Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 454579, 14 pages
doi:10.1155/2012/454579

Research Article

Approximation Theorems for Generalized Complex Kantorovich-Type Operators

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Received 28 April 2012; Accepted 3 September 2012

Academic Editor: Jinyun Yuan

The order of simultaneous approximation and Voronovskaja-type results with quantitative estimate for complex q-Kantorovich polynomials ($q > 0$) attached to analytic functions on compact disks are obtained. In particular, it is proved that for functions analytic in $\{z \in \mathbb{C} : |z| < R\}$, $R > q$, the rate of approximation by the q-Kantorovich operators ($q > 1$) is of order q^{-n} versus $1/n$ for the classical Kantorovich operators.

1. Introduction

For each integer $k \geq 0$, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & \text{if } q \in R^+ \setminus \{1\}, \\ k, & \text{if } q = 1, \end{cases} \quad \text{for } k \in N,\ [0]_q = 0,$$
$$[k]_q! := [1]_q[2]_q \cdots [k]_q \quad \text{for } k \in \mathbb{N},\ [0]! = 1. \tag{1.1}$$

For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{1.2}$$

For fixed $1 \neq q > 0$, we denote the q-derivative $D_q f(z)$ of f by

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{1.3}$$

Let $\mathbb{D}_R$ be a disc $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D}_R)$ the space of all analytic functions on $\mathbb{D}_R$. For $f \in H(\mathbb{D}_R)$ we assume that $f(z) = \sum_{m=0}^{\infty} a_m z^m$.

In several recent papers, convergence properties of complex q-Bernstein polynomials, proposed by Phillips [1], defined by

$$B_{n,q}(f;z) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{j=0}^{n-k-1} \left(1 - q^j x\right) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{n,k}(q;z) \tag{1.4}$$

and attached to an analytic function f in closed disks, were intensively studied by many authors; see [2] and references their in. It is known that the cases $0 < q < 1$ and $q > 1$ are not similar to each other. This difference is caused by the fact that, for $0 < q < 1$, $B_{n,q}$ are positive linear operators on $C[0,1]$ while for $q > 1$, the positivity fails. The lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult than that for $0 < q < 1$. There are few papers [3–6] studying systematically the convergence of the q-Berntsein polynomials in the case $q > 1$. If $q \geq 1$ then qualitative Voronovskaja-type and saturation results for complex q-Bernstein polynomials were obtained in Wang and Wu [5]. Wu [6] studied saturation of convergence on the interval $[0,1]$ for the q-Bernstein polynomials of a continuous function f for arbitrary fixed $q > 1$. On the other hand, Gal [7, 8], Anastassiou and Gal [9, 10], Mahmudov [11–13], and Mahmudov and Gupta [14] obtained quantitative estimates of the convergence and of the Voronovskaja's theorem in compact disks, for different complex Bernstein-Durrmeyer type operators.

The goal of the present note is to extend these type of results to complex Kantorovich operators based on the q-integers, in the case $q > 0$, defined as follows:

$$K_{n,q}(f;z) = \sum_{k=0}^{n} p_{n,k}(q;z) \int_0^1 f\left(\frac{q[k]_q + t}{[n+1]_q}\right) dt. \tag{1.5}$$

Notice that in the case $q = 1$, these operators coincide with the classical Kantorovich operators. For $0 < q \leq 1$ the operator $K_{n,q} : C[0,1] \to C[0,1]$ is positive and for $q > 1$, it is not positive. The problems studied in this paper in the case $q = 1$ were investigated in [2, 9].

We start with the following quantitative estimates of the convergence for complex q-Kantorovich-type operators attached to an analytic function in a disk of radius $R > 1$ and center 0.

Theorem 1.1. *Let $f \in H(\mathbb{D}_R)$.*

(i) *Let $0 < q \le 1$ and $1 \le r < R$. For all $z \in \mathbb{D}_r$ and $n \in \mathbb{N}$, one has*

$$\left|K_{n,q}(f;z) - f(z)\right| \le \frac{3+q^{-1}}{2[n]_q}\sum_{m=1}^{\infty}|a_m|m(m+1)r^m. \tag{1.6}$$

(ii) *Let $1 < q < R < \infty$ and $1 \le r < R/q$. For all $z \in \mathbb{D}_r$ and $n \in \mathbb{N}$, one has*

$$\left|K_{n,q}(f;z) - f(z)\right| \le \frac{2}{[n]_q}\sum_{m=1}^{\infty}|a_m|m(m+1)q^m r^m. \tag{1.7}$$

Remark 1.2. (i) Since $[n]_q \to (1-q)^{-1}$ as $n \to \infty$ in the estimate in Theorem 1.1(i) we do not obtain convergence of $K_{n,q}(f;z)$ to $f(z)$. But this situation can be improved by choosing $0 < q = q_n < 1$ with $q_n \nearrow 1$ as $n \to \infty$. Since in this case $[n]_{q_n} \to \infty$ as $n \to \infty$, from Theorem 1.1(i) we get uniform convergence in $\mathbb{D}_r$.

(ii) Theorem 1.1(ii) says that for functions analytic in $\mathbb{D}_R$, $R > q$, the rate of approximation by the q-Kantorovich operators ($q > 1$) is of order q^{-n} versus $1/n$ for the classical Kantorovich operators.

Let $f \in H(\mathbb{D}_R)$. Let us define

$$L_q(f;z) := \begin{cases} \dfrac{1-2z}{2}f'(z) + \dfrac{(1-z)\left(D_q f(z) - f'(z)\right)}{1-q^{-1}}, & \text{if } |z| < \dfrac{R}{q},\ R > q > 1, \\[2ex] \dfrac{1-2z}{2}f'(z) + \dfrac{z(1-z)}{2}f''(z), & \text{if } |z| < R,\ 0 < q \le 1. \end{cases} \tag{1.8}$$

It is not difficult to show that

$$\begin{aligned} L_q(f;z) &= q(1-z)\sum_{m=1}^{\infty}a_m\frac{[m]_q - m}{q-1}z^{m-1} + \frac{1-2z}{2}\sum_{m=1}^{\infty}a_m m z^{m-1} \\ &= q\sum_{m=1}^{\infty}a_m\left([1]_q + \cdots + [m-1]_q\right)z^{m-1}(1-z) + \frac{1-2z}{2}\sum_{m=1}^{\infty}a_m m z^{m-1}, \quad q > 1. \end{aligned} \tag{1.9}$$

Here we used the identity

$$\frac{[m]_q - m}{q-1} = [1]_q + \cdots + [m-1]_q. \tag{1.10}$$

The next theorem gives Voronovskaja-type result in compact disks, for complex q-Kantorovich operators attached to an analytic function in $\mathbb{D}_R$, $R > 1$ and center 0.

Theorem 1.3. *Let $f \in H(\mathbb{D}_R)$.*

(i) *Let $0 < q \le 1$ and $1 \le r < R$. For all $z \in \mathbb{D}_r$ and $n \in \mathbb{N}$ one has*

$$\left| K_{n,q}(f;z) - f(z) - \frac{1-2z}{2[n+1]_q} f'(z) - \frac{z(1-z)}{2[n+1]_q} f''(z) \right| \le \frac{28 + q^{-1}}{2[n]_q^2} \sum_{m=2}^{\infty} |a_m| m^2 (m-1)^2 r^m. \tag{1.11}$$

(ii) *Let $1 < q < R < \infty$ and $1 \le r < R/q^2$. For all $z \in \mathbb{D}_r$ and $n \in \mathbb{N}$, one has*

$$\left| K_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q} L_q(f;z) \right| \le \frac{14}{[n]_q^2} \sum_{m=2}^{\infty} |a_m| m^2 (m-1)^2 q^{2m} r^m. \tag{1.12}$$

Remark 1.4. (i) In the hypothesis on f in Theorem 1.3(i) choosing $0 < q_n < 1$ with $q_n \nearrow 1$ as $n \to \infty$, it follows that

$$\lim_{n \to \infty} [n+1]_{q_n} \left[K_{n,q_n}(f;z) - f(z) \right] = \frac{1-2z}{2} f'(z) + \frac{z(1-z)}{2} f''(z) \tag{1.13}$$

uniformly in any compact disk included in the open disk $\mathbb{D}_R$.

(ii) Theorem 1.3(ii) gives explicit formulas of Voronovskaja-type for the q-Kantorovich polynomials for $q > 1$.

(iii) Obviously the best order of approximation that can be obtained from the estimate Theorem 1.3(i) is $O(1/[n]_{q_n}^2)$ and $O(1/n^2)$ for $q = 1$, while the order given by Theorem 1.3(ii) is $O(1/q^{2n})$, $q > 1$, which is essentially better.

Next theorem shows that $L_q(f;z)$, $q \ge 1$, is continuous about the parameter q for $f \in H(\mathbb{D}_R)$, $R > 1$.

Theorem 1.5. *Let $R > 1$ and $f \in H(\mathbb{D}_R)$. Then for any r, $0 < r < R$,*

$$\lim_{q \to 1+} L_q(f;z) = L_1(f;z) \tag{1.14}$$

uniformly on $\mathbb{D}_r$.

As an application of Theorem 1.3, we present the order of approximation for complex q-Kantorovich operators.

Theorem 1.6. *Let $1 < q < R$, $1 \le r < R/q^2$ (or $0 < q \le 1$, $1 \le r < R$) and $f \in H(\mathbb{D}_R)$. If f is not a constant function then the estimate*

$$\left\| K_{n,q}(f) - f \right\|_r \ge \frac{1}{[n+1]_q} C_{r,q}(f), \quad n \in \mathbb{N} \tag{1.15}$$

holds, where the constant $C_{r,q}(f)$ depends on f, q and r but is independent of n.

2. Auxiliary Results

Lemma 2.1. *Let $q>0$. For all $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $z\in\mathbb{C}$ one has*

$$K_{n,q}(e_m;z)=\sum_{j=0}^{m}\binom{m}{j}\frac{q^j[n]_q^j}{[n+1]_q^m(m-j+1)}B_{n,q}(e_j;z), \tag{2.1}$$

where $e_m(z)=z^m$.

Proof. The recurrence formula can be derived by direct computation.

$$\begin{aligned} K_{n,q}(e_m;z) &= \sum_{k=0}^{n}p_{n,k}(z)\sum_{j=0}^{m}\int_0^1\binom{m}{j}\frac{q^j[k]_q^j t^{m-j}}{[n+1]_q^m}dt=\sum_{k=0}^{n}p_{n,k}(z)\sum_{j=0}^{m}\binom{m}{j}\frac{q^j[k]_q^j}{[n+1]_q^m(m-j+1)}\\ &=\sum_{j=0}^{m}\binom{m}{j}\frac{q^j[n]_q^j}{[n+1]_q^m(m-j+1)}\sum_{k=0}^{n}\frac{[k]_q^j}{[n]_q^j}p_{n,k}(z)\\ &=\sum_{j=0}^{m}\binom{m}{j}\frac{q^j[n]_q^j}{[n+1]_q^m(m-j+1)}B_{n,q}(e_j;z). \end{aligned} \tag{2.2}$$

□

Lemma 2.2. *For all $z\in\mathbb{D}_r$, $r\geq 1$ one has*

$$\left|K_{n,q}(e_m;z)\right|\leq r^m,\quad n,m\in\mathbb{N}. \tag{2.3}$$

Proof. Indeed, using the inequality $|B_{n,q}(e_j;z)|\leq r^j$ (see [3]), we get

$$\begin{aligned} \left|K_{n,q}(e_m;z)\right| &\leq \sum_{j=0}^{m}\binom{m}{j}\frac{q^j[n]_q^j}{[n+1]_q^m(m-j+1)}\left|B_{n,q}(e_j;z)\right|\\ &\leq\frac{1}{[n+1]_q^m}\sum_{j=0}^{m}\binom{m}{j}q^j[n]_q^j r^m=\left(\frac{1+q[n]_q}{[n+1]_q}\right)^m r^m=r^m. \end{aligned} \tag{2.4}$$

□

Lemma 2.3. *For all n, $m\in\mathbb{N}$, $z\in\mathbb{C}$, $1\neq q>0$ one has*

$$\begin{aligned} K_{n,q}(e_{m+1};z) &= \frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_m;z)+zK_{n,q}(e_m;z)\\ &\quad+\frac{1}{[n+1]_q^{m+1}}\sum_{j=0}^{m+1}\binom{m+1}{j}q^j[n]_q^j\frac{1}{(m-j+2)}\left(1-\frac{j[n+1]_q}{q(m+1)[n]_q}\right)B_{n,q}(e_j;z). \end{aligned} \tag{2.5}$$

Proof. We know that (see [2])

$$\frac{z(1-z)}{[n]_q} D_q B_{n,q}(e_j; z) = B_{n,q}(e_{j+1}; z) - z B_{n,q}(e_j; z). \tag{2.6}$$

Taking the derivative of the formula (2.1) and using the above formula we have

$$\begin{aligned}
\frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m; z) &= \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \frac{z(1-z)}{[n]_q} D_q B_{n,q}(e_j; z) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \left(B_{n,q}(e_{j+1}; z) - z B_{n,q}(e_j; z)\right) \\
&= \sum_{j=1}^{m+1} \binom{m}{j-1} \frac{q^{j-1} [n]_q^{j-1}}{[n+1]_q^m (m-j+2)} B_{n,q}(e_j; z) - z K_{n,q}(e_m; z).
\end{aligned} \tag{2.7}$$

It follows that

$$\begin{aligned}
K_{n,q}(e_{m+1}; z) &= \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m; z) + z K_{n,q}(e_m; z) \\
&\quad + \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{q^j [n]_q^j}{[n+1]_q^{m+1} (m-j+2)} B_{n,q}(e_j; z) \\
&\quad - \sum_{j=1}^{m+1} \binom{m}{j-1} \frac{q^{j-1} [n]_q^{j-1}}{[n+1]_q^m (m-j+2)} B_{n,q}(e_j; z) \\
&= \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m; z) + z K_{n,q}(e_m; z) + \frac{1}{[n+1]_q^{m+1} (m+2)} \\
&\quad + \sum_{j=1}^{m+1} \binom{m+1}{j} \frac{q^{j-1} [n]_q^{j-1}}{[n+1]_q^m (m-j+2)} \frac{(m+1) q [n]_q - j [n+1]_q}{(m+1) [n+1]_q} B_{n,q}(e_j; z) \\
&= \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m; z) + z K_{n,q}(e_m; z) \\
&\quad + \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{q^{j-1} [n]_q^{j-1}}{[n+1]_q^m (m-j+2)} \frac{(m+1) q [n]_q - j [n+1]_q}{(m+1) [n+1]_q} B_{n,q}(e_j; z).
\end{aligned} \tag{2.8}$$

Here we used the identity

$$\binom{m}{j-1} = \binom{m+1}{j} \frac{j}{(m+1)}. \tag{2.9}$$

□

For $m \in \mathbb{N} \cup \{0\}$ define

$$E_{n,m}(z) := \begin{cases} K_{n,q}(e_m; z) - e_m(z) - \dfrac{1-2z}{2[n+1]_q} m z^{m-1} - \dfrac{z(1-z)}{2[n+1]_q} m(m-1) z^{m-2}, & \text{if } 0 < q \le 1, \\[2ex] K_{n,q}(e_m; z) - e_m(z) - \dfrac{1-2z}{2[n+1]_q} m z^{m-1} - \displaystyle\sum_{j=1}^{m-1} [j]_q \dfrac{q z^{m-1}(1-z)}{[n+1]_q}, & \text{if } q > 1. \end{cases} \tag{2.10}$$

Here it is assumed that $\sum_{j=1}^{0} [j]_q = 0$.

Lemma 2.4. *Let n, $m \in \mathbb{N}$.*

(a) *If $0 < q < 1$, one has the following recurrence formula:*

$$\begin{aligned} E_{n,m}(z) = {} & \frac{z(1-z)}{[n]_q} D_q\left(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)\right) + z E_{n,m-1}(z) \\ & + \left(\frac{m-1}{[n+1]_q} - \frac{[m-1]_q}{[n]_q} \right) z^{m-1}(1-z) - \frac{1-2z}{2[n+1]_q} z^{m-1} \\ & + \frac{1}{[n+1]_q^m} \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j; z). \end{aligned} \tag{2.11}$$

(b) *If $q > 1$, one has*

$$\begin{aligned} E_{n,m}(z) = {} & \frac{z(1-z)}{[n]_q} D_q\left(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)\right) + z E_{n,m-1}(z) + \frac{[m-1]_q}{[n]_q [n+1]_q} z^{m-1}(1-z) \\ & - \frac{1-2z}{2[n+1]_q} z^{m-1} + \frac{1}{[n+1]_q^m} \sum_{j=0}^{m} \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j; z). \end{aligned} \tag{2.12}$$

Proof. We give the proof for the case $q > 1$. The case $0 < q < 1$ is similar to that of $q > 1$.

(b) It is immediate that $E_{n,m}(z)$ is a polynomial of degree less than or equal to m and that $E_{n,0}(z) = E_{n,1}(z) = 0$.

Using the formula (2.5), we get

$$\begin{aligned}E_{n,m}(z) &= \frac{z(1-z)}{[n]_q}D_q\left(K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right)+\frac{[m-1]_q}{[n]_q}z^{m-1}(1-z)\\&\quad+z\left(E_{n,m-1}(z)+\frac{1-2z}{2[n+1]_q}(m-1)z^{m-2}+\sum_{j=1}^{m-2}[j]_q\frac{qz^{m-2}(1-z)}{[n+1]_q}\right)\\&\quad-\frac{1-2z}{2[n+1]_q}mz^{m-1}-\sum_{j=1}^{m-1}[j]_q\frac{qz^{m-1}(1-z)}{[n+1]_q}\\&\quad+\frac{1}{[n+1]_q^m}\sum_{j=0}^{m}\binom{m}{j}\frac{q^j[n]_q^j}{(m-j+1)}\frac{mq[n]_q-j[n+1]_q}{mq[n]_q}B_{n,q}(e_j;z).\end{aligned}\tag{2.13}$$

A simple calculation leads us to the following relationship:

$$\begin{aligned}E_{n,m}(z) &= \frac{z(1-z)}{[n]_q}D_q\left(K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right)+zE_{n,m-1}(z)+\frac{[m-1]_q}{[n]_q[n+1]_q}z^{m-1}(1-z)\\&\quad-\frac{1-2z}{2[n+1]_q}z^{m-1}+\frac{1}{[n+1]_q^m}\sum_{j=0}^{m}\binom{m}{j}\frac{q^j[n]_q^j}{(m-j+1)}\frac{mq[n]_q-j[n+1]_q}{mq[n]_q}B_{n,q}(e_j;z),\end{aligned}\tag{2.14}$$

which is the desired recurrence formula. □

Remark 2.5. Lemmas 2.3 and 2.4 are true in the case $q = 1$. In the formulae, we have to replace q-derivative by the ordinary derivative.

3. Proofs of the Main Results

We give proofs for the case $q > 1$. The case $0 < q < 1$ and $q = 1$ are similar to that of $q > 1$.

Proof of Theorem 1.1. The use of the above recurrence we obtain the following relationship:

$$\begin{aligned}K_{n,q}(e_m;z)-e_m(z) &= \frac{z(1-z)}{[n]_q}D_qK_{n,q}(e_{m-1};z)+z\left(K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right)\\&\quad+\frac{1}{[n+1]_q^m}\sum_{j=0}^{m}\binom{m}{j}\frac{q^j[n]_q^j}{(m-j+1)}\frac{mq[n]_q-j[n+1]_q}{mq[n]_q}B_{n,q}(e_j;z).\end{aligned}\tag{3.1}$$

We can easily estimate the sum in the above formula as follows:

$$\left|\frac{1}{[n+1]_q^m}\sum_{j=0}^{m}\binom{m}{j}q^j[n]_q^j\frac{1}{(m-j+1)}\left(1-\frac{j}{m}-\frac{j}{mq[n]_q}\right)B_{n,q}(e_j;z)\right|$$
$$\leq \frac{1}{[n+1]_q^m}\left(\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{m}{m-j}\frac{q^j[n]_q^j}{m-j+1}\left|1-\frac{j}{m}-\frac{j}{mq[n]_q}\right|\right)\left|B_{n,q}(e_j;z)\right|+\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^m}r^m$$
$$\leq \frac{2m\left(q[n]_q+1\right)^{m-1}+q^{m-1}[n]_q^{m-1}}{[n+1]_q^m}r^m \leq \frac{2m+1}{[n+1]_q}r^m. \tag{3.2}$$

It is known that by a linear transformation, the Bernstein inequality in the closed unit disk becomes

$$\left|P_m'(z)\right| \leq \frac{m}{qr}\|P_m\|_{qr}, \quad \forall |z| \leq qr,\ r \geq 1, \tag{3.3}$$

(where $\|P_m\|_{qr} = \max\{|P_m(z)| : |z| \leq qr\}$) which combined with the mean value theorem in complex analysis implies

$$\left|D_q(P_m;z)\right| = \left|\frac{P_m(qz)-P_m(z)}{qz-z}\right| \leq \left\|P_m'\right\|_{qr} \leq \frac{m}{qr}\|P_m\|_{qr}, \tag{3.4}$$

for all $|z| \leq r$, where $P_m(z)$ is a complex polynomial of degree $\leq m$. From the above recurrence formula (3.1), we get

$$\left|K_{n,q}(e_m;z)-e_m(z)\right| \leq \frac{|z||1-z|}{[n]_q}\left|D_qK_{n,q}(e_{m-1};z)\right| + |z|\left|K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right| + \frac{2m+1}{[n+1]_q}r^m$$
$$\leq \frac{r(1+r)}{[n]_q}\frac{m-1}{qr}\left\|K_{n,q}(e_{m-1})\right\|_{qr} + r\left|K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right| + \frac{2m+1}{[n+1]_q}r^m$$
$$\leq r\left|K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right| + \frac{2(m-1)}{[n]_q}q^{m-1}r^m + \frac{2m+1}{[n+1]_q}r^m$$
$$\leq r\left|K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right| + \frac{4m}{[n]_q}q^m r^m. \tag{3.5}$$

By writing the last inequality for $m = 1, 2, \ldots$, we easily obtain, step by step, the following:

$$|K_n(e_m;z) - e_m(z)| \le \frac{4m}{[n]_q} q^m r^m + r\frac{4(m-1)}{[n]_q} q^{m-1} r^{m-1} + r^2 \frac{4(m-2)}{[n]_q} q^{m-2} r^{m-2} + \cdots + r^{m-1}\frac{4}{[n]_q} qr$$
$$= \frac{4}{[n]_q} q^m r^m (m + m - 1 + \cdots + 1) \le \frac{2m(m+1)}{[n]_q} q^m r^m. \tag{3.6}$$

Since $K_{n,q}(f;z)$ is analytic in $\mathbb{D}_R$, we can write

$$K_{n,q}(f;z) = \sum_{m=0}^{\infty} a_m K_{n,q}(e_m;z), \quad z \in \mathbb{D}_R, \tag{3.7}$$

which together with (3.6) immediately implies for all $|z| \le r$

$$\left|K_{n,q}(f;z) - f(z)\right| \le \sum_{m=0}^{\infty} |a_m| \left|K_{n,q}(e_m;z) - e_m(z)\right| \le \frac{2}{[n]_q} \sum_{m=1}^{\infty} |c_m| m(m+1)(qr)^m. \tag{3.8}$$

□

Proof of Theorem 1.3. A simple calculation and the use of the recurrence formula (2.5) lead us to the following relationship:

$$\begin{aligned}
E_{n,m}(z) = {} & \frac{z(1-z)}{[n]_q} D_q\left(K_{n,q}(e_{m-1};z) - e_{m-1}(z)\right) + zE_{n,m-1}(z) + \frac{[m-1]_q}{[n]_q[n+1]_q} z^{m-1}(1-z) \\
& + \frac{1}{[n+1]_q}\left(z^m - B_{n,q}(e_m;z)\right) + \frac{1}{[n+1]_q}\left(1 - \frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}\right) B_{n,q}(e_m;z) \\
& + \frac{1}{2[n+1]_q}\left(\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}} - 1\right) B_{n,q}(e_{m-1};z) + \frac{1}{2[n+1]_q}\left(B_{n,q}(e_{m-1};z) - z^{m-1}\right) \\
& - \frac{(m-1)q^{m-2}[n]_q^{m-2}}{2[n+1]_q^m} B_{n,q}(e_{m-1};z) \\
& + \frac{1}{[n+1]_q^m} \sum_{j=0}^{m-2} \binom{m}{j} \frac{q^j[n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j;z) \\
:= {} & \sum_{k=1}^{9} I_k.
\end{aligned} \tag{3.9}$$

Firstly, we estimate I_3, I_8. It is clear that

$$\begin{aligned} |I_3| &\le \frac{[m-1]_q}{[n]_q[n+1]_q} r^{m-1}(1+r), \\ |I_8| &\le \frac{(m-1)}{2[n+1]_q^2}\left|B_{n,q}(e_{m-1};z)\right| \le \frac{(m-1)}{2[n+1]_q^2} r^{m-1}. \end{aligned} \tag{3.10}$$

Secondly, using the known inequality

$$1-\prod_{k=1}^{m} x_k \le \sum_{k=1}^{m}(1-x_k), \quad 0 \le x_k \le 1, \tag{3.11}$$

to estimate I_5, I_6, I_9.

$$\begin{aligned} |I_5| &\le \frac{1}{[n+1]_q}\left(1-\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}\right)\left|B_{n,q}(e_m;z)\right| \le \frac{m-1}{[n+1]_q^2} r^m, \\ |I_6| &\le \frac{1}{2[n+1]_q}\left(1-\frac{q^{m-1}[n]_q^{m-1}}{[n+1]_q^{m-1}}\right)\left|B_{n,q}(e_{m-1};z)\right| \le \frac{m-1}{2[n+1]_q^2} r^{m-1}, \\ |I_9| &\le \frac{1}{[n+1]_q^m}\sum_{j=0}^{m-2}\binom{m-2}{j}\frac{m(m-1)}{(m-j)(m-j-1)}\frac{q^j[n]_q^j}{(m-j+1)}\left(1-\frac{j}{m}-\frac{j}{mq[n]_q}\right) r^j \\ &\le \frac{2m(m-1)[n+1]_q^{m-2}}{[n+1]_q^m} r^m = \frac{2m(m-1)}{[n+1]_q^2} r^m. \end{aligned} \tag{3.12}$$

Finally, we estimate I_4, I_7. We use [2, Theorem 1.1.2]

$$\begin{aligned} |I_4|+|I_7| &\le \frac{1}{[n+1]_q}\left|z^m - B_{n,q}(e_m;z)\right| + \frac{1}{2[n+1]_q}\left|B_n(e_{m-1};z)-z^{m-1}\right| \\ &\le \frac{2[m-1]_q(m-1)}{[n]_q[n+1]_q} r^m + \frac{[m-2]_q(m-2)}{[n]_q[n+1]_q} r^{m-1}. \end{aligned} \tag{3.13}$$

Using (3.6), (3.10), (3.12), and (3.13) in (3.9) finally we have ($m \geq 3$)

$$\begin{aligned}
|E_{n,m}(z)| \leq{}& \frac{r(1+r)}{[n]_q}\left|D_q\left(K_{n,q}(e_{m-1};z)-e_{m-1}(z)\right)\right| + r|E_{n,m-1}(z)| + \frac{[m-1]_q}{[n]_q[n+1]_q}r^{m-1}(1+r) \\
&+ \frac{2[m-1]_q(m-1)}{[n]_q[n+1]_q}r^m + \frac{m-1}{[n+1]_q^2}r^m + \frac{m-1}{2[n+1]_q^2}r^{m-1} + \frac{[m-2]_q(m-2)}{[n]_q[n+1]_q}r^{m-1} \\
&+ \frac{(m-1)}{2[n+1]_q^2}r^{m-1} + \frac{2m(m-1)}{[n+1]_q^2}r^m \\
\leq{}& \frac{r(1+r)}{[n]_q}\frac{m-1}{qr}\left\|K_{n,q}(e_{m-1})-e_{m-1}\right\|_{qr} + r|E_{n,m-1}(z)| + \frac{10m[m-1]_q}{[n]_q^2}r^m \\
\leq{}& \frac{(m-1)(1+r)}{[n]_q}\frac{2(m-1)m}{[n]_q}q^{2(m-1)}r^{m-1} + r|E_{n,m-1}(z)| + \frac{10m[m-1]_q}{[n]_q^2}r^m \\
\leq{}& r|E_{n,m-1}(z)| + \frac{4m(m-1)^2}{[n]_q^2}q^{2m}r^m + \frac{10m(m-1)}{[n]_q^2}q^m r^m \\
\leq{}& r|E_{n,m-1}(z)| + \frac{14m^2(m-1)^2}{[n]_q^2}q^{2m}r^m.
\end{aligned} \tag{3.14}$$

As a consequence, we get

$$|E_{n,m}(z)| \leq \frac{14m^2(m-1)^2}{[n]_q^2}q^{2m}r^m. \tag{3.15}$$

This inequality combined with

$$\left|K_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q}L_q(f,z)\right| \leq \sum_{m=1}^{\infty}|a_m||E_{n,m}(z)| \tag{3.16}$$

immediately implies the required estimate in statement.

Note that since $f^{(4)} = \sum_{m=4}^{\infty} a_m m(m-1)(m-2)(m-3)z^{m-4}$ and the series is absolutely convergent for all $|z| < R$, it easily follows the finiteness of the involved constants in the statement. □

Proof of Theorem 1.6. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$, we get

$$K_{n,q}(f;z) - f(z) = \frac{1}{[n+1]_q}\left\{L_q(f;z) + [n+1]_q\left(K_{n,q}(f;z) - f(z) - \frac{1}{[n+1]_q}L_q(f;z)\right)\right\}. \tag{3.17}$$

We apply

$$\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r \tag{3.18}$$

to get

$$\|K_{n,q}(f)-f\|_r \geq \frac{1}{[n+1]_q}\left\{\|L_q(f;z)\|_r - [n+1]_q\left\|K_{n,q}(f;z)-f(z)-\frac{1}{[n+1]_q}L_q(f;z)\right\|_r\right\}. \tag{3.19}$$

Because by hypothesis f is not a constant in $\mathbb{D}_R$, it follows $\|L_q(f;z)\|_r > 0$. Indeed, assuming the contrary, it follows that $L_q(f;z) = 0$ for all $z \in \overline{\mathbb{D}}_R$ that is

$$\begin{gathered}\sum_{m=1}^{\infty} a_m\left(\frac{1}{2}-z\right)mz^{m-1} + \sum_{m=1}^{\infty} a_m \sum_{j=1}^{m-1}[j]_q z^{m-1}(1-z) = 0,\\ \frac{1}{2}a_1 + a_1 + \sum_{m=1}^{\infty}\left(\frac{1}{2}(m+1)a_{m+1} - a_m + a_{m+1}\sum_{j=1}^{m}[j]_q - a_m\sum_{j=1}^{m-1}[j]_q\right)z^m = 0\end{gathered} \tag{3.20}$$

for all $z \in \overline{\mathbb{D}}_R \setminus \{0\}$. Thus $a_m = 0$, $m = 1,2,3,\ldots$. Thus, f is constant, which is contradiction with the hypothesis.

Now, by Theorem 1.3, we have

$$\begin{aligned}&[n+1]_q\left|K_{n,q}(f;z)-f(z)-\frac{1}{[n+1]_q}L_q(f;z)\right|\\ &\quad\leq \frac{[n+1]_q}{[n]_q}\frac{14}{[n]_q}\sum_{m=2}^{\infty}|a_m|m^2(m-1)^2q^{2m}r^m \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.21}$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n \geq n_1$ we have

$$\|L_q(f;z)\|_r - [n+1]_q\left\|K_{n,q}(f;z)-f(z)-\frac{1}{[n+1]_q}L_q(f;z)\right\|_r \geq \frac{1}{2}\|L_q(f;z)\|_r, \tag{3.22}$$

which implies

$$\|K_{n,q}(f)-f\|_r \geq \frac{1}{[n+1]_q}\frac{1}{2}\|L_q(f;z)\|_r, \quad \forall n \geq n_1. \tag{3.23}$$

For $1 \leq n \leq n_1 - 1$, we have

$$\|K_{n,q}(f)-f\|_r \geq \frac{1}{[n+1]_q}\left([n+1]_q\|K_{n,q}(f)-f\|_r\right) = \frac{1}{[n+1]_q}M_{r,n}(f) > 0, \tag{3.24}$$

which finally implies that

$$\|K_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f), \tag{3.25}$$

for all n, with $C_{r,q}(f) = \min\{M_{r,1}(f), \ldots, M_{r,n_1-1}(f), (1/2)\|L_q(f;z)\|_r\}$. □

Proof of Theorem 1.5. Proof is similar to that of Theorem 1.3 [5]. □

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