), for $\eta = e_i, u = Tx$ and $\xi = e_{ij}/\sqrt{2}$, $1 \le i < j \le d$, we have

$$\begin{aligned}\left\|\omega\varphi_\xi^{2r}\left(\frac{\partial}{\partial\xi}\right)^{2r} B_{n,d}(f,\mathbf{x})\right\| &= \left\|\omega_T\varphi_\eta^{2r}\left(\frac{\partial}{\partial\eta}\right)^{2r} B_{n,d}(f_T,T\mathbf{x})\right\| \\ &\le C\left\|\omega_T\varphi_\eta^{2r}\left(\frac{\partial}{\partial\eta}\right)^{2r} (f_T)(\mathbf{u})\right\| \\ &\le C\left\|\omega\varphi_\xi^{2r}\left(\frac{\partial}{\partial\xi}\right)^{2r} f\right\|_p.\end{aligned} \tag{2.20}$$

Similar to the discussion in the proof of Lemma 2.2., we need only to prove the case of $\xi = e_2$, that is,

$$\left\|\omega\varphi_{e_2}^{2r}\left(\frac{\partial}{\partial x_2}\right)^{2r} B_{n,d}(f,\mathbf{x})\right\| \le C\left\|\omega\varphi_{e_2}^{2r}\left(\frac{\partial}{\partial x_2}\right)^{2r} f\right\|. \tag{2.21}$$

The steps to prove (2.21) are similar to those to prove the inequality (2.12). Hence, the proof of Lemma 2.3 is complete □

3. Proof of Theorem

We will prove Theorem 1.2 in the followings. For $\xi = e_2$ and for all $g \in W_\phi^{r,\infty}(S)$, it follows from Lemmas 2.2 and 2.3 that

$$\begin{aligned}&\left\|\omega\varphi_{e_2}^{2r}\left(\frac{\partial}{\partial x_2}\right)^{2r} B_{n,d}(f,x)\right\| \\ &\le \left\|\omega\varphi_{e_2}^{2r}\left(\frac{\partial}{\partial x_2}\right)^{2r} B_{n,d}(f-g,x)\right\| + \left\|\omega\varphi_{e_2}^{2r}\left(\frac{\partial}{\partial x_1}\right)^{2r} B_{n,d}(g,x)\right\| \\ &\le C\left\{n^r\|\omega(f-g)\| + \left\|\omega\varphi_{e_2}^{2r}\left(\frac{\partial}{\partial x_2}\right)^{2r} g\right\|\right\}.\end{aligned} \tag{3.1}$$

From the definition of K-functional and (1.14), we obtain

$$\begin{aligned}\left\|\omega\varphi_{e_2}^{2r}\left(\frac{\partial}{\partial x_2}\right)^{2r}B_{n,d}(f,x)\right\| &\leq Cn^r K_r^{e_2}\left(f;n^{-r}\right)_\omega \\ &\leq Cn^r\Omega_r^{e_2}\left(f,\frac{1}{n}\right)_\omega \\ &\leq Cn^{r-\alpha}.\end{aligned} \tag{3.2}$$

Similarly, the case of $\xi = e_i$, $i = 1, 3, 4, \ldots, d$ can also be proved. If $\xi = ((e_i - e_j)/\sqrt{2})$ $1 \leq i < j \leq d$, it is not difficult to obtain

$$\left\|\omega\varphi_{\xi}^{2r}\left(\frac{\partial}{\partial \xi}\right)^{2r}B_{n,d}(f,x)\right\| = \left\|\omega_T\varphi_{\eta}^{2r}\left(\frac{\partial}{\partial \eta}\right)^{2r}B_{n,d}(f_T,u)\right\| \leq Cn^{r-\alpha}, \tag{3.3}$$

by assuming $\eta = e_i, u = Tx$. The proof of Theorem 1.2 is complete.

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