Research Article

Almost Periodic Solutions to Dynamic Equations on Time Scales and Applications

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We first introduce the concept of admitting an exponential dichotomy to a class of linear dynamic equations on time scales and study the existence and uniqueness of almost periodic solution and its expression form to this class of linear dynamic equations on time scales. Then, as an application, using these concepts and results, we establish sufficient conditions for the existence and exponential stability of almost periodic solution to a class of Hopfield neural networks with delays. Finally, two examples and numerical simulations given to illustrate our results are plausible and meaningful.

1. Introduction

In recent years, researches in many fields on time scales have received much attention. The theory of calculus on time scales (see [1, 2] and references cited therein) was initiated by Hilger in his Ph.D. thesis in 1988 [3] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has recently received much attention since his fundamental work. It has been created in order to unify the study of differential and difference equations. Also, the existence of almost periodic, asymptotically almost periodic, pseudo-almost periodic solutions is among the most attractive topics in qualitative theory of differential equations and difference equations due to their applications, especially in biology, economics, and physics [4–20].

Motivated by the above, based on the theory of almost periodic functions on time scales in our previous work [21, 22], we first introduce the concept of admitting an exponential dichotomy to a class of linear dynamic equations on time scales and study the existence and uniqueness of almost periodic solution and its expression form to this class of linear dynamic equations on time scales. Then, as an application, using these concepts,

results, the fixed point theorem and differential inequality techniques, we establish sufficient conditions for the existence and exponential stability of almost periodic solution to a class of Hopfield neural networks with delays. Finally, two examples given to illustrate our results are plausible and meaningful to unify continuous and discrete models.

The organization of this paper is as follows. In Section 2, we introduce some notations and state some preliminary results needed in the later sections. In Section 3, we introduce the concepts of admitting an exponential dichotomy to a class of linear dynamic equations on time scales under which the existence, uniqueness, and expression form of an almost periodic solution are obtained. Furthermore, some fundamental conditions of admitting an exponential dichotomy to linear dynamic equations are also derived. In Section 4, as an application of our results, we study the existence and exponential stability of the almost periodic solutions of a class of Hopfield neural networks with delays, finally, we give two examples and numerical simulations to show that our unification of continuous and discrete situations is effective.

2. Preliminaries

In this section, we will first recall some basic definitions, lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t. \tag{2.1}$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that $|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$ for all $s \in U$.

Let y be right-dense continuous, if $Y^{\Delta}(t) = y(t)$, then we define the delta integral by $\int_a^t y(s)\Delta s = Y(t) - Y(a)$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{ for all } t \in \mathbb{T}\}$.

An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called regressive provided that $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$ and the class of all such regressive and rd-continuous functions is denoted, similar to the above scalar case, by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$.

If r is a regressive function, then the generalized exponential function e_r is defined by $e_r(t,s) = \exp\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\}$ for all $s,t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\text{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases} \tag{2.2}$$

Definition 2.1 (see [1, 2]). Let $p,q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu pq, \qquad \ominus p = -\frac{p}{1+\mu p}, \qquad p \ominus q = p \oplus (\ominus q). \tag{2.3}$$

If A is a matrix, then we let A^* denote its conjugate transpose.

Lemma 2.2 (see [1, 2]). *If $A, B \in \mathcal{R}$ are matrix-value functions on $\mathbb{T}$, then*

    (i) $e_0(t,s) \equiv I$ and $e_A(t,t) \equiv I$,

    (ii) $e_A(\sigma(t),s) = (I + \mu(t)A(t))e_A(t,s)$,

    (iii) $e_A^{-1}(t,s) = e_{\ominus A^*}^*(t,s)$,

    (iv) $e_A(t,s) = e_A^{-1}(s,t) = e_{\ominus A^*}^*(s,t)$,

    (v) $e_A(t,s)e_A(s,r) = e_A(t,r)$,

    (vi) $e_A(t,s)e_B(t,s) = e_{A \oplus B}(t,s)$ if $e_A(t,s)$ and $B(t)$ commute.

Lemma 2.3 (see [1, 2]). *If $A \in \mathcal{R}$ and $a,b,c \in \mathbb{T}$, then $[e_A(c,\cdot)]^\Delta = -[e_A(c,\cdot)]^\sigma A$ and*

$$\int_a^b e_A(c,\sigma(t))A(t)\Delta t = e_A(c,a) - e_A(c,b). \tag{2.4}$$

Lemma 2.4 (see [1, 2]). *Let $A \in \mathcal{R}$ be an $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-contibuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^\Delta = -A^*(t)y^\sigma + f(t), \quad y(t_0) = y_0 \tag{2.5}$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$. Moreover, this solution is given by $y(t) = e_{\ominus A^*}(t,t_0)y_0 + \int_{t_0}^t e_{\ominus A^*}(t,\tau)f(\tau)\Delta\tau$.

For convenience, $\mathbb{E}^n$ denotes $\mathbb{R}^n$ or $\mathbb{C}^n$, $AP(\mathbb{T})$ denote the set of all almost periodic $n \times m$-matrix-valued functions on $\mathbb{T}$.

Definition 2.5 (see [22]). A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \ \forall t \in \mathbb{T}\} \neq \{0\}. \tag{2.6}$$

Definition 2.6 (see [22]). Let $\mathbb{T}$ be an almost periodic time scale. A continuous $n \times m$-matrix-valued function f on $\mathbb{T}$ is called almost periodic on $\mathbb{T}$ if the ε-translation set of f

$$E\{\varepsilon, f\} = \{\tau \in \Pi : \|f(t + \tau) - f(t)\| < \varepsilon, \forall t \in \mathbb{T}\} \tag{2.7}$$

is relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau(\varepsilon) \in E\{\varepsilon, f\}$ such that

$$\|f(t + \tau) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{T}. \tag{2.8}$$

τ is called the ε-translation number of f and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$, where $\|\cdot\|$ is a matrix norm on $\mathbb{T}$, (say, e.g., if $A = (a_{ij}(t))_{n \times m}$, then we can take e.g., $\|A\| = \sup_{t \in \mathbb{T}} (\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}(t)|^2)^{1/2})$.

For convenience, let $\alpha = \{\alpha_n\}$, $\beta = \{\beta_n\}$ be two sequences. Then $\beta \subset \alpha$ means that β is a subsequence of α; $\alpha + \beta = \{\alpha_n + \beta_n\}$; $-\alpha = \{-\alpha_n\}$; and α and β are common subsequences of α' and β', respectively, means that $\alpha_n = \alpha'_{n(k)}$ and $\beta_n = \beta'_{n(k)}$ for some given function $n(k)$.

Let f, g be $n \times m$-matrix-valued functions on $\mathbb{T}$, we will introduce the translation operator T, $T_\alpha f = g$ means that $g(t) = \lim_{n \to +\infty} f(t + \alpha_n)$ and is written only when the limit exists.

Definition 2.7. Let f be an $n \times m$-matrix-valued function on $\mathbb{T}$, $H(f) = \{g$: there exits $\alpha \subset \Pi$ such that $T_\alpha f = g$ exists uniformly on $\mathbb{T}\}$ is called the hull of f.

Similar to the proof of Theorem 3.14 in [22], one can easily get a more general version of the following.

Theorem 2.8. *Let f be an $n \times m$-matrix-valued function on $\mathbb{T}$, if for any $\alpha' \subset \Pi$, there exists $\alpha \subset \alpha'$ such that $T_\alpha f(t)$ exists uniformly on $\mathbb{T}$, then $f(t)$ is almost periodic.*

Also, from Theorem 3.30 in [22], one can easily show a more general version of the following.

Lemma 2.9. *An $n \times m$-matrix-valued function f is almost periodic on $\mathbb{T}$, if and only if for every pair of sequences $\alpha', \beta' \subseteq \Pi$, there exist common subsequences $\alpha \subset \alpha', \beta \subset \beta'$ such that*

$$T_{\alpha+\beta} f(t) = T_\alpha T_\beta f(t). \tag{2.9}$$

3. Almost Periodic Dynamic Equations on Time Scales

Consider the linear almost periodic equation

$$x^\Delta = -A^*(t)x^\sigma + f(t) \tag{3.1}$$

and its associated homogeneous equation

$$x^\Delta = -A^*(t)x^\sigma, \tag{3.2}$$

where $A(t)$ is an almost periodic matrix function and $f(t)$ is an almost periodic vector function.

Definition 3.1. If $B \in H(A^*)$, we say that

$$y^\Delta = -B(t)y^\sigma \tag{3.3}$$

is a homogeneous equation in the hull of (3.1).

Definition 3.2. If $B \in H(A^*)$, and $g \in H(f)$, we say that

$$y^\Delta = -B(t)y^\sigma + g(t) \tag{3.4}$$

is an equation in hull of (3.1).

Definition 3.3. Let $A(t)$ be $n \times n$ rd-continuous matrix function on $\mathbb{T}$, the linear equation

$$x^\Delta = -A^*(t)x^\sigma \tag{3.5}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants K, α, projection P and the fundamental solution matrix $X(t)$ of (3.5), satisfying

$$\left| X(t)PX^{-1}(s) \right| \leq Ke_{\ominus\alpha}(t,\sigma(s)), \quad s,t \in \mathbb{T}, t \geq \sigma(s),$$
$$\left| X(t)(I-P)X^{-1}(s) \right| \leq Ke_{\ominus\alpha}(\sigma(s),t), \quad s,t \in \mathbb{T}, t \leq \sigma(s). \tag{3.6}$$

From Theorem 2.8 and Lemma 2.9 for $\beta = -\alpha$ and Lemma 2.4, one can also easily get the following Favard theorem.

Lemma 3.4. *If $A(t)$ is almost periodic matrix function and $x(t)$ is an almost periodic solution to the homogeneous linear almost periodic equation $x^\Delta = -A^*(t)x^\sigma$, then $\inf_{t\in\mathbb{T}}|x(t)| > 0$ or $x(t) \equiv 0$.*

Similar to the proof of Lemma 4.16 in [22], one can easily show the following lemma.

Lemma 3.5. *Suppose that $A(t)$ is an almost periodic matrix function and (3.2) satisfies an exponential dichotomy (3.6), then for every $B(t) \in H(A^*)$, (3.3) satisfies an exponential dichotomy with same projection P and same constants K, α.*

Similar to the proof of Lemma 4.17 in [22], one can easily show the following:

Lemma 3.6. *If the homogeneous equation (3.2) satisfies an exponential dichotomy (3.6), then (3.2) has only one bounded solution $x(t) \equiv 0$.*

Lemma 3.7. *If the homogeneous equation (3.2) satisfies an exponential dichotomy (3.6), then all equations in the hull of (3.2) have only one bounded solution $x(t) \equiv 0$.*

Proof. By Lemma 3.5, all equations in the hull of (3.2) satisfy an exponential dichotomy (3.6), according to Lemma 3.6, all equations in the hull of (3.2) have only one bounded solution $x(t) \equiv 0$. This completes the proof. $\qquad\square$

By Lemmas 3.4 and 3.7, one can easily have the existence and uniqueness theorem for an almost periodic solution to (3.1):

Theorem 3.8. *Let $A(t)$ be an almost periodic matrix function, $f(t)$ is an almost periodic vector function. If (3.2) admits an exponential dichotomy, then (3.1) has a unique almost periodic solution*

$$x(t) = \int_{-\infty}^{t} X(t) P X^{-1}(s) f(s) \Delta s - \int_{t}^{+\infty} X(t)(I - P) X^{-1}(s) f(s) \Delta s, \tag{3.7}$$

where $X(t)$ is the fundamental solution matrix of (3.2).

Similar to the proof of Lemma 2.15 in [21], one can easily show the following.

Lemma 3.9. *Let $c_i(t)$ be an almost periodic function on $\mathbb{T}$, where $c_i(t) > 0$, $-c_i(t) \in \mathcal{R}^+$, $t \in \mathbb{T}$, $i = 1, 2, \ldots, n$ and $\min_{1 \leq i \leq n}\{\inf_{t \in \mathbb{T}} c_i(t)\} = \tilde{m} > 0$, then the linear equation*

$$x^{\Delta} = \mathrm{diag}(-c_1(t), -c_2(t), \ldots, -c_n(t)) x^{\sigma} \tag{3.8}$$

admits an exponential dichotomy on $\mathbb{T}$.

4. Applications

In the real world, both continuous and discrete systems are very important in implementation and applications. Therefore, it is meaningful to study almost periodic problems on time scales which can unify the continuous and discrete situations.

In this section, we consider the following model for the delayed Hopfield neutral networks (HNNs):

$$x_i^{\Delta} = -c_i(t) x_i^{\sigma} + \sum_{j=1}^{n} b_{ij}(t) g_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) + I_i(t), \quad i = 1, 2, \ldots, n, \tag{4.1}$$

in which n is the number of units in a neural network, $x_i(t)$ is the state vector of the ith unit at the time t, $c_i(t)$ represents the rate at time t with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $g_j(x_j)$ denotes the conversion of the membrane potential of the jth unit into its firing rate, $b_{ij}(t)$ denotes the strength of the jth unit on the ith unit at time $t - \tau_{ij}(t)$, $\tau_{ij}(t) \geq 0$ corresponds to the transmission delay of the ith unit along the axon of the jth unit at time t, and $I_i(t)$ denotes the external bias on the ith unit at time t.

It is well known that the HNNs have been successfully applied to signal and image processing, pattern recognition, and optimization. Hence, they have been the object of intensive analysis by numerous authors in recent years. In particular, there have been extensive results on the problem of the existence and stability of periodic solutions of system (4.1) in the literature. We refer the reader to [23–30] and the references cited therein. In order to unify continuous and discrete situations, by using results in Sections 3 and 4, one can discuss almost periodic problems on time scales.

The main purpose of this section is to give the conditions for the existence and exponential stability of the almost periodic solutions for system (4.1). By applying fixed point theorem and differential inequality techniques, we derive some new sufficient conditions ensuring the existence, uniqueness, and exponential stability of the almost periodic solution on time scales. Our results are new even if the time scale $\mathbb{T} = h\mathbb{Z}$ and other types of almost periodic time scales such as $\cup_{k=0}^{\infty}[2k, 2k+1]$, $k \in \mathbb{Z}$. Numerical examples and simulations are given to illustrate our feasible results and effectiveness of our methods.

For (4.1), we assume that $t - \tau_{ij}(t) \in \mathbb{T}$ for $t \in \mathbb{T}$ and $c_i > 0$, $I_i, b_{ij}, \tau_{ij} : \mathbb{T} \to \mathbb{R}$ are almost periodic functions, where c_i and $\tau \geq 0$ are constants, $i, j = 1, 2, \ldots, n$, and we use the following notation:

$$\inf_{t \in \mathbb{T}} c_i(t) := \underline{c_i}, \qquad \sup_{t \in \mathbb{T}}|b_{ij}(t)| := \overline{b}_{ij}, \qquad \sup_{t \in \mathbb{T}}|I_i(t)| := \overline{I}_i,$$

$$\tau =: \max_{1 \leq i,j \leq n}\left\{\sup_{t \in \mathbb{T}}\tau_{ij}(t)\right\}, \qquad i, j = 1, 2, \ldots, n. \tag{4.2}$$

We also assume that the following condition (H_0) holds.

(H_0) for each $j \in \{1, 2, \ldots, n\}$, $g_j : \mathbb{R} \to \mathbb{R}$ is Lipschitz with Lipschitz constant L_j, that is,

$$|g_j(u_j) - g_j(v_j)| \leq L_j|u_j - v_j|, \quad \forall u_j, v_j \in \mathbb{R}. \tag{4.3}$$

For convenience, we will use $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ to denote a column vector, in which the symbol $(\cdot)^T$ denotes the transpose of a vector. We let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$, and define $\|x\| = \max_{1 \leq i \leq n}|x_i|$. For matrix $A = (a_{ij})_{n \times n}$, A^T denotes the transpose of A, A^{-1} denotes the inverse of A, $|A|$ denotes the absolute-value matrix given by $|A| = (|a_{ij}|)_{n \times n}$, and $\varrho(A)$ denotes the spectral radius of A. A matrix or vector $A \geq 0$ means that all entries of A are greater than or equal to zero. $A > 0$ is defined similarly. For matrices or vectors A and B, $A \geq B$ (resp. $A > B$) means that $A - B \geq 0$ (resp. $A - B > 0$). Let

$$D = \mathrm{diag}\left(\underline{c_1}, \underline{c_2}, \ldots, \underline{c_n}\right), \qquad \overline{E} = \left(\overline{b}_{ij}\right)_{n \times n}, \qquad L = \mathrm{diag}(L_1, L_2, \ldots, L_n). \tag{4.4}$$

Definition 4.1 (see [31]). For $A \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$, $t \in \mathbb{T}$ and $F \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$, $V \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+)$, we call $\Delta^r V(t, A(t))$ and $\Delta_r V(t, A(t))$ the right upper and right lower derivatives of the function V at $(t, A(t))$, respectively, if

$$
\Delta^r V(t, A(t)) = \begin{cases} \dfrac{V(\sigma(t), A(\sigma(t))) - V(t, A(t))}{\mu(t)}, & \sigma(t) > t, \\[4mm] \limsup_{s \to t^+} \dfrac{V(s, A(t) + (s-t)F(t, A(t))) - V(t, A(t))}{s - t}, & \sigma(t) = t, \end{cases}
$$

$$
\Delta_r V(t, A(t)) = \begin{cases} \dfrac{V(\sigma(t), A(\sigma(t))) - V(t, A(t))}{\mu(t)}, & \sigma(t) > t, \\[4mm] \liminf_{s \to t^+} \dfrac{V(s, A(t) + (s-t)F(t, A(t))) - V(t, A(t))}{s - t}, & \sigma(t) = t. \end{cases}
\tag{4.5}
$$

Lemma 4.2. *Let $f \in C(\mathbb{T}, \mathbb{R})$ is Δ-differentiable at t. Then*

$$
\Delta^r |f(t)| \le \operatorname{sign}(f^\sigma(t)) f^\Delta(t), \quad \text{where } f^\sigma(t) = f(\sigma(t)).
\tag{4.6}
$$

Proof. Case (i). If t is a right dense point, that is, $\sigma(t) = t$.

$$
\Delta^r |f(t)| \le \operatorname{sign}(f(t)) f^\Delta(t) = \operatorname{sign}(f^\sigma(t)) f^\Delta(t).
\tag{4.7}
$$

Case (ii). If t is a right scattered point, that is, $\sigma(t) > t$. If $f(t) f^\sigma(t) > 0$, one can easily have $\operatorname{sign}(f(t)) = \operatorname{sign}(f^\sigma(t))$, so we can obtain

$$
\begin{aligned}
\Delta^r |f(t)| &= \frac{|f^\sigma(t)| - |f(t)|}{\mu(t)} = \frac{\operatorname{sign}(f^\sigma(t)) f^\sigma(t) - \operatorname{sign}(f(t)) f(t)}{\mu(t)} \\[2mm]
&= \operatorname{sign}(f^\sigma(t)) \left(\frac{f^\sigma(t) - f(t)}{\mu(t)} \right) = \operatorname{sign}(f^\sigma(t)) f^\Delta(t).
\end{aligned}
\tag{4.8}
$$

If $f(t) f^\sigma(t) \le 0$, then one can get $|f(t)| \ge \operatorname{sign}(f^\sigma(t)) f(t)$. Then

$$
\begin{aligned}
\Delta^r |f(t)| &= \frac{|f^\sigma(t)| - |f(t)|}{\mu(t)} = \frac{\operatorname{sign}(f^\sigma(t)) f^\sigma(t) - |f(t)|}{\mu(t)} \\[2mm]
&\le \frac{\operatorname{sign}(f^\sigma(t)) f^\sigma(t) - \operatorname{sign}(f^\sigma(t)) f(t)}{\mu(t)} \\[2mm]
&= \operatorname{sign}(f^\sigma(t)) \left(\frac{f^\sigma(t) - f(t)}{\mu(t)} \right) = \operatorname{sign}(f^\sigma(t)) f^\Delta(t).
\end{aligned}
\tag{4.9}
$$

Therefore, by (4.7), (4.8), (4.9), one can get

$$\Delta^r |f(t)| \leq \text{sign}(f^\sigma(t)) f^\Delta(t). \tag{4.10}$$

This completes the proof. $\qquad\square$

As usual, we introduce the phase space $C([t_0 - \tau, t_0] \cap \mathbb{T}, \mathbb{R}^n)$ as a Banach space of continuous mappings from $[t_0 - \tau, t_0] \cap \mathbb{T}$ to $\mathbb{R}^n$ equipped with the supremum norm defined by $\|\varphi\| = \max_{1 \leq i \leq n} \sup_{t \in [t_0 - \tau, t_0] \cap \mathbb{T}} |\varphi_i(t)|$ for all $\varphi = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T \in C([t_0 - \tau, t_0] \cap \mathbb{T}, \mathbb{R}^n)$.

The initial conditions associated with system (4.1) are of the following form:

$$x_i(s) = \varphi_i(s), \quad s \in [t_0 - \tau, t_0] \cap \mathbb{T}, \quad i = 1, 2, \ldots, n, \tag{4.11}$$

where $\varphi = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T \in C([t_0 - \tau, t_0] \cap \mathbb{T}, \mathbb{R}^n)$, $t_0 \in \mathbb{T}$.

Definition 4.3. The almost periodic solution $z^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ of (4.1) is said to be globally exponentially stable, if there exists a positive α such that for any $\delta \in [t_0 - \tau_0, t_0] \cap \mathbb{T}$, there exists $N = N(\delta) \geq 1$ such that for any solution $z = (x_1, x_2, \ldots, x_n)^T$ of (4.1), it is valid that

$$\|z - z^*\| \leq N \|z_0 - z^*\| e_{\ominus \alpha}(t, \delta), \quad t \in [t_0, +\infty) \cap \mathbb{T}, \tag{4.12}$$

where $z_0(s), s \in [t_0 - \tau_0, t_0] \cap \mathbb{T}$ is the initial condition.

Definition 4.4 (see [30]). A real $n \times n$ matrix $K = (k_{ij})_{n \times n}$ is said to be an M-matrix if $k_{ij} \leq 0$, $i, j = 1, 2, \ldots, n, i \neq j$, and $K^{-1} \geq 0$.

Lemma 4.5 (see [30]). *Let $A \geq 0$ be an $n \times n$ matrix and $\varrho(A) < 1$, then $(E_n - A)^{-1} \geq 0$, where E_n denotes the identity matrix of size n.*

In the following, we will show the existence and uniqueness of almost periodic solution to (4.1) on time scales.

Theorem 4.6. *Let (H_0) hold and $\varrho(D^{-1}\overline{E}L) < 1$. Then, there exists exactly one almost periodic solution of system (4.1).*

Proof. Let $X = \{\phi \mid \phi = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^T$, where $\phi_i : \mathbb{T} \to \mathbb{R}$ is an almost periodic function, $i = 1, 2, \ldots, n\}$. Then X is a Banach space with the norm defined by $\|\phi\|_X = \sup_{t \in \mathbb{T}} \max_{1 \leq i \leq n} |\phi_i(t)|$.

To proceed further, we need to introduce an auxiliary equation

$$x_i^\Delta = -c_i(t) x_i^\sigma + \sum_{j=1}^n b_{ij}(t) g_j(\phi_j(t - \tau_{ij}(t))) + I_i(t), \quad i = 1, 2, \ldots, n, \tag{4.13}$$

where $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^T \in X$. Notice that $\tau_{ij}(t), b_{ij}(t)$ and $I_i(t)$ are almost periodic functions, by Lemma 3.9 and Theorem 3.8, we know that the auxiliary (4.13) has exactly one almost periodic solution

$$
\begin{aligned}
x^{\phi}(t) &= \left(x_1^{\phi}(t), x_2^{\phi}(t), \ldots, x_n^{\phi}(t) \right)^T \\
&= \left(\int_{-\infty}^{t} e_{-c_1}(t, s) \left[\sum_{j=1}^{n} b_{1j}(s) g_j(\phi_j(t - \tau_{1j}(s))) + I_1(s) \right] \Delta s, \ldots, \right. \\
&\qquad \left. \int_{-\infty}^{t} e_{-c_n}(t, s) \left[\sum_{j=1}^{n} b_{nj}(s) g_j(\phi_j(t - \tau_{nj}(s))) + I_n(s) \right] \Delta s \right)^T .
\end{aligned}
\tag{4.14}
$$

Define a mapping $\Phi : X \to X$ by setting $\Phi(\phi(t)) = x^{\phi}(t)$ for all X. Let $\phi, \psi \in X$, then by (H_0), we have

$$
\begin{aligned}
&|\Phi(\phi(t)) - \Phi(\psi(t))| \\
&= \left(|(\Phi(\phi(t)) - \Phi(\psi(t)))_1|, |(\Phi(\phi(t)) - \Phi(\psi(t)))_2|, \ldots, |(\Phi(\phi(t)) - \Phi(\psi(t)))_n| \right)^T \\
&= \left(\left| \int_{-\infty}^{t} e_{-c_1}(t, s) \left[\sum_{j=1}^{n} b_{1j}(s) (g_j(\phi_j(t - \tau_{1j}(s))) - g_j(\psi_j(t - \tau_{1j}(s)))) \right] \Delta s \right|, \ldots, \right. \\
&\qquad \left. \left| \int_{-\infty}^{t} e_{-c_n}(t, s) \left[\sum_{j=1}^{n} b_{nj}(s) (g_j(\phi_j(t - \tau_{nj}(s))) - g_j(\psi_j(t - \tau_{nj}(s)))) \right] \Delta s \right| \right)^T \\
&\leq \left(\int_{-\infty}^{t} e_{-\underline{c_1}}(t, \sigma(s)) \left[\sum_{j=1}^{n} \overline{b}_{1j} L_j |\phi_j(s - \tau_{1j}(s)) - \psi_j(s - \tau_{1j}(s))| \right] \Delta s, \ldots, \right. \\
&\qquad \left. \int_{-\infty}^{t} e_{-\underline{c_n}}(t, \sigma(s)) \left[\sum_{j=1}^{n} \overline{b}_{nj} L_j |\phi_j(s - \tau_{nj}(s)) - \psi_j(s - \tau_{nj}(s))| \right] \Delta s \right)^T \\
&\leq \left(\sum_{j=1}^{n} \underline{c_1}^{-1} \overline{b}_{1j} L_j \sup_{t \in \mathbb{T}} |\phi_j(t) - \psi_j(t)|, \ldots, \sum_{j=1}^{n} \underline{c_n}^{-1} \overline{b}_{nj} L_j \sup_{t \in \mathbb{T}} |\phi_j(t) - \psi_j(t)| \right)^T ,
\end{aligned}
\tag{4.15}
$$

which implies that

$$
\left(\sup_{t \in \mathbb{T}} \left| \left(\Phi(\phi(t)) - \Phi(\psi(t)) \right)_1 \right|, \sup_{t \in \mathbb{T}} \left| \left(\Phi(\phi(t)) - \Phi(\psi(t)) \right)_2 \right|, \ldots, \sup_{t \in \mathbb{T}} \left| \left(\Phi(\phi(t)) - \Phi(\psi(t)) \right)_n \right| \right)^T
$$
$$
\leq \left(\sum_{j=1}^{n} \underline{c_1}^{-1} \overline{b}_{1j} L_j \sup_{t \in \mathbb{T}} \left| \phi_j(t) - \psi_j(t) \right|, \ldots, \sum_{j=1}^{n} \underline{c_n}^{-1} \overline{b}_{nj} L_j \sup_{t \in \mathbb{T}} \left| \phi_j(t) - \psi_j(t) \right| \right)^T
$$
$$
\leq F \left(\sup_{t \in \mathbb{T}} \left| \phi_1(t) - \psi_1(t) \right|, \ldots, \sup_{t \in \mathbb{T}} \left| \phi_n(t) - \psi_n(t) \right| \right)^T,
$$

(4.16)

where $F = D^{-1} \overline{E} L$. Let m be a positive integer. Then, from (4.16), we get

$$
\left(\sup_{t \in \mathbb{T}} \left| \left(\Phi^m(\phi(t)) - \Phi^m(\psi(t)) \right)_1 \right|, \ldots, \sup_{t \in \mathbb{T}} \left| \left(\Phi^m(\phi(t)) - \Phi^m(\psi(t)) \right)_n \right| \right)^T
$$
$$
= \left(\sup_{t \in \mathbb{T}} \left| \left(\Phi\left(\Phi^{m-1}(\phi(t)) \right) - \Phi\left(\Phi^{m-1}(\psi(t)) \right) \right)_1 \right|, \ldots, \right.
$$
$$
\left. \sup_{t \in \mathbb{T}} \left| \left(\Phi\left(\Phi^{m-1}(\phi(t)) \right) - \Phi\left(\Phi^{m-1}(\psi(t)) \right) \right)_n \right| \right)^T
$$
$$
\leq F \left(\sup_{t \in \mathbb{T}} \left| \left(\Phi^{m-1}(\phi(t)) - \Phi^{m-1}(\psi(t)) \right)_1 \right|, \ldots, \sup_{t \in \mathbb{T}} \left| \left(\Phi^{m-1}(\phi(t)) - \Phi^{m-1}(\psi(t)) \right)_n \right| \right)^T
$$
$$
\vdots
$$
$$
\leq F^m \left(\sup_{t \in \mathbb{T}} \left| \left(\phi(t) - \psi(t) \right)_1 \right|, \ldots, \sup_{t \in \mathbb{T}} \left| \left(\phi(t) - \psi(t) \right)_n \right| \right)^T
$$
$$
= F^m \left(\sup_{t \in \mathbb{T}} \left| \phi_1(t) - \psi_1(t) \right|, \ldots, \sup_{t \in \mathbb{T}} \left| \phi_n(t) - \psi_n(t) \right| \right)^T.
$$

(4.17)

Since $\varrho(F) < 1$, we obtain $\lim_{m \to +\infty} F^m = 0$, which implies that there exist a positive integer N and a positive constant $r < 1$ such that

$$
F^N = \left(D^{-1} \overline{E} L \right)^N = \left(h_{ij} \right)_{n \times n}, \quad \sum_{j=1}^{n} h_{ij} \leq r, \quad i = 1, 2, \ldots, n.
$$

(4.18)

In view of (4.17), (4.18), we have

$$\left| \left(\Phi^N (\phi(t)) - \Phi^N (\psi(t)) \right)_i \right| \leq \sup_{t \in \mathbb{T}} \left| \left(\Phi^N (\phi(t)) - \Phi^N (\psi(t)) \right)_i \right|$$

$$\leq \sum_{j=1}^{n} h_{ij} \sup_{t \in \mathbb{T}} \left| \phi_j(t) - \psi_j(t) \right| \qquad (4.19)$$

$$\leq \sup_{t \in \mathbb{T}} \max_{1 \leq j \leq n} \left| \phi_j(t) - \psi_j(t) \right| \sum_{j=1}^{n} h_{ij} \leq r \| \phi - \psi \|_X$$

for all $t \in \mathbb{T}$, $i = 1, 2, \ldots, n$. It follows that

$$\left\| \Phi^N (\phi(t)) - \Phi^N (\psi(t)) \right\|_X = \sup_{t \in \mathbb{T}} \max_{1 \leq i \leq n} \left| \left(\Phi^N (\phi(t)) - \Phi^N (\psi(t)) \right)_i \right|$$

$$\leq r \| \phi - \psi \|_X. \qquad (4.20)$$

This implies that the mapping $\Phi^N : X \to X$ is a contraction mapping.

By the fixed point theorem of Banach space, Φ possesses a unique fixed point Z^* in X such that $\Phi Z^* = Z^*$. We know from (4.14) that Z^* satisfies system (4.1), and therefore, it is the unique almost periodic solution of system (4.1). This completes the proof. $\qquad\square$

Next, we will establish a result for the exponential stability of the almost periodic solution of system (4.1).

Theorem 4.7. *Suppose that all the conditions of Theorem 4.6 hold. Then system* (4.1) *has exactly one almost periodic solution* $Z^*(t)$. *Moreover,* $Z^*(t)$ *is globally exponentially stable.*

Proof. Since $\varrho(D^{-1}\overline{E}L) < 1$, it follows from Theorem 4.6 that system (4.1) has a unique almost periodic solution $Z^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$. Let $Z(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ be an arbitrary solution of system (4.1) and define $y(t) = Z(t) - Z^*(t)$. Then, set

$$f_j(t, y_j(t - \tau_{ij}(t))) = g_j\left(y_j(t - \tau_{ij}(t)) + x_j^*(t - \tau_{ij}(t)) \right) - g_j\left(x_j^*(t - \tau_{ij}(t)) \right) \quad j = 1, 2, \ldots, n, \qquad (4.21)$$

we get

$$y_i^\Delta = -c_i(t) y_i^\sigma + \sum_{j=1}^{n} b_{ij}(t) f_j(t, y_j(t - \tau_{ij}(t))), \quad i = 1, 2, \ldots, n. \qquad (4.22)$$

Thus, for $i = 1, 2, \ldots, n$, by Lemma 4.2, we have

$$
\Delta^r |y_i(t)| \le \operatorname{sign}(y_i^\sigma) y_i^\Delta(t) \le -\underline{c_i}|y_i^\sigma| + \sum_{j=1}^n |b_{ij}(t) f_j(t, y_j(t - \tau_{ij}(t)))|
$$

$$
\le -\underline{c_i}|y_i^\sigma| + \sum_{j=1}^n \overline{b}_{ij} L_j \sup_{s \in [t-\tau, t] \cap \mathbb{T}} |y_j(s)| = -\underline{c_i}|y_i^\sigma| + \sum_{j=1}^n \overline{b}_{ij} L_j \overline{y}_j(t),
$$

$$(4.23)$$

where $\overline{y}_j(t) = \sup_{s \in [t-\tau, t] \cap \mathbb{T}} |y_j(s)|$, $j = 1, 2, \ldots, n$. Again from $\varrho(D^{-1}\overline{E}L) < 1$, it follows from Lemma 4.5 that $E_n - D^{-1}\overline{E}L$ is an M-matrix, we obtain that there exist a constant $\sigma_0 > 0$ and a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T > (0, 0, \ldots, 0)^T$ such that $(E_n - D^{-1}\overline{E}L)\xi > (\sigma_0, \sigma_0, \ldots, \sigma_0)$. Therefore, $\xi_i - \sum_{j=1}^n \underline{c_i}^{-1} \overline{b}_{ij} L_j \xi_j > \sigma_0$, $i = 1, 2, \ldots, n$, which imply that $-\underline{c_i}\xi_i + \sum_{j=1}^n \overline{b}_{ij} L_j \xi_j < -\underline{c_i}\sigma_0$, $i = 1, 2, \ldots, n$. Hence, we can choose a positive constant $\alpha < 1$ such that

$$
\alpha \xi_i + \left[-\underline{c_i}\xi_i + \sum_{j=1}^n \overline{b}_{ij} L_j \xi_j e_\alpha(\tau, 0) \right] < 0, \quad i = 1, 2, \ldots, n. \tag{4.24}
$$

Choose a constant $\beta > 1$ such that

$$
\beta \xi_i e_{\ominus \alpha}(t, \delta) > 1, \quad \forall t \in [t_0 - \tau, t_0] \cap \mathbb{T}, \ \delta \in [t_0 - \tau, t_0] \cap \mathbb{T}, \ i = 1, 2, \ldots, n. \tag{4.25}
$$

For any $\varepsilon > 0$, let

$$
Z_i(t) = \beta \xi_i \left[\sum_{j=1}^n \overline{y}_j(t_0) + \varepsilon \right] e_{\ominus \alpha}(t, \delta), \quad i = 1, 2, \ldots, n. \tag{4.26}
$$

From (4.24), (4.26), noticing that $(1 + \mu(t)(\ominus \alpha)) < 1$, we obtain

$$
\Delta_r Z_i(t) = (\ominus \alpha) \beta \xi_i \left[\sum_{j=1}^n \overline{y}_j(t_0) + \varepsilon \right] e_{\ominus \alpha}(t, \delta)
$$

$$
> \left[-\underline{c_i}\xi_i + \sum_{j=1}^n \overline{b}_{ij} L_j \xi_j e_\alpha(\tau, 0) \right] \beta \left[\sum_{j=1}^n \overline{y}_j(t_0) + \varepsilon \right] e_{\ominus \alpha}(t, \delta)(1 + \mu(t)(\ominus \alpha))
$$

$$
= -\underline{c_i}\beta \xi_i \left[\sum_{j=1}^n \overline{y}_j(t_0) + \varepsilon \right] e_{\ominus \alpha}(\sigma(t), \delta) \tag{4.27}
$$

$$
+ \sum_{j=1}^n \left[\overline{b}_{ij} L_j \xi_j \beta \left(\sum_{j=1}^n \overline{y}_j(t_0) + \varepsilon \right) e_{\ominus \alpha}(\sigma(t), \delta) e_\alpha(\tau, 0) \right]
$$

$$
> -\underline{c_i} Z_i(\sigma(t)) + \sum_{j=1}^n \overline{b}_{ij} L_j \overline{Z}_j(t), \quad i = 1, 2, \ldots, n,
$$

where $\overline{Z}_j(t) = \sup_{s \in [\sigma(t) - \tau, \sigma(t)] \cap \mathbb{T}} Z_j(s)$, $j = 1, 2, \ldots, n$. In view of (4.25) and (4.26), for $i = 1, 2, \ldots, n$, we have

$$
\begin{aligned}
Z_i(t) &= \beta \xi_i \left[\sum_{j=1}^{n} \overline{y}_j(t_0) + \varepsilon \right] e_{\ominus \alpha}(t, \delta) \\
&> \sum_{j=1}^{n} \overline{y}_j(t_0) + \varepsilon > |y_i(t)|, \quad \forall t \in [t_0 - \tau, t_0] \cap \mathbb{T}.
\end{aligned}
\tag{4.28}
$$

We claim that

$$
|y_i(t)| < Z_i(t) \quad \forall t \in [t_0, +\infty) \cap \mathbb{T}, \ i = 1, 2, \ldots, n.
\tag{4.29}
$$

Contrarily, there must exist $i \in \{1, 2, \ldots, n\}$ and $t_i > 0$ such that

$$
|y_i(t_i)| \geq Z_i(t_i), \quad |y_j(t)| < Z_j(t), \quad \forall t \in [t_0 - \tau, t_i) \cap \mathbb{T}, \ j = 1, 2, \ldots, n,
\tag{4.30}
$$

which implies that

$$
|y_i(t_i)| - Z_i(t_i) \geq 0, \quad |y_j(t)| - Z_j(t) < 0, \quad \forall t \in [t_0 - \tau, t_i) \cap \mathbb{T}, \ j = 1, 2, \ldots, n.
\tag{4.31}
$$

$\square$

Case I. If $\rho(t_i)$ is right dense point, that is, $\sigma(\rho(t_i)) = \rho(t_i) = t_i$, we have

$$
\begin{aligned}
0 \leq \Delta^r \big(|y_i(\rho(t_i))| - Z_i(\rho(t_i)) \big) &= \limsup_{h \to 0^-} \frac{\big[|y_i(t_i + h)| - Z_i(t_i + h) \big] - \big[|y_i(t_i)| - Z_i(t_i) \big]}{h} \\
&\leq \limsup_{h \to 0^-} \frac{|y_i(t_i + h)| - |y_i(t_i)|}{h} - \liminf_{h \to 0^-} \frac{Z_i(t_i + h) - Z_i(t_i)}{h} \\
&= \Delta^r |y_i(t_i)| - \Delta_r Z_i(t_i).
\end{aligned}
\tag{4.32}
$$

Case II. If $\rho(t_i)$ is right scattered point, that is, $\sigma(\rho(t_i)) = t_i > \rho(t_i)$, we obtain

$$
\begin{aligned}
0 \leq \Delta^r \big(|y_i(\rho(t_i))| - Z_i(\rho(t_i)) \big) &= \frac{|y_i(t_i)| - |y_i(\rho(t_i))|}{\mu(\rho(t_i))} - \frac{Z_i(t_i) - Z_i(\rho(t_i))}{\mu(\rho(t_i))} \\
&= \Delta^r |y_i(\rho(t_i))| - \Delta_r Z_i(\rho(t_i)).
\end{aligned}
\tag{4.33}
$$

Thus,

$$
\Delta^r |y_i(\rho(t_i))| \geq \Delta_r Z_i(\rho(t_i)).
\tag{4.34}
$$

From (4.23), (4.27), (4.30), we can obtain

$$
\begin{aligned}
\Delta^r \big| y_i(\rho(t_i)) \big| &\leq -\underline{c_i} |y_i(t_i)| + \sum_{j=1}^{n} \overline{b}_{ij} L_j \overline{y}_j(\rho(t_i)) \\
&\leq -\underline{c_i} Z_i(t_i) + \sum_{j=1}^{n} \overline{b}_{ij} L_j \overline{y}_j(\rho(t_i)) < \Delta_r Z_i(\rho(t_i)),
\end{aligned}
\tag{4.35}
$$

which contradicts (4.34). Hence, (4.29) holds. Letting $\varepsilon \to 0^+$ and $M = n\max_{1\leq i\leq n}\{\beta\xi_i + 1\}$, we have from (4.26) and (4.29) that

$$
\begin{aligned}
\big| x_i(t) - x_i^*(t) \big| = \big| y_i(t) \big| &\leq \beta\xi_i \sum_{j=1}^{n} \overline{y}_j(t_0) e_{\ominus\alpha}(t,\delta) \leq \beta\xi_i \|Z - Z^*\| n e_{\ominus\alpha}(t,\delta) \\
&< \max_{1\leq i\leq n}\{\beta\xi_i + 1\} n \|Z - Z^*\| e_{\ominus\alpha}(t,\delta) \\
&\leq M\|Z - Z^*\| e_{\ominus\alpha}(t,\delta), \quad i = 1,2,\ldots,n
\end{aligned}
\tag{4.36}
$$

for all $t \in [t_0, +\infty) \cap \mathbb{T}$. This completes the proof.

Example 4.8. Let $\mathbb{T} = \mathbb{Z}$, consider the following HNN:

$$
\begin{aligned}
x_1^\Delta &= -c_1(t)x_1^\sigma + 0.04(\sin t)g_1\big(x_1\big(t - \sin^2 t\big)\big) + 0.016(\cos t)g_2\big(x_2\big(t - 2\sin^2 t\big)\big) + 2\cos t, \\
x_2^\Delta &= -c_2(t)x_2^\sigma + 0.02(\sin 2t)g_1\big(x_1\big(t - 3\cos^2 t\big)\big) + 0.014(\cos 4t)g_2\big(x_2\big(t - 4\sin^2 t\big)\big) + \sin t, \\
x_3^\Delta &= -c_3(t)x_3^\sigma + 0.013(\sin 2t)g_3\big(x_3\big(t - \cos^2 t\big)\big) + \cos^2 t,
\end{aligned}
\tag{4.37}
$$

where $g_1(x) = g_2(x) = g_3(x) = (1/2)(|x+1|-|x-1|)$ and $c_1(t) = 1+0.05\sin t$, $c_2(t) = 1+0.03\cos t$, $c_3(t) = 1 + 0.01\sin t$.

Note that $\underline{c_1} = 0.05$, $\underline{c_2} = 0.07$, $\underline{c_3} = 0.09$, $L_1 = L_2 = L_3 = 1$, and $\overline{b}_{11} = 0.04$, $\overline{b}_{12} = 0.016$, $\overline{b}_{21} = 0.02$, $\overline{b}_{22} = 0.014$, $\overline{b}_{33} = 0.013$, $\overline{b}_{13} = \overline{b}_{23} = \overline{b}_{31} = \overline{b}_{32} = 0$. Then we have

$$
D = \left(c_i^{-1}\overline{b}_{ij}L_j\right)_{3\times3} = \begin{pmatrix} 0.8000 & 0.3200 & 0 \\ 0.2857 & 0.2000 & 0 \\ 0 & 0 & 0.1444 \end{pmatrix}
\tag{4.38}
$$

and $\varrho(D) = 0.9259 < 1$. Thus, from Theorems 4.6 and 4.7, (4.37) has exactly one almost periodic solution and it is exponentially stable. We can take the initial value $\varphi_1(s) = -0.3$, $\varphi_3(s) = -0.1$, $\varphi_3(s) = -0.4$, $s \in [-4,0] \cap \mathbb{Z}$, we can give the following numerical simulation figures to show our results are plausible and effective on time scales (see Figures 1 and 2).
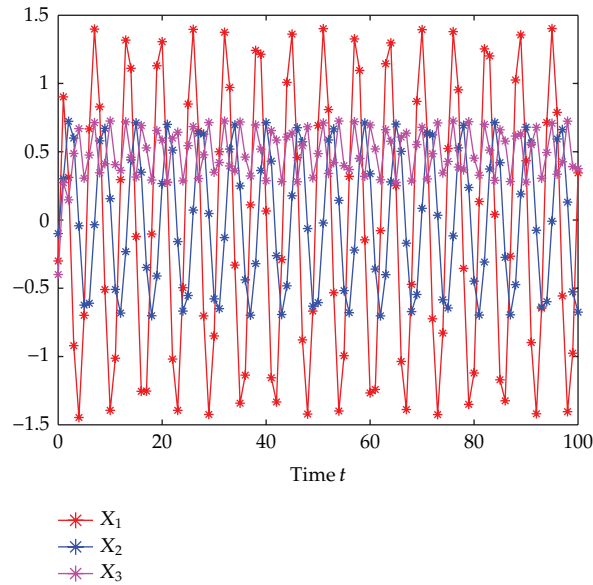
Figure 1: Transient response of state variables $x_1, x_2,$ and x_3 in Example 4.8.
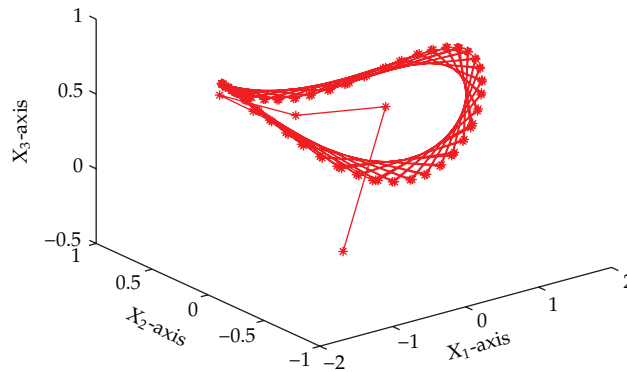

Figure 2: Phase response of state variables $x_1, x_2,$ and x_3 in Example 4.8.

Example 4.9. Let $\mathbb{T} = \mathbb{R}$, consider the following HNN:

$$x_1^\Delta = -c_1(t)x_1^\sigma + \frac{1}{2}(\sin t)g_1\left(x_1\left(t - \sin^2 t\right)\right) + \frac{1}{18}(\cos t)g_2\left(x_2\left(t - 2\sin^2 t\right)\right) + 2\cos t,$$

$$x_2^\Delta = -c_2(t)x_2^\sigma + 2(\sin 2t)g_1\left(x_1\left(t - 3\cos^2 t\right)\right) + \frac{1}{2}(\cos 4t)g_2\left(x_2\left(t - 4\sin^2 t\right)\right) + \sin t,$$

$$x_3^\Delta = -c_3(t)x_3^\sigma + \frac{6}{7}(\sin 2t)g_3\left(x_3\left(t - \cos^2 t\right)\right) + \cos^2 t,$$

$$(4.39)$$

where $g_1(x) = g_2(x) = g_3(x) = (1/2)(|x + 1| - |x - 1|)$ and $c_1(t) = 2 + \sin t$, $c_2(t) = 2 + \cos t$, $c_3(t) = 2 + \sin t$.
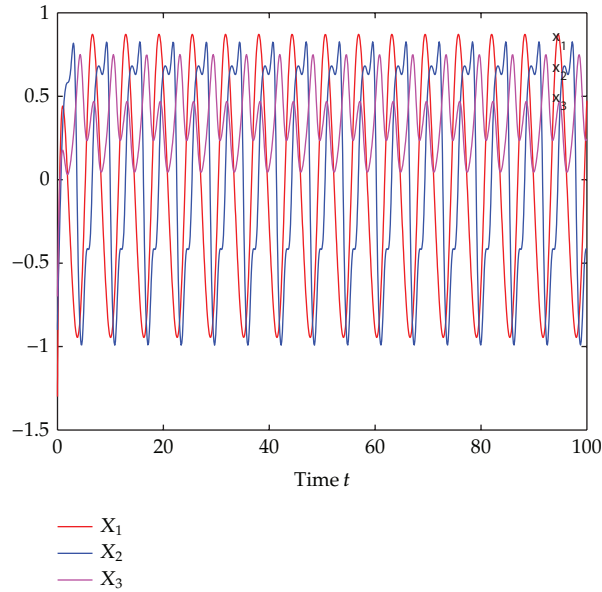
Figure 3: Transient response of state variables x_1, x_2 and x_3 in Example 4.9.
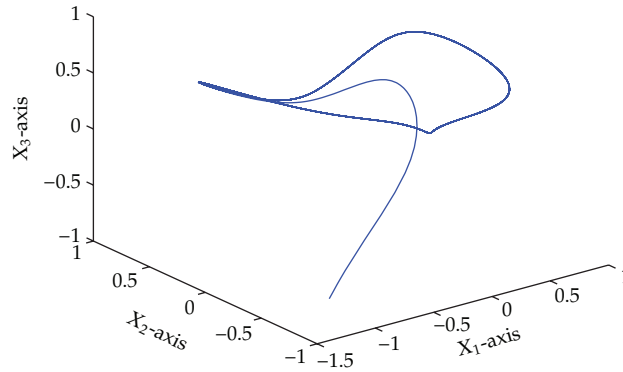

Figure 4: Phase response of state variables x_1, x_2 and x_3 in Example 4.9.

Note that $c_1 = c_2 = c_3 = L_1 = L_2 = L_3 = 1$, and $\bar{b}_{11} = 1/2, \bar{b}_{12} = 1/18, \bar{b}_{21} = 2, \bar{b}_{22} = 1/2,$ $\bar{b}_{33} = 6/7, \bar{b}_{13} = \bar{b}_{23} = \bar{b}_{31} = \bar{b}_{32} = 0$. Then we have

$$D = \left(c_i^{-1} \bar{b}_{ij} L_j \right)_{3\times3} = \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{18} & 0 \\ 2 & \dfrac{1}{2} & 0 \\ 0 & 0 & \dfrac{6}{7} \end{pmatrix} \tag{4.40}$$

and $\rho(D) = 0.8571 < 1$. Thus, from Theorems 4.6 and 4.7, (4.39) has exactly one almost periodic solution and it is exponentially stable. We can take the initial value $\varphi_1(s) = -1.3,$

$\varphi_3(s) = -0.9$, $\varphi_3(s) = -0.7$, $s \in [-4, 0]$, we can give the following numerical simulation figures to show our results are plausible and effective on time scales (see Figures 3 and 4).

5. Conclusion

The existence and uniqueness of almost periodic solution and its expression form to a class of linear dynamic equations on time scales are obtained. As an application, sufficient conditions for the existence and exponential stability of almost periodic solution to a class of Hopfield neural networks with delays are established. To the best of our knowledge, the results presented here have not appeared in the related literature. In fact, both continuous and discrete systems are very important in implementation and applications. But it is troublesome to study the existence and stability of almost periodic solutions for continuous and discrete systems, respectively. Therefore, it is meaningful to study that on timescales which can unify the continuous and discrete situations. Also, the results and methods used in this paper can be used to study many other types neural networks and population models.

Acknowledgments

References

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäauser, Boston, Mass, USA, 2001.

[2] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäauser, Boston, Mass, USA, 2003.

[3] S. Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten [Ph.D. thesis]*, Universität, Würzburg, 1988, in German.

[4] L. G. Deysach and G. R. Sell, "On the existence of almost periodic motions," *The Michigan Mathematical Journal*, vol. 12, pp. 87–95, 1965.

[5] R. K. Miller, "Almost periodic differential equations as dynamical systems with applications to the existence of A. P. solutions," *Journal of Differential Equations*, vol. 1, no. 3, pp. 337–345, 1965.

[6] G. Seifert, "Almost periodic solutions and asymptotic stability," *Journal of Mathematical Analysis and Applications*, vol. 21, no. 1, pp. 136–149, 1968.

[7] G. Seifert, "On uniformly almost periodic sets of functions for almost periodic differential equations," *The Tôhoku Mathematical Journal*, vol. 34, no. 2, pp. 301–309, 1982.

[8] S. Bochner, "A new approach to almost periodicity," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 48, pp. 2039–2043, 1962.

[9] A. M. Fink, *Almost Periodic Differential Equations*, vol. 377 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1974.

[10] A. M. Fink and G. Seifert, "Liapunov functions and almost periodic solutions for almost periodic systems," *Journal of Differential Equations*, vol. 5, pp. 307–313, 1969.

[11] D. Cheban and C. Mammana, "Invariant manifolds, global attractors and almost periodic solutions of nonautonomous difference equations," *Nonlinear Analysis*, vol. 56, no. 4, pp. 465–484, 2004.

[12] C. Corduneanu, *Almost Periodic Functions*, Chelsea Publishing, New York, NY, USA, 2nd edition, 1989.

[13] G. M. N'Guerekata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic Publishers, New York, NY, USA, Plenum Press, London, UK, 2001.

[14] C. Y. Zhang, *Almost Periodic Type Functions and Ergodicity*, Science Press, Kluwer Academic Publishers, New York, NY, USA, 2003.

[15] C. Y. Zhang, "Pseudo-almost-periodic solutions of some differential equations," *Journal of Mathematical Analysis and Applications*, vol. 181, no. 1, pp. 62–76, 1994.

[16] A. M. Fink and J. A. Gatica, "Positive almost periodic solutions of some delay integral equations," *Journal of Differential Equations*, vol. 83, no. 1, pp. 166–178, 1990.

[17] Y. Hino, N. V. Minh, J. S. Shin, and T. Naito, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, Taylor & Francis, New York, NY, USA, 2002.

[18] N. Boukli-Hacene and K. Ezzinbi, "Weighted pseudo almost periodic solutions for some partial functional differential equations," *Nonlinear Analysis*, vol. 71, no. 9, pp. 3612–3621, 2009.

[19] E. H. A. Dads, P. Cieutat, and K. Ezzinbi, "The existence of pseudo-almost periodic solutions for some nonlinear differential equations in Banach spaces," *Nonlinear Analysis*, vol. 69, no. 4, pp. 1325–1342, 2008.

[20] Y. K. Li and X. Fan, "Existence and globally exponential stability of almost periodic solution for Cohen-Grossberg BAM neural networks with variable coefficients," *Applied Mathematical Modelling*, vol. 33, no. 4, pp. 2114–2120, 2009.

[21] Y. K. Li and C. Wang, "Almost periodic functions on time scales and applications," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 727068, 20 pages, 2011.

[22] Y. K. Li and C. Wang, "Uniformly almost periodic functions and almost periodic solutions to dynamic equations on time scales," *Abstract and Applied Analysis*, vol. 2011, Article ID 341520, 22 pages, 2011.

[23] J. J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 79, no. 8, pp. 2554–2558, 1982.

[24] J. Hopfied, "Neurons with graded response have collective computational properties like those of two-state neurons," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 81, no. 10, pp. 3088–3092, 1984.

[25] Y. K. Li and L. H. Lu, "Global exponential stability and existence of periodic solution of Hopfield-type neural networks with impulses," *Physics Letters A*, vol. 333, no. 1-2, pp. 62–71, 2004.

[26] Q. Dong, K. Matsui, and X. Huang, "Existence and stability of periodic solutions for Hopfield neural network equations with periodic input," *Nonlinear Analysis*, vol. 49, no. 4, pp. 471–479, 2002.

[27] S. Guo and L. Huang, "Periodic solutions in an inhibitory two-neuron network," *Journal of Computational and Applied Mathematics*, vol. 161, no. 1, pp. 217–229, 2003.

[28] Z. Liu and L. Liao, "Existence and global exponential stability of periodic solution of cellular neural networks with time-varying delays," *Journal of Mathematical Analysis and Applications*, vol. 290, no. 1, pp. 247–262, 2004.

[29] S. Guo and L. Huang, "Stability analysis of a delayed Hopfield neural network," *Physical Review E*, vol. 67, no. 6, Article ID 061902, 7 pages, 2003.

[30] B. Liu and L. Huang, "Existence and exponential stability of almost periodic solutions for Hopfield neural networks with delays," *Neurocomputing*, vol. 68, pp. 196–207, 2005.

[31] S. Hong, "Stability criteria for set dynamic equations on time scales," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3444–3457, 2010.