Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 382094, 12 pages
doi:10.1155/2012/382094

Research Article

Common Fixed Point Results Using Generalized Altering Distances on Orbitally Complete Ordered Metric Spaces

Hemant Kumar Nashine,[1] Zoran Kadelburg,[2] and Zorana Golubović[3]

[1] *Department of Mathematics, Disha Institute of Management and Technology, Satya Vihar, Vidhansabha-Chandrakhuri Marg, Mandir Hasaud, Raipur 492101, India*
[2] *Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Beograd, Serbia*
[3] *Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia*

Correspondence should be addressed to Zoran Kadelburg, kadelbur@matf.bg.ac.rs

Received 9 December 2011; Accepted 21 March 2012

Academic Editor: Giuseppe Marino

We prove the existence of common fixed points for three relatively asymptotically regular mappings defined on an orbitally complete ordered metric space using orbital continuity of one of the involved maps. We furnish a suitable example to demonstrate the validity of the hypotheses of our results.

1. Introduction and Preliminaries

Browder and Petryshyn introduced the concept of asymptotic regularity of a self-map at a point in a metric space.

Definition 1.1 (see [1]). A self-map $\mathcal{T}$ on a metric space $(\mathcal{X}, d)$ is said to be asymptotically regular at a point $x \in \mathcal{X}$ if $\lim_{n\to\infty} d(\mathcal{T}^n x, \mathcal{T}^{n+1} x) = 0$.

Recall that the set $\mathcal{O}(x_0; \mathcal{T}) = \{\mathcal{T}^n x_0 : n = 0, 1, 2, \ldots\}$ is called the orbit of the self-map $\mathcal{T}$ at the point $x_0 \in \mathcal{X}$.

Definition 1.2 (see [2]). A metric space $(\mathcal{X}, d)$ is said to be $\mathcal{T}$-orbitally complete if every Cauchy sequence contained in $\mathcal{O}(x; \mathcal{T})$ (for some x in $\mathcal{X}$) converges in $\mathcal{X}$.

Here, it can be pointed out that every complete metric space is $\mathcal{T}$-orbitally complete for any $\mathcal{T}$, but a $\mathcal{T}$-orbitally complete metric space need not be complete.

Definition 1.3 (see [1]). A self-map $\mathcal{T}$ defined on a metric space $(\mathcal{X}, d)$ is said to be orbitally continuous at a point z in $\mathcal{X}$ if for any sequence $\{x_n\} \subset \mathcal{O}(x; \mathcal{T})$ (for some $x \in \mathcal{X}$), $x_n \to z$ as $n \to \infty$ implies $\mathcal{T}x_n \to \mathcal{T}z$ as $n \to \infty$.

Clearly, every continuous self-mapping of a metric space is orbitally continuous, but not conversely.

Sastry et al. [3] extended the above concepts to two and three mappings and employed them to prove common fixed point results for commuting mappings. In what follows, we collect such definitions for three maps.

Definition 1.4 (see [3]). Let $\mathcal{S}, \mathcal{T}, \mathcal{R}$ be three self-mappings defined on a metric space $(\mathcal{X}, d)$.

(1) If for a point $x_0 \in \mathcal{X}$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{R}x_{2n+1} = \mathcal{S}x_{2n}$, $\mathcal{R}x_{2n+2} = \mathcal{T}x_{2n+1}$, $n = 0, 1, 2, \ldots$, then the set $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R}) = \{\mathcal{R}x_n : n = 1, 2, \ldots\}$ is called the orbit of $(\mathcal{S}, \mathcal{T}, \mathcal{R})$ at x_0.

(2) The space $(\mathcal{X}, d)$ is said to be $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-orbitally complete at x_0 if every Cauchy sequence in $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$ converges in $\mathcal{X}$.

(3) The map $\mathcal{R}$ is said to be orbitally continuous at x_0 if it is continuous on $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$.

(4) The pair $(\mathcal{S}, \mathcal{T})$ is said to be asymptotically regular (in short a.r.) with respect to $\mathcal{R}$ at x_0 if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{R}x_{2n+1} = \mathcal{S}x_{2n}$, $\mathcal{R}x_{2n+2} = \mathcal{T}x_{2n+1}$, $n = 0, 1, 2, \ldots$, and $d(\mathcal{R}x_n, \mathcal{R}x_{n+1}) \to 0$ as $n \to \infty$.

On the other side, Khan et al. [4] introduced the notion of an altering distance function, which is a control function that alters distance between two points in a metric space. This notion has been used by several authors to establish fixed point results in a number of subsequent works, some of which are noted in [5–9]. In [5], Choudhury introduced the concept of a generalized altering distance function in three variables which was further generalized by Rao et al. [10] to four variables and is defined as follows.

Definition 1.5 (see [10]). A function $\psi : [0, +\infty)^4 \to [0, +\infty)$ is said to be a generalized altering distance function if

(i) ψ is continuous,

(ii) ψ is nondecreasing in each variable,

(iii) $\psi(t_1, t_2, t_3, t_4) = 0 \Leftrightarrow t_1 = t_2 = t_3 = t_4 = 0$.

$\mathcal{F}_4$ will denote the set of all functions ψ satisfying conditions (i)–(iii).

Simple examples of generalized altering distance functions with four variables are

$$\begin{aligned}
\psi(t_1, t_2, t_3, t_4) &= k \max\{t_1, t_2, t_3, t_4\}, \qquad k > 0; \\
\psi(t_1, t_2, t_3, t_4) &= \frac{\max\{t_1, t_2, t_3, t_4\}}{1 + \max\{t_1, t_2, t_3, t_4\}}; \\
\psi(t_1, t_2, t_3, t_4) &= t_1^p + t_2^q + t_3^r + t_4^s, \qquad p, q, r, s \geq 1.
\end{aligned} \tag{1.1}$$

On the other hand, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [11] who presented its applications to matrix equations. Subsequently, Nieto and Rodríguez-López [12] extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, several authors obtained many fixed point theorems in ordered metric spaces. For more details see [13–20] and the references cited therein.

In this paper, an attempt has been made to derive some common fixed point theorems for three relatively asymptotically regular mappings defined on an orbitally complete ordered metric space, using orbital continuity of one of the involved maps and conditions involving a generalized altering distance function. The presented theorems generalize, extend, and improve some recent results given in [7, 14, 21, 22]. In the hypotheses, we have considered the space as not necessarily complete, the maps $\mathcal{R}, \mathcal{S}$, and $\mathcal{T}$ as not necessarily continuous and the range of $\mathcal{S}$ and $\mathcal{T}$ may not be contained in the range of $\mathcal{R}$.

2. Results

2.1. Notations and Definitions

First, we introduce some further notations and definitions that will be used later.

If $(\mathcal{X}, \preccurlyeq)$ is a partially ordered set, then $x, y \in \mathcal{X}$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be well ordered if every two elements of $\mathcal{K}$ are comparable. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is such that, for $x, y \in \mathcal{X}$, $x \preccurlyeq y$ implies $\mathcal{T}x \preccurlyeq \mathcal{T}y$, then the mapping $\mathcal{T}$ is said to be nondecreasing.

Definition 2.1. Let $(\mathcal{X}, \preccurlyeq)$ be a partially ordered set and $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$.

(1) [23] The pair $(\mathcal{S}, \mathcal{T})$ is called weakly increasing if $\mathcal{S}x \preccurlyeq \mathcal{T}\mathcal{S}x$ and $\mathcal{T}x \preccurlyeq \mathcal{S}\mathcal{T}x$ for all $x \in \mathcal{X}$.

(2) [24] The pair $(\mathcal{S}, \mathcal{T})$ is called partially weakly increasing if $\mathcal{S}x \preccurlyeq \mathcal{T}\mathcal{S}x$ for all $x \in \mathcal{X}$.

(3) [24] The mapping $\mathcal{S}$ is called a weak annihilator of $\mathcal{T}$ if $\mathcal{S}\mathcal{T}x \preccurlyeq x$ for all $x \in \mathcal{X}$.

(4) [24] The mapping $\mathcal{S}$ is called dominating if $x \preccurlyeq \mathcal{S}x$ for each $x \in \mathcal{X}$.

Note that none of two weakly increasing mappings need to be nondecreasing. There exist some examples to illustrate this fact in [23]. Obviously, the pair $(\mathcal{S}, \mathcal{T})$ is weakly increasing if and only if the ordered pairs $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{S})$ are partially weakly increasing. Following is an example of an ordered pair $(\mathcal{S}, \mathcal{T})$ which is partially weakly increasing but not weakly increasing.

Example 2.2 (see [24]). Let $\mathcal{X} = [0, 1]$ be endowed with usual ordering.

(1) Let $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ be defined by $\mathcal{S}x = x^2$ and $\mathcal{T}x = \sqrt{x}$. Clearly, $(\mathcal{S}, \mathcal{T})$ is partially weakly increasing. But $\mathcal{T}x = \sqrt{x} \npreceq x = \mathcal{S}\mathcal{T}x$ for $x \in (0, 1)$ implies that $(\mathcal{T}, \mathcal{S})$ is not partially weakly increasing.

(2) Let $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ be defined by $\mathcal{S}x = x^2$ and $\mathcal{T}x = x^3$. Obviously, $\mathcal{S}\mathcal{T}x = x^6 \leq x$ for all $x \in \mathcal{X}$. Thus $\mathcal{S}$ is a weak annihilator of $\mathcal{T}$.

(3) Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be defined by $\mathcal{S}x = \sqrt[n]{x}$. Since $x \leq \sqrt[n]{x} = \mathcal{S}x$ for all $x \in \mathcal{X}$, $\mathcal{S}$ is a dominating map.

Definition 2.3 (see [25, 26]). Let $(\mathcal{X}, d)$ be a metric space and $f, g : \mathcal{X} \to \mathcal{X}$. The mappings f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Definition 2.4. Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preccurlyeq)$ is called an ordered metric space if

(i) $(\mathcal{X}, d)$ is a metric space,

(ii) $(\mathcal{X}, \preccurlyeq)$ is a partially ordered set.

2.2. Main Results

The first main result is as follows.

Theorem 2.5. *Let $(\mathcal{X}, d, \preccurlyeq)$ be an ordered metric space. Let $\mathcal{S}, \mathcal{T}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ be given mappings satisfying*

$$\Psi_1(d(\mathcal{S}x, \mathcal{T}y)) \leq \varphi_1(M[\mathcal{S}, \mathcal{T}, \mathcal{R}](x, y)) - \varphi_2(M[\mathcal{S}, \mathcal{T}, \mathcal{R}](x, y)), \tag{2.1}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$ (for some x_0) such that $\mathcal{R}x$ and $\mathcal{R}y$ are comparable, where

$$M[\mathcal{S}, \mathcal{T}, \mathcal{R}](x, y) = \left(d(\mathcal{R}x, \mathcal{R}y), d(\mathcal{R}x, \mathcal{S}x), d(\mathcal{R}y, \mathcal{T}y), \frac{1}{2}[d(\mathcal{R}x, \mathcal{T}y) + d(\mathcal{R}y, \mathcal{S}x)]\right), \tag{2.2}$$

and φ_1 and φ_2 are generalized altering distance functions (in $\mathcal{F}_4$) and $\Psi_1(t) = \varphi_1(t, t, t, t)$. We assume the following hypotheses:

(i) *$(\mathcal{S}, \mathcal{T})$ is a.r. with respect to $\mathcal{R}$ at $x_0 \in \mathcal{X}$;*

(ii) *$\mathcal{X}$ is $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-orbitally complete at x_0;*

(iii) *$(\mathcal{R}, \mathcal{S})$ and $(\mathcal{R}, \mathcal{T})$ are partially weakly increasing;*

(iv) *$\mathcal{S}$ and $\mathcal{T}$ are dominating maps;*

(v) *$\mathcal{S}$ and $\mathcal{T}$ are weak annihilators of $\mathcal{R}$;*

(vi) *for a nondecreasing sequence $\{x_n\}$, $x_n \preccurlyeq y_n$ for all n and $y_n \to u$ as $n \to \infty$ imply that $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$.*

Assume either

(a) *$\mathcal{S}$ and $\mathcal{R}$ are compatible; $\mathcal{S}$ or $\mathcal{R}$ is orbitally continuous at x_0 or*

(b) *$\mathcal{T}$ and $\mathcal{R}$ are compatible; $\mathcal{T}$ or $\mathcal{R}$ is orbitally continuous at x_0.*

Then $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$, and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$ is well ordered if and only if it is a singleton.

Proof. Since $(\mathcal{S}, \mathcal{T})$ is a.r. with respect to $\mathcal{R}$ at x_0 in $\mathcal{X}$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$\mathcal{S}x_{2n-2} = \mathcal{R}x_{2n-1}, \quad \mathcal{T}x_{2n-1} = \mathcal{R}x_{2n}, \quad \forall n \in \mathbb{N}. \tag{2.3}$$

By the given assumptions, $x_{2n-2} \preccurlyeq \mathcal{S}x_{2n-2} = \mathcal{R}x_{2n-1} \preccurlyeq \mathcal{S}\mathcal{R}x_{2n-1} \preccurlyeq x_{2n-1}$, and $x_{2n-1} \preccurlyeq \mathcal{T}x_{2n-1} = \mathcal{R}x_{2n} \preccurlyeq \mathcal{T}\mathcal{R}x_{2n} \preccurlyeq x_{2n}$. Thus, for all $n \geq 1$, we have

$$x_n \preccurlyeq x_{n+1}. \tag{2.4}$$

In view of (i), we have

$$\lim_{n\to\infty} d(\mathcal{R}x_n, \mathcal{R}x_{n+1}) = 0. \tag{2.5}$$

Now, we assert that $\{\mathcal{R}x_n\}$ is a Cauchy sequence in the metric space $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$.

From (2.5), it will be sufficient to prove that $\{\mathcal{R}x_{2n}\}$ is a Cauchy sequence. We proceed by negation and suppose that $\{\mathcal{R}x_{2n}\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(i)\}$ and $\{n(i)\}$ such that for all positive integers i,

$$n(i) > m(i) > i, \qquad d\left(\mathcal{R}x_{2m(i)}, \mathcal{R}x_{2n(i)}\right) \geq \varepsilon, \qquad d\left(\mathcal{R}x_{2m(i)}, \mathcal{R}x_{2n(i)-2}\right) < \varepsilon. \tag{2.6}$$

From (2.6) and using the triangular inequality, we get

$$\begin{aligned}\varepsilon &\leq d\left(\mathcal{R}x_{2m(i)}, \mathcal{R}x_{2n(i)}\right)\\ &\leq d\left(\mathcal{R}x_{2m(i)}, \mathcal{R}x_{2n(i)-2}\right) + d\left(\mathcal{R}x_{2n(i)-2}, \mathcal{R}x_{2n(i)-1}\right) + d\left(\mathcal{R}x_{2n(i)-1}, \mathcal{R}x_{2n(i)}\right)\\ &< \varepsilon + d\left(\mathcal{R}x_{2n(i)-2}, \mathcal{R}x_{2n(i)-1}\right) + d\left(\mathcal{R}x_{2n(i)-1}, \mathcal{R}x_{2n(i)}\right).\end{aligned} \tag{2.7}$$

Letting $i \to \infty$ in the above inequality and using (2.5), we obtain

$$\lim_{i\to\infty} d\left(\mathcal{R}x_{2m(i)}, \mathcal{R}x_{2n(i)}\right) = \varepsilon. \tag{2.8}$$

Again, the triangular inequality gives us

$$\left|d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2m(i)-1}\right) - d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2m(i)}\right)\right| \leq d\left(\mathcal{R}x_{2m(i)-1}, \mathcal{R}x_{2m(i)}\right). \tag{2.9}$$

Letting $i \to \infty$ in the above inequality and using (2.5) and (2.8), we get

$$\lim_{i\to\infty} d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2m(i)-1}\right) = \varepsilon. \tag{2.10}$$

Similarly, we have

$$\lim_{i\to\infty} d\left(\mathcal{R}x_{2n(i)+1}, \mathcal{R}x_{2m(i)-1}\right) = \varepsilon. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2m(i)}\right) &\leq d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2n(i)+1}\right) + d\left(\mathcal{R}x_{2n(i)+1}, \mathcal{R}x_{2m(i)}\right)\\ &= d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2n(i)+1}\right) + d\left(\mathcal{T}x_{2n(i)}, \mathcal{S}x_{2m(i)-1}\right).\end{aligned} \tag{2.12}$$

Then, from (2.5), (2.8), and the continuity of Ψ_1, we get by letting $i \to \infty$ in (2.1)

$$\Psi_1(\varepsilon) \leq \lim_{i\to\infty} \Psi_1\left(d\left(\mathcal{S}x_{2m(i)-1}, \mathcal{T}x_{2n(i)}\right)\right). \tag{2.13}$$

Now, using the considered contractive condition (2.1) for $x = x_{2m(i)-1}$ and $y = x_{2n(i)}$, we have

$$\begin{aligned} &\Psi_1\left(d\left(\mathcal{S}x_{2m(i)-1}, \mathcal{T}x_{2n(i)}\right)\right) \\ &\quad \leq \varphi_1\Bigg(d\left(\mathcal{R}x_{2m(i)-1}, \mathcal{R}x_{2n(i)}\right), d\left(\mathcal{R}x_{2m(i)-1}, \mathcal{R}x_{2m(i)}\right), d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2n(i)+1}\right), \\ &\qquad\qquad \frac{1}{2}\left[d\left(\mathcal{R}x_{2m(i)-1}, \mathcal{R}x_{2n(i)+1}\right) + d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2m(i)}\right)\right]\Bigg) \\ &\qquad - \varphi_2\Bigg(d\left(\mathcal{R}x_{2m(i)-1}, \mathcal{R}x_{2n(i)}\right), d\left(\mathcal{R}x_{2m(i)-1}, \mathcal{R}x_{2m(i)}\right), d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2n(i)+1}\right), \\ &\qquad\qquad \frac{1}{2}\left[d\left(\mathcal{R}x_{2m(i)-1}, \mathcal{R}x_{2n(i)+1}\right) + d\left(\mathcal{R}x_{2n(i)}, \mathcal{R}x_{2m(i)}\right)\right]\Bigg). \end{aligned} \tag{2.14}$$

Then, from (2.5), (2.10), (2.11), and the continuity of φ_1 and φ_2, we get by letting $i \to \infty$ in the above inequality

$$\lim_{i\to\infty} \Psi_1\left(d\left(\mathcal{S}x_{2m(i)-1}, \mathcal{T}x_{2n(i)}\right)\right) \leq \varphi_1(\varepsilon, 0, 0, \epsilon) - \varphi_2(\varepsilon, 0, 0, \epsilon) \leq \Psi_1(\varepsilon) - \varphi_2(\varepsilon, 0, 0, \epsilon). \tag{2.15}$$

Now, combining (2.13) with the above inequality, we get

$$\Psi_1(\varepsilon) \leq \Psi_1(\varepsilon) - \varphi_2(\varepsilon, 0, 0, \epsilon), \tag{2.16}$$

which implies that $\varphi_2(\varepsilon, 0, 0, \epsilon) = 0$, which is a contradiction since $\varepsilon > 0$. Hence $\{\mathcal{R}x_n\}$ is a Cauchy sequence in $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$. Since $\mathcal{X}$ is $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-orbitally complete at x_0, there exists some $z \in \mathcal{X}$ such that $\mathcal{R}x_n \to z$ as $n \to \infty$.

Finally, we prove the existence of a common fixed point of the three mappings $\mathcal{S}, \mathcal{T}$, and $\mathcal{R}$.

We have

$$\begin{aligned} \mathcal{R}x_{2n+1} &= \mathcal{S}x_{2n} \longrightarrow z \quad \text{as } n \longrightarrow \infty, \\ \mathcal{R}x_{2n+2} &= \mathcal{T}x_{2n+1} \longrightarrow z \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.17}$$

Suppose that (a) holds. Since $\{\mathcal{S}, \mathcal{R}\}$ are compatible, we have

$$\lim_{n\to\infty} \mathcal{S}\mathcal{R}x_{2n+2} = \lim_{n\to\infty} \mathcal{R}\mathcal{S}x_{2n+2} = \mathcal{R}z. \tag{2.18}$$

Also, $x_{2n+1} \preccurlyeq \mathcal{T}x_{2n+1} = \mathcal{R}x_{2n+2}$. Now

$$\begin{aligned}
&\Psi_1(d(\mathcal{S}\mathcal{R}x_{2n+2}, \mathcal{T}x_{2n+1})) \\
&\quad \leq \psi_1\bigg(d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{R}x_{2n+1}), d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{S}\mathcal{R}x_{2n+2}), d(\mathcal{R}x_{2n+1}, \mathcal{T}x_{2n+1}), \\
&\qquad \frac{1}{2}[d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{T}x_{2n+1}) + d(\mathcal{S}\mathcal{R}x_{2n+2}, \mathcal{R}x_{2n+1})]\bigg) \\
&\quad - \psi_2\bigg(d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{R}x_{2n+1}), d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{S}\mathcal{R}x_{2n+2}), d(\mathcal{R}x_{2n+1}, \mathcal{T}x_{2n+1}), \\
&\qquad \frac{1}{2}[d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{T}x_{2n+1}) + d(\mathcal{S}\mathcal{R}x_{2n+2}, \mathcal{R}x_{2n+1})]\bigg).
\end{aligned} \tag{2.19}$$

Assume that $\mathcal{R}$ is orbitally continuous. Passing to the limit as $n \to \infty$, we obtain

$$\begin{aligned}
\Psi_1(d(\mathcal{R}z, z)) &\leq \psi_1(d(\mathcal{R}z, z), 0, 0, d(\mathcal{R}z, z)) - \psi_2(d(\mathcal{R}z, z), 0, 0, d(\mathcal{R}z, z)) \\
&\leq \Psi_1(d(\mathcal{R}z, z)) - \psi_2(d(\mathcal{R}z, z), 0, 0, d(\mathcal{R}z, z)),
\end{aligned} \tag{2.20}$$

so $\psi_2(d(\mathcal{R}z, z), 0, 0, d(\mathcal{R}z, z)) = 0$, which implies that

$$\mathcal{R}z = z. \tag{2.21}$$

Now, $x_{2n+1} \preccurlyeq \mathcal{T}x_{2n+1}$ and $\mathcal{T}x_{2n+1} \to z$ as $n \to \infty$, so by the assumption we have $x_{2n+1} \preccurlyeq z$ and (2.1) becomes

$$\begin{aligned}
\Psi_1(d(\mathcal{S}z, \mathcal{T}x_{2n+1})) &\leq \psi_1\bigg(d(\mathcal{R}z, \mathcal{R}x_{2n+1}), d(\mathcal{S}z, \mathcal{R}z), d(\mathcal{T}x_{2n+1}, \mathcal{R}x_{2n+1}), \\
&\qquad \frac{1}{2}[d(\mathcal{R}z, \mathcal{T}x_{2n+1}) + d(\mathcal{S}z, \mathcal{R}x_{2n+1})]\bigg) \\
&\quad - \psi_2\bigg(d(\mathcal{R}z, \mathcal{R}x_{2n+1}), d(\mathcal{S}z, \mathcal{R}z), d(\mathcal{T}x_{2n+1}, \mathcal{R}x_{2n+1}), \\
&\qquad \frac{1}{2}[d(\mathcal{R}z, \mathcal{T}x_{2n+1}) + d(\mathcal{S}z, \mathcal{R}x_{2n+1})]\bigg).
\end{aligned} \tag{2.22}$$

Passing to the limit as $n \to \infty$ in the above inequality and using (2.21), it follows that

$$\begin{aligned}
\Psi_1(d(\mathcal{S}z, z)) &\leq \psi_1\left(0, d(\mathcal{S}z, z), 0, \frac{1}{2}d(\mathcal{S}z, z)\right) - \psi_2\left(0, d(\mathcal{S}z, z), 0, \frac{1}{2}d(\mathcal{S}z, z)\right) \\
&\leq \Psi_1(d(\mathcal{S}z, z)) - \psi_2\left(0, d(\mathcal{S}z, z), 0, \frac{1}{2}d(\mathcal{S}z, z)\right),
\end{aligned} \tag{2.23}$$

which holds unless $\psi_2(0, d(\mathcal{S}z, z), 0, (1/2)d(\mathcal{S}z, z)) = 0$, so

$$\mathcal{S}z = z. \tag{2.24}$$

Now, since $x_{2n} \preccurlyeq \mathcal{S}x_{2n}$ and $\mathcal{S}x_{2n} \to z$ as $n \to \infty$ implies that $x_{2n} \preccurlyeq z$, from (2.1)

$$\begin{aligned}
&\Psi_1(d(\mathcal{S}x_{2n}, \mathcal{T}z)) \\
&\quad \le \varphi_1\left(d(\mathcal{R}x_{2n}, \mathcal{R}z), d(\mathcal{R}x_{2n}, \mathcal{S}x_{2n}), d(\mathcal{R}z, \mathcal{T}z), \frac{1}{2}(d(\mathcal{R}x_{2n}, \mathcal{T}z) + d(\mathcal{S}x_{2n}, \mathcal{R}z))\right) \\
&\qquad - \varphi_2\left(d(\mathcal{R}x_{2n}, \mathcal{R}z), d(\mathcal{R}x_{2n}, \mathcal{S}x_{2n}), d(\mathcal{R}z, \mathcal{T}z), \frac{1}{2}(d(\mathcal{R}x_{2n}, \mathcal{T}z) + d(\mathcal{S}x_{2n}, \mathcal{R}z))\right).
\end{aligned} \tag{2.25}$$

Passing to the limit as $n \to \infty$, we have

$$\begin{aligned}
\Psi_1(d(z, \mathcal{T}z)) &\le \varphi_1(0, 0, d(z, \mathcal{T}z), d(z, \mathcal{T}z)) - \varphi_2(d(z, \mathcal{T}z), 0, 0, d(z, \mathcal{T}z)) \\
&\le \Psi_1(d(z, \mathcal{T}z)) - \varphi_2(0, 0, d(z, \mathcal{T}z), d(z, \mathcal{T}z)),
\end{aligned} \tag{2.26}$$

which gives that

$$z = \mathcal{T}z. \tag{2.27}$$

Therefore, $\mathcal{S}z = \mathcal{T}z = \mathcal{R}z = z$, hence z is a common fixed point of $\mathcal{R}, \mathcal{S}$, and $\mathcal{T}$. The proof is similar when $\mathcal{S}$ is orbitally continuous.

Similarly, the result follows when condition (b) holds.

Now, suppose that the set of common fixed points of $\mathcal{S}, \mathcal{T}$, and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$ is well ordered. We claim that it cannot contain more than one point. Assume to the contrary that $\mathcal{S}u = \mathcal{T}u = \mathcal{R}u = u$ and $\mathcal{S}v = \mathcal{T}v = \mathcal{R}v = v$ but $u \ne v$. By supposition, we can replace x by u and y by v in (2.1) to obtain

$$\begin{aligned}
\Psi_1(d(u, v)) &= \Psi_1(d(\mathcal{S}u, \mathcal{T}v)) \\
&\le \varphi_1\left(d(\mathcal{R}u, \mathcal{R}v), d(\mathcal{R}u, \mathcal{S}u), d(\mathcal{R}v, \mathcal{T}v), \frac{1}{2}[d(\mathcal{R}u, \mathcal{T}v) + d(\mathcal{S}u, \mathcal{R}v)]\right) \\
&\quad - \varphi_2\left(d(\mathcal{R}u, \mathcal{R}v), d(\mathcal{R}u, \mathcal{S}u), d(\mathcal{R}v, \mathcal{T}v), \frac{1}{2}[d(\mathcal{R}u, \mathcal{T}v) + d(\mathcal{S}u, \mathcal{R}v)]\right) \\
&= \varphi_1(d(u, v), 0, 0, d(u, v)) - \varphi_2(d(u, v), 0, 0, d(u, v)) \\
&< \Psi_1(d(u, v)),
\end{aligned} \tag{2.28}$$

a contradiction. Hence, $u = v$. The converse is trivial. □

Now, it is easy to state a corollary of Theorem 2.5 involving a contraction of integral type.

Corollary 2.6. *Let $\mathcal{S}, \mathcal{T}$, and $\mathcal{R}$ satisfy the conditions of Theorem 2.5, except that condition* (2.1) *is replaced by the following: there exists a positive Lebesgue integrable function u on $\mathbb{R}_+$ such that $\int_0^{\varepsilon} u(t)dt > 0$ for each $\varepsilon > 0$ and that*

$$\int_0^{\Psi_1(d(\mathcal{S}x, \mathcal{T}y))} u(t)dt \le \int_0^{\varphi_1(M[\mathcal{S}, \mathcal{T}, \mathcal{R}](x, y))} u(t)dt - \int_0^{\varphi_2(M[\mathcal{S}, \mathcal{T}, \mathcal{R}](x, y))} u(t)dt. \tag{2.29}$$

Then, $\mathcal{S}, \mathcal{T}$, and $\mathcal{R}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$, and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$ is well ordered if and only if it is a singleton.

Remark 2.7. If we take

$$\psi_1(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}, \qquad \psi_2(t_1, t_2, t_3, t_4) = (1-k)\max\{t_1, t_2, t_3, t_4\}, \tag{2.30}$$

for $k \in (0,1)$, then $\Psi_1(t) = t$ for all $t \geq 0$, and the contractive condition (2.1) becomes

$$d(\mathcal{S}x, \mathcal{T}y) \leq k \max\left\{ d(\mathcal{R}x, \mathcal{R}y), d(\mathcal{R}x, \mathcal{S}x), d(\mathcal{R}y, \mathcal{T}y), \frac{1}{2}\left[d(\mathcal{R}x, \mathcal{T}y) + d(\mathcal{R}y, \mathcal{S}x)\right] \right\}, \tag{2.31}$$

which corresponds to the contraction given by Theorem 2.1 in [24] by taking $\psi(t) = t$ and $\varphi(t) = (1-k)t$. Hence, the result of Abbas et al. [24] is covered by Theorem 2.5 for three maps.

Other results could be derived for other choices of ψ_1 and ψ_2.

As consequences of Theorem 2.5, we may state the following corollaries.

Corollary 2.8. *Let $(\mathcal{X}, d, \preccurlyeq)$ be an ordered metric space. Let $\mathcal{T}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ be given mappings satisfying*

$$\begin{aligned} \Psi_1(d(\mathcal{T}x, \mathcal{T}y)) \leq{} & \psi_1\left(d(\mathcal{R}x, \mathcal{R}y), d(\mathcal{R}x, \mathcal{T}x), d(\mathcal{R}y, \mathcal{T}y), \frac{1}{2}\left[d(\mathcal{R}x, \mathcal{T}y) + d(\mathcal{R}y, \mathcal{T}x)\right] \right) \\ & - \psi_2\left(d(\mathcal{R}x, \mathcal{R}y), d(\mathcal{R}x, \mathcal{T}x), d(\mathcal{R}y, \mathcal{T}y), \frac{1}{2}\left[d(\mathcal{R}x, \mathcal{T}y) + d(\mathcal{R}y, \mathcal{T}x)\right] \right), \end{aligned} \tag{2.32}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{T}, \mathcal{T}, \mathcal{R})}$ such that $\mathcal{R}x$ and $\mathcal{R}y$ are comparable, where ψ_1 and ψ_2 are generalized altering distance functions (in $\mathcal{F}_4$) and $\Psi_1(t) = \psi_1(t, t, t, t)$. We assume the following hypotheses:

(i) *$\mathcal{T}$ is a.r. with respect to $\mathcal{R}$ at x_0;*

(ii) *$\mathcal{X}$ is $(\mathcal{T}, \mathcal{R})$-orbitally complete at x_0;*

(iii) *$\mathcal{T}$ or $\mathcal{R}$ is orbitally continuous at x_0;*

(iv) *$(\mathcal{T}, \mathcal{R})$ is partially weakly increasing;*

(v) *$\mathcal{T}$ is a dominating map;*

(vi) *$\mathcal{T}$ is a weak annihilator of $\mathcal{R}$;*

(vii) *$\mathcal{T}$ and $\mathcal{R}$ are compatible.*

Let for a nondecreasing sequence $\{x_n\}$ with $x_n \preccurlyeq y_n$ for all n, $y_n \to u$ as $n \to \infty$ imply that $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$.

Then $\mathcal{T}$ and $\mathcal{R}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{T}$ and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{T}, \mathcal{T}, \mathcal{R})}$ is well ordered if and only if it is a singleton.

Corollary 2.9. *Let $(\mathcal{X}, d, \preccurlyeq)$ be an ordered metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying*

$$\begin{aligned}\Psi_1(d(\mathcal{T}x,\mathcal{T}y)) \le \varphi_1\Big(d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), \frac{1}{2}[d(x,\mathcal{T}y)+d(y,\mathcal{T}x)]\Big) \\ - \varphi_2\Big(d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), \frac{1}{2}[d(x,\mathcal{T}y)+d(y,\mathcal{T}x)]\Big),\end{aligned} \tag{2.33}$$

for all $(x,y) \in \overline{\mathcal{O}(x_0;\mathcal{T})}$ such that x and y are comparable, where φ_1 and φ_2 are generalized altering distance functions (in $\mathcal{F}_4$) and $\Psi_1(t) = \varphi_1(t,t,t,t)$. We assume the following hypotheses:

(i) *$\mathcal{T}$ is a.r. at some point x_0 of $\mathcal{X}$;*

(ii) *$\mathcal{X}$ is $\mathcal{T}$-orbitally complete at x_0;*

(iii) *$\mathcal{T}$ is a dominating map.*

Let for a nondecreasing sequence $\{x_n\}$ with $x_n \preccurlyeq y_n$ for all n, $y_n \to u$ as $n \to \infty$ imply that $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$.

Then $\mathcal{T}$ has a fixed point. Moreover, the set of fixed points of $\mathcal{T}$ in $\overline{\mathcal{O}(x_0;\mathcal{T})}$ is well ordered if and only if it is a singleton.

We present an example showing the usage of our results.

Example 2.10. Let the set $\mathcal{X} = [0,+\infty)$ be equipped with the usual metric d and the order defined by

$$x \preccurlyeq y \iff x \ge y. \tag{2.34}$$

Consider the following self-mappings on $\mathcal{X}$:

$$\mathcal{R}x = 6x, \qquad \mathcal{S}x = \begin{cases} \frac{1}{2}x, & 0 \le x \le \frac{1}{2}, \\ x, & x > \frac{1}{2}, \end{cases} \qquad \mathcal{T}x = \begin{cases} \frac{1}{3}x, & 0 \le x \le \frac{1}{3}, \\ x, & x > \frac{1}{3}. \end{cases} \tag{2.35}$$

Take $x_0 = 1/2$. Then it is easy to show that

$$\mathcal{O}(x_0;\mathcal{S},\mathcal{T},\mathcal{R}) \subset \left\{\frac{1}{2^k \cdot 3^l} : k,l \in \mathbb{N}\right\}, \qquad \overline{\mathcal{O}(x_0;\mathcal{S},\mathcal{T},\mathcal{R})} = \mathcal{O}(x_0;\mathcal{S},\mathcal{T},\mathcal{R}) \cup \{0\} \tag{2.36}$$

and all the conditions (i)–(vi) and (a)-(b) of Theorem 2.5 are fulfilled. Take $\varphi_1(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$ and $\varphi_2(t_1,t_2,t_3,t_4) = (5/6)\max\{t_1,t_2,t_3,t_4\}$. Then contractive condition (2.1) takes the form

$$\left|\frac{1}{2}x - \frac{1}{3}y\right| \le \frac{1}{6}\max\left\{|6x-6y|, \frac{11}{2}x, \frac{17}{3}y, \frac{1}{2}\left[\left|6x-\frac{1}{3}y\right| + \left|6y - \frac{1}{2}x\right|\right]\right\}, \tag{2.37}$$

for $x, y \in \mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$. Using substitution $y = tx$, $t > 0$, the last inequality reduces to

$$|3-2t| \leq \max\left\{6|1-t|, \frac{11}{2}, \frac{17}{3}t, \frac{1}{2}\left[\left|6-\frac{1}{3}t\right| + \left|6t-\frac{1}{2}\right|\right]\right\}, \tag{2.38}$$

and can be checked by discussion on possible values for $t > 0$. Hence, all the conditions of Theorem 2.5 are satisfied and $\mathcal{S}, \mathcal{T}, \mathcal{R}$ have a common fixed point (which is 0).

Remark 2.11. It was shown by examples in [22] that (in similar situations)

(1) if the contractive condition is satisfied just on $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$, there might not exist a (common) fixed point;

(2) under the given hypotheses (common), fixed point might not be unique in the whole space $\mathcal{X}$.

Acknowledgment

The authors thank the referees for their careful reading of the text and for suggestions that helped to improve the exposition of the paper. The second and third authors are thankful to the Ministry of Science and Technological Development of Serbia.

References

[1] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[2] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, pp. 267–273, 1974.

[3] K. P. R. Sastry, S. V. R. Naidu, I. H. N. Rao, and K. P. R. Rao, "Common fixed points for asymptotically regular mappings," *Indian Journal of Pure and Applied Mathematics*, vol. 15, no. 8, pp. 849–854, 1984.

[4] M. S. Khan, M. Swaleh, and S. Sessa, "Fixed point theorems by altering distances between the points," *Bulletin of the Australian Mathematical Society*, vol. 30, no. 1, pp. 1–9, 1984.

[5] B. S. Choudhury, "A common unique fixed point result in metric spaces involving generalised altering distances," *Mathematical Communications*, vol. 10, no. 2, pp. 105–110, 2005.

[6] P. N. Dutta and B. S. Choudhury, "A generalisation of contraction principle in metric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 406368, 8 pages, 2008.

[7] H. K. Nashine, "New fixed point theorems for mappings satisfying generalized weakly contractive condition with weaker control functions," *Annales Polonici Mathematici*, vol. 104, pp. 109–119, 2012.

[8] S. Radenovic, Z. Kadelburg, D. Jandrlic, and A. Jandrlic, "Some results on weak contraction maps," *Bulletin of the Iranian Mathematical Society*. In press.

[9] Q. Zhang and Y. Song, "Fixed point theory for generalized φ-weak contractions," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 75–78, 2009.

[10] K. P. R. Rao, G. R. Babu, and D. V. Babu, "Common fixed point theorems through generalized altering distance functions," *Mathematical Communications*, vol. 13, no. 1, pp. 67–73, 2008.

[11] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[12] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[13] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, "Generalized contractions in partially ordered metric spaces," *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.

[14] J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3403–3410, 2009.

[15] H. K. Nashine and I. Altun, "Fixed point theorems for generalized weakly contractive condition in ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2011, Article ID 132367, 20 pages, 2011.

[16] H. K. Nashine and B. Samet, "Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 6, pp. 2201–2209, 2011.

[17] H. K. Nashine, B. Samet, and J. K. Kim, "Fixed point results for contractions involving generalized altering distances in ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2011, article 5, 2011.

[18] H. K. Nashine, B. Samet, and C. Vetro, "Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 712–720, 2011.

[19] H. K. Nashine and W. Shatanawi, "Coupled common fixed point theorems for a pair of commuting mappings in partially ordered complete metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1984–1993, 2011.

[20] D. O'Regan and A. Petruşel, "Fixed point theorems for generalized contractions in ordered metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1241–1252, 2008.

[21] I. Altun and H. Simsek, "Some fixed point theorems on ordered metric spaces and application," *Fixed Point Theory and Applications*, vol. 2010, Article ID 621469, 17 pages, 2010.

[22] G. V. R. Babu and P. D. Sailaja, "A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces," *Thai Journal of Mathematics*, vol. 9, no. 1, pp. 1–10, 2011.

[23] I. Altun, B. Damjanović, and D. Djorić, "Fixed point and common fixed point theorems on ordered cone metric spaces," *Applied Mathematics Letters*, vol. 23, no. 3, pp. 310–316, 2010.

[24] M. Abbas, T. Nazir, and S. Radenović, "Common fixed points of four maps in partially ordered metric spaces," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1520–1526, 2011.

[25] G. Jungck, "Commuting mappings and fixed points," *The American Mathematical Monthly*, vol. 83, no. 4, pp. 261–263, 1976.

[26] G. Jungck, "Compatible mappings and common fixed points," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771–779, 1986.