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Research Article

Logarithmically Improved Blow up Criterion for Smooths Solution to the 3D Micropolar Fluid Equations

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Blow-up criteria of smooth solutions for the 3D micropolar fluid equations are investigated. Logarithmically improved blow-up criteria are established in the Morrey-Campanto space.

1. Introduction

This paper concerns the initial value problem for the micropolar fluid equations in $\mathbb{R}^3$

$$\begin{gathered}\partial_t u-(\mu+\chi)\Delta u+u\cdot\nabla u+\nabla p-\chi\nabla\times w=0,\\ \partial_t w-\gamma\Delta w-\kappa\nabla\nabla\cdot w+2\chi w+u\cdot\nabla w-\chi\nabla\times u=0,\\ \nabla\cdot u=0\end{gathered}\tag{1.1}$$

with the initial value

$$t=0:\quad u=u_0(x),\quad w=w_0(x),\tag{1.2}$$

where $u(t,x)$, $w(t,x)$, and $p(t,x)$ stand for the velocity field, microrotation field, and the scalar pressure, respectively. And $\nu>0$ is the Newtonian kinetic viscosity, $\kappa>0$ is the dynamics micro-rotation viscosity, and $\alpha,\beta,\gamma>0$ are the angular viscosity (see, i.e., Lukaszewicz [1]).

The micropolar fluid equations were first proposed by Eringen [2]. It is a type of fluids which exhibits the micro-rotational effects and micro-rotational inertia and can be viewed as a non-Newtonian fluid. Physically, it may represent adequately the fluids consisting of bar-like elements. Certain anisotropic fluids, for example, liquid crystals that are made up of dumbbell molecules, are of the same type. For more background, we refer to [1] and references therein. Besides their physical applications, the micropolar fluid equations are also mathematically significant. Fundamental mathematical issues such as the global regularity of their solutions have generated extensive research, and many interesting results have been obtained (see [3–8]). Regularity criterion of weak solutions to (1.1) and (1.2) in terms of the pressure was obtained (see [4]). Gala [5] established a Serrin-type regularity criterion for the weak solutions to (1.1) and (1.2) in Morrey-Campanato space. Wang and Chen [7] established the regularity criteria of weak solutions to (1.1) and (1.2) via the derivative of the velocity in one direction. A new logarithmically improved blow-up criterion of smooth solutions to (1.1) and (1.2) in an appropriate homogeneous Besov space is established by Wang and Yuan [8].

If $\kappa = 0$ and $w = 0$, then (1.1) reduces to be the Navier-Stokes equations. Besides its physical applications, the Navier-Stokes equations are also mathematically significant. In the last century, Leray [9] and Hopf [10] constructed weak solutions to the Navier-Stokes equations. The solution is called the Leray-Hopf weak solution. Later on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier-Stokes equations. Different criteria for regularity of the weak solutions have been proposed and many interesting results are established (see [11–26]). Regularity criteria of weak solutions to the Navier-Stokes equations in Morrey space were obtained in [13, 21].

The main aim of this paper is to establish two logarithmically blow-up criteria of smooth solution to (1.1), (1.2). Our results state as follows.

Theorem 1.1. *Let $u_0, w_0 \in H^m(\mathbb{R}^3)$ $(m \geq 3)$ with $\nabla \cdot u_0 = 0$. Assume that (u, w) is a smooth solution to* (1.1) *and* (1.2) *on [0, T). If u satisfies*

$$\int_0^T \frac{\|u(t)\|_{\dot{M}_{2,3/r}}^{2/(1-r)}}{1+\ln(e+\|u(t)\|_{L^\infty})}dt < \infty, \quad 0 < r < 1, \tag{1.3}$$

then the solution (u, w) can be extended beyond $t = T$.

We have the following corollary immediately.

Corollary 1.2. *Let $u_0, w_0 \in H^m(\mathbb{R}^3)$ $(m \geq 3)$ with $\nabla \cdot u_0 = 0$. Assume that (u, w) is a smooth solution to* (1.1) *and* (1.2) *on [0, T). Suppose that T is the maximal existence time, then*

$$\int_0^T \frac{\|u(t)\|_{\dot{M}_{2,3/r}}^{2/(1-r)}}{1+\ln(e+\|u(t)\|_{L^\infty})}dt = \infty, \quad 0 < r < 1. \tag{1.4}$$

Theorem 1.3. *Let $u_0, w_0 \in H^m(\mathbb{R}^3)$ $(m \geq 3)$ with $\nabla \cdot u_0 = 0$. Assume that (u, w) is a smooth solution to* (1.1) *and* (1.2) *on [0, T). If u satisfies*

$$\int_0^T \frac{\|\nabla u(t)\|_{\dot{M}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|\nabla u(t)\|_{L^\infty})}dt < \infty, \quad 0 < r < 1, \tag{1.5}$$

then the solution (u, w) can be extended beyond $t = T$.

One has the following corollary immediately.

Corollary 1.4. *Let $u_0, w_0 \in H^m(\mathbb{R}^3)$ $(m \geq 3)$ with $\nabla \cdot u_0 = 0$. Assume that (u, w) is a smooth solution to* (1.1) *and* (1.2) *on [0, T). Suppose that T is the maximal existence time, then*

$$\int_0^T \frac{\|\nabla u(t)\|_{\dot{M}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|\nabla u(t)\|_{L^\infty})} dt = \infty, \quad 0 < r < 1. \tag{1.6}$$

The paper is organized as follows. We first state some important inequalities in Section 2, which play an important role in the proof of our main result. Then, we prove the main result in Section 3 and Section 4, respectively.

2. Preliminaries

Firstly, we recall the definition and some properties of the space that we are going to use. The space plays an important role in studying the regularity of solutions to nonlinear differential equations.

Definition 2.1. For $1 < p \leq q \leq +\infty$, the Morrey-Campanato space $\dot{M}_{p,q}$ is defined by

$$\dot{M}_{p,q} = \left\{ f \in L^p_{\mathrm{loc}}\left(\mathbb{R}^3\right) \mid \|f\|_{\dot{M}_{p,q}} = \sup_{x\in\mathbb{R}^3} \sup_{R>0} R^{3/q-3/p} \|f\|_{L^p(B(x,R))} < \infty \right\}, \tag{2.1}$$

where $B(x, R)$ denotes the ball of center x with radius R.

It is easy to verify that $\dot{M}_{p,q}$ is a Banach space under the norm $\|\cdot\|_{\dot{M}_{p,q}}$. Furthermore, it is easy to check the following:

$$\|f(\lambda\cdot)\|_{\dot{M}_{p,q}} = \lambda^{-3/q}\|f\|_{\dot{M}_{p,q}}, \quad \lambda > 0. \tag{2.2}$$

Morrey-Campanato spaces can be seen as a complement to L^p spaces. In fact, for $p \leq q$, one has

$$L^q = \dot{M}_{q,q} \subset \dot{M}_{p,q}. \tag{2.3}$$

one has the following comparison between Lorentz spaces and Morrey-Campanato spaces: for $p \geq 2$,

$$L^{3/r}\left(\mathbb{R}^3\right) \subset L^{3/r,\infty}\left(\mathbb{R}^3\right) \subset \dot{M}_{p,3/r}\left(\mathbb{R}^3\right), \tag{2.4}$$

where $L^{p,\infty}$ denotes the usual Lorentz (weak L^p) space.

In the proof of our main result, we need the following lemma which was given in [27].

Lemma 2.2. *For $0 \leq r < 3/2$, the space $\dot{Z}_r$ is defined as the space of $f(x) \in L^2_{\text{loc}}(\mathbb{R}^3)$ such that*

$$\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}} \leq 1} \|fg\|_{L^2} < \infty. \tag{2.5}$$

Then $f \in \dot{M}_{2,3/r}$ if and only if $f \in \dot{Z}_r$ with equivalence of norms. And the fact that

$$L^2 \bigcap \dot{H}^1 \subset \dot{B}^r_{2,1} \subset \dot{H}^r, \quad 0 < r < 1, \tag{2.6}$$

one has

$$\dot{X}_r \subset \dot{M}_{2,3/r}, \tag{2.7}$$

where $\dot{X}_r$ denotes the pointwise multiplier space from $\dot{H}^r$ to L^2.

We need the following lemma that is basically established in [28]. For completeness, the proof will be also sketched here.

Lemma 2.3. *For $0 < r < 1$, the inequality*

$$\|f\|_{\dot{B}^r_{2,1}} \leq C\|f\|^{1-r}_{L^2} \|\nabla f\|^r_{L^2} \tag{2.8}$$

holds, where C is a positive constant that depends on r.

Proof. It follows from the definition of Besov spaces that

$$\begin{aligned}
\|f\|_{\dot{B}^r_{2,1}} &= \sum_{i\in\mathbb{Z}} 2^{ir} \|\Delta_i f\|_{L^2} \\
&\leq \sum_{i\leq j} 2^{ir} \|\Delta_i f\|_{L^2} + \sum_{i>j} 2^{i(r-1)} 2^i \|\Delta_i f\|_{L^2} \\
&\leq \left(\sum_{i\leq j} 2^{2ir}\right)^{1/2} \left(\sum_{i\leq j} \|\Delta_i f\|^2_{L^2}\right)^{1/2} + \left(\sum_{i\leq j} 2^{2i(r-1)}\right)^{1/2} \left(\sum_{i>j} 2^{2i} \|\Delta_i f\|^2_{L^2}\right)^{1/2} \\
&\leq C\left(2^{jr}\|f\|_{L^2} + 2^{j(r-1)}\|f\|_{\dot{H}^1}\right) \\
&= C\left(2^{jr}A^{-r} + 2^{j(r-1)}A^{1-r}\right)\|f\|^{1-r}_{L^2}\|f\|^r_{\dot{H}^1},
\end{aligned} \tag{2.9}$$

where $A = (\|f\|_{\dot{H}^1})/(\|f\|_{L^2})$. Choosing j such that $1/2 \leq 2^{jr}A^{-r} \leq 1$, from (2.9) we get

$$\begin{aligned}
\|f\|_{\dot{B}^r_{2,1}} &\leq \left(1 + 2^{j(r-1)}A^{1-r}\right)\|f\|^{1-r}_{L^2}\|f\|^r_{\dot{H}^1} \\
&\leq C\left(1 + \left(\frac{1}{2}\right)^{-1/r}\right) \|f\|^{1-r}_{L^2}\|\nabla f\|^r_{L^2}.
\end{aligned} \tag{2.10}$$

Therefore, we have completed the proof of Lemma 2.3. □

The following Lemma comes from [29].

Lemma 2.4. *Assume that* $1 < p < \infty$. *For* $f, g \in W^{m,p}$, *and* $1 < q \le \infty,\ 1 < r < \infty$, *one has*

$$\|\nabla^{\alpha}(fg) - f\nabla^{\alpha}g\|_{L^p} \le C\Big(\|\nabla f\|_{L^{q_1}}\left\|\nabla^{\alpha-1}g\right\|_{L^{r_1}} + \|g\|_{L^{q_2}}\|\nabla^{\alpha}f\|_{L^{r_2}}\Big), \tag{2.11}$$

where $1 \le \alpha \le m$ *and* $1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2$.

In order to prove Theorem 1.1, we need the following interpolation inequalities in three space dimensions.

Lemma 2.5. *In three space dimensions, the following inequalities*

$$\begin{gathered} \|\nabla f\|_{L^4} \le C\|f\|_{L^2}^{1/8}\left\|\nabla^2 f\right\|_{L^2}^{7/8} \\ \|f\|_{L^4} \le C\|f\|_{L^2}^{3/4}\left\|\nabla^3 f\right\|_{L^2}^{1/4} \\ \left\|\nabla^2 f\right\|_{L^4} \le C\|f\|_{L^2}^{1/12}\left\|\nabla^3 f\right\|_{L^2}^{11/12} \\ \left\|\nabla^2 f\right\|_{L^2} \le C\|f\|_{L^2}^{1/3}\left\|\nabla^3 f\right\|_{L^2}^{2/3} \end{gathered} \tag{2.12}$$

hold.

3. Proof of Theorem 1.1

Proof. Multiplying the first equation of (1.1) by u and integrating with respect to x over $\mathbb{R}^3$, using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + (\mu+\chi)\|\nabla u(t)\|_{L^2}^2 = \chi\int_{\mathbb{R}^3}(\nabla\times w)\cdot u\,dx \tag{3.1}$$

Similarly, we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \gamma\|\nabla w(t)\|_{L^2}^2 + \kappa\|\nabla\cdot w\|_{L^2}^2 + 2\chi\|w\|_{L^2}^2 = \chi\int_{\mathbb{R}^3}(\nabla\times u)\cdot w\,dx. \tag{3.2}$$

Summing up (3.1) and (3.2), we deduce thats

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\Big) + (\mu+\chi)\|\nabla u(t)\|_{L^2}^2 \\ &\quad + \gamma\|\nabla w(t)\|_{L^2}^2 + \kappa\|\nabla\cdot w\|_{L^2}^2 + 2\chi\|w\|_{L^2}^2 \\ &= \chi\int_{\mathbb{R}^3}(\nabla\times w)\cdot u\,dx + \chi\int_{\mathbb{R}^3}(\nabla\times u)\cdot w\,dx. \end{aligned} \tag{3.3}$$

Using integration by parts and Cauchy's inequality, we obtain

$$\chi\int_{\mathbb{R}^3}(\nabla\times w)\cdot u\,dx+\chi\int_{\mathbb{R}^3}(\nabla\times u)\cdot w\,dx\leq\chi\|\nabla u\|_{L^2}^2+\chi\|w\|_{L^2}^2. \tag{3.4}$$

Combining (3.3) and (3.4) yields

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right)+\mu\|\nabla u(t)\|_{L^2}^2+\gamma\|\nabla w(t)\|_{L^2}^2+\kappa\|\nabla\cdot w\|_{L^2}^2+\chi\|w\|_{L^2}^2\leq 0. \tag{3.5}$$

Integrating with respect to t, we have

$$\begin{aligned}&\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2+2\int_0^t\left(\mu\|\nabla u(\tau)\|_{L^2}^2+\gamma\|\nabla w(\tau)\|_{L^2}^2\right)d\tau\\&\quad+2\kappa\int_0^t\|\nabla\cdot w(\tau)\|_{L^2}^2d\tau+2\chi\int_0^t\|w(\tau)\|_{L^2}^2d\tau\leq\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2.\end{aligned} \tag{3.6}$$

Taking ∇ to the first equation of (1.1), then multiplying the resulting equation by ∇u and using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2+(\mu+\chi)\left\|\nabla^2u(t)\right\|_{L^2}^2=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\nabla u\,dx+\chi\int_{\mathbb{R}^3}\nabla(\nabla\times w)\nabla u\,dx. \tag{3.7}$$

Similarly, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla w(t)\|_{L^2}^2+\gamma\left\|\nabla^2w(t)\right\|_{L^2}^2+\kappa\|\nabla\cdot\nabla w\|_{L^2}^2+2\chi\|\nabla w\|_{L^2}^2\\&\quad=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla w)\cdot\nabla w\,dx+\chi\int_{\mathbb{R}^3}\nabla(\nabla\times u)\cdot\nabla w\,dx.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\right)+(\mu+\chi)\left\|\nabla^2u(t)\right\|_{L^2}^2\\&\quad+\gamma\left\|\nabla^2w(t)\right\|_{L^2}^2+\kappa\|\nabla\nabla\cdot w\|_{L^2}^2+2\chi\|\nabla w\|_{L^2}^2\\&\quad=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\nabla u\,dx+\chi\int_{\mathbb{R}^3}\nabla(\nabla\times w)\nabla u\,dx\\&\qquad-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla w)\nabla w\,dx+\chi\int_{\mathbb{R}^3}\nabla(\nabla\times u)\nabla w\,dx.\end{aligned} \tag{3.9}$$

Using integration by parts and Cauchy's inequality, we obtain

$$\chi\int_{\mathbb{R}^3}\nabla(\nabla\times w)\cdot\nabla u\,dx+\chi\int_{\mathbb{R}^3}\nabla(\nabla\times u)\cdot\nabla w\,dx\leq\chi\left\|\nabla^2u\right\|_{L^2}^2+\chi\|\nabla w\|_{L^2}^2. \tag{3.10}$$

Using Hölder's inequality, (2.8), and Young's inequality, we obtain

$$\begin{aligned}
&-\int_{\mathbb{R}^3} \nabla(u\cdot\nabla u)\nabla u\,dx \\
&\quad\le \|\nabla u\|_{L^2}\|\nabla u\nabla u\|_{L^2} \\
&\quad\le C\|\nabla u\|_{\dot{M}_{2,3/r}}\|\nabla u\|_{\dot{B}^r_{2,1}}\|\nabla u\|_{L^2} \\
&\quad\le C\|\nabla u\|_{\dot{M}_{2,3/r}}\|\nabla u\|^{2-r}_{\dot{B}^r_{2,1}}\left\|\nabla^2 u\right\|^r_{L^2} \\
&\quad\le \frac{\mu}{2}\left\|\nabla^2 u(t)\right\|^2_{L^2} + C\|\nabla u\|^{2/(2-r)}_{\dot{M}_{2,3/r}}\|\nabla u\|^2_{L^2}.
\end{aligned} \tag{3.11}$$

Similarly, we have the following estimate:

$$\begin{aligned}
&-\int_{\mathbb{R}^3} \nabla(u\cdot\nabla w)\nabla w\,dx \\
&\quad\le \|\nabla w\|_{L^2}\|\nabla u\nabla w\|_{L^2} \\
&\quad\le C\|\nabla u\|_{\dot{M}_{2,3/r}}\|\nabla w\|_{\dot{B}^r_{2,1}}\|\nabla w\|_{L^2} \\
&\quad\le C\|\nabla u\|_{\dot{M}_{2,3/r}}\|\nabla w\|^{2-r}_{\dot{B}^r_{2,1}}\left\|\nabla^2 w\right\|^r_{L^2} \\
&\quad\le \frac{\gamma}{2}\left\|\nabla^2 w(t)\right\|^2_{L^2} + C\|\nabla u\|^{2/(2-r)}_{\dot{M}_{2,3/r}}\|\nabla w\|^2_{L^2}.
\end{aligned} \tag{3.12}$$

Combining (3.9)-(3.12) yields

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u(t)\|^2_{L^2} + \|\nabla w(t)\|^2_{L^2}\right) + \mu\left\|\nabla^2 u(t)\right\|^2_{L^2} + \gamma\left\|\nabla^2 w\right\|^2_{L^2} + \kappa\|\nabla\nabla\cdot w\|^2_{L^2} + \chi\|\nabla w\|^2_{L^2} \\
&\quad\le C\|\nabla u\|^{2/(2-r)}_{\dot{M}_{2,3/r}}\left(\|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2}\right) \\
&\quad\le C\frac{\|\nabla u\|^{2/(2-r)}_{\dot{M}_{2,3/r}}}{1+\ln(e+\|\nabla u\|_{L^\infty})}\left(\|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2}\right)(1+\ln(e+\|\nabla u\|_{L^\infty})) \\
&\quad\le C\frac{\|\nabla u\|^{2/(2-r)}_{\dot{M}_{2,3/r}}}{1+\ln(e+\|\nabla u\|_{L^\infty})}\left(\|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2}\right)\left(1+\ln\left(e+\left\|\nabla^3 u\right\|_{L^2} + \left\|\nabla^3 w\right\|_{L^2}\right)\right),
\end{aligned} \tag{3.13}$$

where we have used

$$H^2\left(\mathbb{R}^3\right) \hookrightarrow L^\infty\left(\mathbb{R}^3\right). \tag{3.14}$$

For any $T_0 \le t \le T$, we set

$$\vartheta(t) = \sup_{T_0\le\tau\le t}\left(\left\|\nabla^3 u(\tau)\right\|^2_{L^2} + \left\|\nabla^3 w(\tau)\right\|^2_{L^2}\right). \tag{3.15}$$

Thus, from (3.13), we have

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\right)+\mu\left\|\nabla^2 u(t)\right\|_{L^2}^2+\gamma\left\|\nabla^2 w\right\|_{L^2}^2+\kappa\|\nabla\nabla\cdot w\|_{L^2}^2+\chi\|\nabla w\|_{L^2}^2\\&\quad\leq C\frac{\|\nabla u\|_{\dot{M}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|\nabla u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)(1+\ln(e+\vartheta(t))),\quad \forall T_0\leq t<T.\end{aligned}\tag{3.16}$$

It follows from (3.8) and Gronwall's inequality that

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\\&\quad\leq\left(\|\nabla u(T_0)\|_{L^2}^2+\|\nabla w(T_0)\|_{L^2}^2\right)\exp\left\{C(1+\ln(e+\vartheta(t)))\int_{T_0}^t\frac{\|\nabla u(\tau)\|_{\dot{M}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|\nabla u\|_{L^\infty})}d\tau\right\}\\&\quad\leq C_0\exp\{C\varepsilon[1+\ln(e+\vartheta(t))]\}\\&\quad\leq C_0\exp\{2C\varepsilon[\ln(e+\vartheta(t))]\}\\&\quad\leq C_0(e+\vartheta(t))^{2C\varepsilon},\end{aligned}\tag{3.17}$$

provided that

$$\int_{T_0}^t\frac{\|\nabla u(\tau)\|_{\dot{M}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|\nabla u\|_{L^\infty})}d\tau<\varepsilon\ll 1,\tag{3.18}$$

where $C_0=\|\nabla u(T_0)\|_{L^2}^2+\|\nabla w(T_0)\|_{L^2}^2$.

Applying ∇^m to the first equation of (1.1), then multiplying the resulting equation by $\nabla^m u$ and using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla^m u(t)\|_{L^2}^2+(\mu+\chi)\left\|\nabla^{m+1}u(t)\right\|_{L^2}^2=-\int_{\mathbb{R}^3}\nabla^m(u\cdot\nabla u)\nabla^m u\,dx+\chi\int_{\mathbb{R}^3}\nabla^m(\nabla\times w)\nabla^m u\,dx.\tag{3.19}$$

Likewise, from the second equation of (1.1), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^m w(t)\|_{L^2}^2+\gamma\left\|\nabla^{m+1}w(t)\right\|_{L^2}^2+\kappa\|\nabla^m\nabla\cdot w\|_{L^2}^2+2\chi\|\nabla^m w(t)\|_{L^2}^2\\&\quad=-\int_{\mathbb{R}^3}\nabla^m(u\cdot\nabla w)\nabla^m w\,dx+\chi\int_{\mathbb{R}^3}\nabla^m(\nabla\times u)\nabla^m w\,dx.\end{aligned}\tag{3.20}$$

Using $\nabla \cdot u = 0$ and (3.19) and (3.20), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^m u(t)\|_{L^2}^2 + \|\nabla^m w(t)\|_{L^2}^2\right) + (\mu+\chi)\left\|\nabla^{m+1}u(t)\right\|_{L^2}^2 \\
&\quad + \gamma\left\|\nabla^{m+1}w(t)\right\|_{L^2}^2 + \kappa\|\nabla^m \nabla\cdot w\|_{L^2}^2 + 2\chi\|\nabla^m w(t)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3}[\nabla^m(u\cdot\nabla u) - u\cdot\nabla\nabla^m u]\nabla^m u\,dx + \chi\int_{\mathbb{R}^3}\nabla^m(\nabla\times w)\nabla^m u\,dx \\
&\quad - \int_{\mathbb{R}^3}[\nabla^m(u\cdot\nabla w) - u\cdot\nabla\nabla^m w]\nabla^m w\,dx + \chi\int_{\mathbb{R}^3}\nabla^m(\nabla\times u)\nabla^m w\,dx.
\end{aligned} \tag{3.21}$$

In what follows, for simplicity, we will set $m = 3$.

By Hölder's inequality, (2.11), (2.12), and Young's inequality, we obtain

$$\begin{aligned}
&-\int_{\mathbb{R}^3}\left[\nabla^3(u\cdot\nabla u) - u\cdot\nabla\nabla^3 u\right]\nabla^3 u dx \\
&\quad\le \left\|\nabla^3(u\cdot\nabla u) - u\cdot\nabla\nabla^3 u\right\|_{L^2}\left\|\nabla^3 u\right\|_{L^2} \\
&\quad\le C\|\nabla u\|_{L^4}\left\|\nabla^3 u\right\|_{L^4}\left\|\nabla^3 u\right\|_{L^2} \\
&\quad\le C\|\nabla u\|_{L^2}^{3/4}\left\|\nabla^4 u\right\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{1/12}\left\|\nabla^4 u\right\|_{L^2}^{11/12}\|\nabla u\|_{L^2}^{1/3}\left\|\nabla^4 u\right\|_{L^2}^{2/3} \\
&\quad\le C\|\nabla u\|_{L^2}^{7/6}\left\|\nabla^4 u\right\|_{L^2}^{11/6} \\
&\quad\le \frac{\mu}{4}\left\|\nabla^4 u\right\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{14} \\
&\quad\le \frac{\mu}{4}\left\|\nabla^4 u\right\|_{L^2}^2 + C(e+\vartheta(t))^{14C\varepsilon},
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&-\int_{\mathbb{R}^3}\left[\nabla^3(u\cdot\nabla w) - u\cdot\nabla\nabla^3 w\right]\nabla^3 w\,dx \\
&\quad\le \left\|\nabla^3(u\cdot\nabla w) - u\cdot\nabla\nabla^3 w\right\|_{L^2}\left\|\nabla^3 w\right\|_{L^2} \\
&\quad\le C\|\nabla u\|_{L^4}\left\|\nabla^3 w\right\|_{L^4}\left\|\nabla^3 w\right\|_{L^2} + \|\nabla w\|_{L^4}\left\|\nabla^3 u\right\|_{L^4}\left\|\nabla^3 w\right\|_{L^2} \\
&\quad\le C\|\nabla u\|_{L^2}^{3/4}\left\|\nabla^4 u\right\|_{L^2}^{1/4}\|\nabla w\|_{L^2}^{1/12}\left\|\nabla^4 w\right\|_{L^2}^{11/12}\|\nabla w\|_{L^2}^{1/3}\left\|\nabla^4 w\right\|_{L^2}^{2/3} \\
&\qquad + C\|\nabla w\|_{L^2}^{3/4}\left\|\nabla^4 w\right\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{1/12}\left\|\nabla^4 u\right\|_{L^2}^{11/12}\|\nabla w\|_{L^2}^{1/3}\left\|\nabla^4 w\right\|_{L^2}^{2/3} \\
&\quad\le \frac{\mu}{4}\left\|\nabla^4 u\right\|_{L^2}^2 + \frac{\gamma}{2}\left\|\nabla^4 w\right\|_{L^2}^2 + C\|\nabla u\|_{L^2}^9\|\nabla w\|_{L^2}^5 + C\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{13} \\
&\quad\le \frac{\mu}{4}\left\|\nabla^4 u\right\|_{L^2}^2 + \frac{\gamma}{2}\left\|\nabla^4 w\right\|_{L^2}^2 + C(e+\vartheta(t))^{14C\varepsilon}.
\end{aligned} \tag{3.23}$$

It follows from integration by parts and Cauchy's inequality that

$$\chi\int_{\mathbb{R}^3}\nabla^3(\nabla\times w)\nabla^3 u\,dx + \chi\int_{\mathbb{R}^3}\nabla^3(\nabla\times u)\nabla^3 w\,dx \le \chi\left\|\nabla^4 u(t)\right\|_{L^2}^2 + \chi\left\|\nabla^3 w(t)\right\|_{L^2}^2. \tag{3.24}$$

Combining (3.21)-(3.24) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^m u(t)\|_{L^2}^2+\|\nabla^m w(t)\|_{L^2}^2\right)+(\mu+\chi)\left\|\nabla^{m+1}u(t)\right\|_{L^2}^2\\
&\qquad+\gamma\left\|\nabla^{m+1}w(t)\right\|_{L^2}^2+\kappa\|\nabla^m\nabla\cdot w\|_{L^2}^2+2\chi\|\nabla^m w(t)\|_{L^2}^2\\
&\leq C(e+\vartheta(t))^{14C\varepsilon},\qquad \forall T_0\leq t<T.
\end{aligned}\tag{3.25}$$

Taking ε small enough yields

$$\frac{d}{dt}\left(\left\|\nabla^3 u\right\|_{L^2}^2+\left\|\nabla^3 w\right\|_{L^2}^2\right)\leq C(e+\vartheta(t)),\qquad T_0\leq t<T,\tag{3.26}$$

for all $T_0 \leq t < T$.

Integrating (3.26) with respect to time from T_0 to τ, we have

$$e+\left\|\nabla^3 u(\tau)\right\|_{L^2}^2+\left\|\nabla^3 w(\tau)\right\|_{L^2}^2\leq e+\left\|\nabla^3 u(T_0)\right\|_{L^2}^2+\left\|\nabla^3 w(T_0)\right\|_{L^2}^2+C_2\int_{T_0}^{\tau}(e+\vartheta(s))ds.\tag{3.27}$$

Owing to (3.27), we get

$$e+\vartheta(t)\leq e+\left\|\nabla^3 u(T_0)\right\|_{L^2}^2+\left\|\nabla^3 w(T_0)\right\|_{L^2}^2+C_2\int_{T_0}^{t}(e+\vartheta(\tau))d\tau\tag{3.28}$$

For all $T_0 \leq t < T$, with help of Gronwall inequality and (3.28), we have

$$e+\left\|\nabla^3 u(t)\right\|_{L^2}^2+\left\|\nabla^3 w(t)\right\|_{L^2}^2\leq C,\tag{3.29}$$

where C depends on $\|\nabla u(T_0)\|_{L^2}^2+\|\nabla w(T_0)\|_{L^2}^2$. From (3.29) and (3.5), we know that $(u,w)\in L^\infty(0,T;H^3(\mathbb{R}^3))$. Thus, (u,w) can be extended smoothly beyond $t=T$. We have completed the proof of Theorem 1.1. □

4. Proof of Theorem 1.3

We start to estimate every term on the right of (3.9). Using integration by parts, Hölder inequality, (2.8) and Young inequality, we obtain

$$\begin{aligned} -\int_{\mathbb{R}^3} &\nabla(u\cdot\nabla u)\nabla u\,dx \\ &\leq \left\|\nabla^2 u\right\|_{L^2}\|u\nabla u\|_{L^2} \\ &\leq C\|u\|_{\dot{M}_{2,3/r}}\|\nabla u\|_{\dot{B}^r_{2,1}}\left\|\nabla^2 u\right\|_{L^2} \\ &\leq C\|u\|_{\dot{M}_{2,3/r}}\|\nabla u\|^{1-r}_{\dot{B}^r_{2,1}}\left\|\nabla^2 u\right\|^{1+r}_{L^2} \\ &\leq \frac{\mu}{2}\left\|\nabla^2 u(t)\right\|^2_{L^2} + C\|u\|^{2/(1-r)}_{\dot{M}_{2,3/r}}\|\nabla u\|^2_{L^2}. \end{aligned} \tag{4.1}$$

Similarly, we have the following estimate

$$\begin{aligned} -\int_{\mathbb{R}^3} &\nabla(u\cdot\nabla w)\nabla w\,dx \\ &\leq \left\|\nabla^2 w\right\|_{L^2}\|u\nabla w\|_{L^2} \\ &\leq C\|u\|_{\dot{M}_{2,3/r}}\|\nabla w\|_{\dot{B}^r_{2,1}}\left\|\nabla^2 w\right\|_{L^2} \\ &\leq C\|u\|_{\dot{M}_{2,3/r}}\|\nabla w\|^{1-r}_{\dot{B}^r_{2,1}}\left\|\nabla^2 w\right\|^{1+r}_{L^2} \\ &\leq \frac{\gamma}{2}\left\|\nabla^2 w(t)\right\|^2_{L^2} + C\|u\|^{2/(1-r)}_{\dot{M}_{2,3/r}}\|\nabla w\|^2_{L^2}. \end{aligned} \tag{4.2}$$

Thus from (3.9), (3.10), (4.1), and (4.2), we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\|\nabla u(t)\|^2_{L^2} + \|\nabla w(t)\|^2_{L^2}\right) + \mu\left\|\nabla^2 u(t)\right\|^2_{L^2} + \gamma\left\|\nabla^2 w\right\|^2_{L^2} + \kappa\|\nabla\nabla\cdot w\|^2_{L^2} + \chi\|\nabla w\|^2_{L^2} \\ &\quad\leq C\frac{\|u\|^{2/(1-r)}_{\dot{M}_{2,3/r}}}{1+\ln(e+\|u\|_{L^\infty})}\left(\|\nabla u\|^2_{L^2} + \|\nabla w\|^2_{L^2}\right)(1+\ln(e+\vartheta(t))), \quad \forall T_0 \leq t < T. \end{aligned} \tag{4.3}$$

It follows from (4.3) and Gronwall's inequality that

$$\begin{aligned} &\|\nabla u(t)\|^2_{L^2} + \|\nabla w(t)\|^2_{L^2} \\ &\quad\leq \left(\|\nabla u(T_0)\|^2_{L^2} + \|\nabla w(T_0)\|^2_{L^2}\right)\exp\left\{C(1+\ln(e+\vartheta(t)))\int_{T_0}^{t}\frac{\|u(\tau)\|^{2/(1-r)}_{\dot{M}_{2,3/r}}}{1+\ln(e+\|u\|_{L^\infty})}d\tau\right\} \\ &\quad\leq C_0\exp\{C\varepsilon[1+\ln(e+\vartheta(t))]\} \\ &\quad\leq C_0\exp\{2C\varepsilon[\ln(e+\vartheta(t))]\} \\ &\quad\leq C_0(e+\vartheta(t))^{2C\varepsilon}, \end{aligned} \tag{4.4}$$

provided that

$$\int_{T_0}^{t} \frac{\|u(\tau)\|_{\dot{M}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|u\|_{L^\infty})} d\tau < \varepsilon \ll 1, \tag{4.5}$$

where $C_0 = \|\nabla u(T_0)\|_{L^2}^2 + \|\nabla w(T_0)\|_{L^2}^2$.

From (4.4), H^m estimate for Theorem 1.3 is same as that for Theorem 1.1. Thus, Theorem 1.3 is proved.

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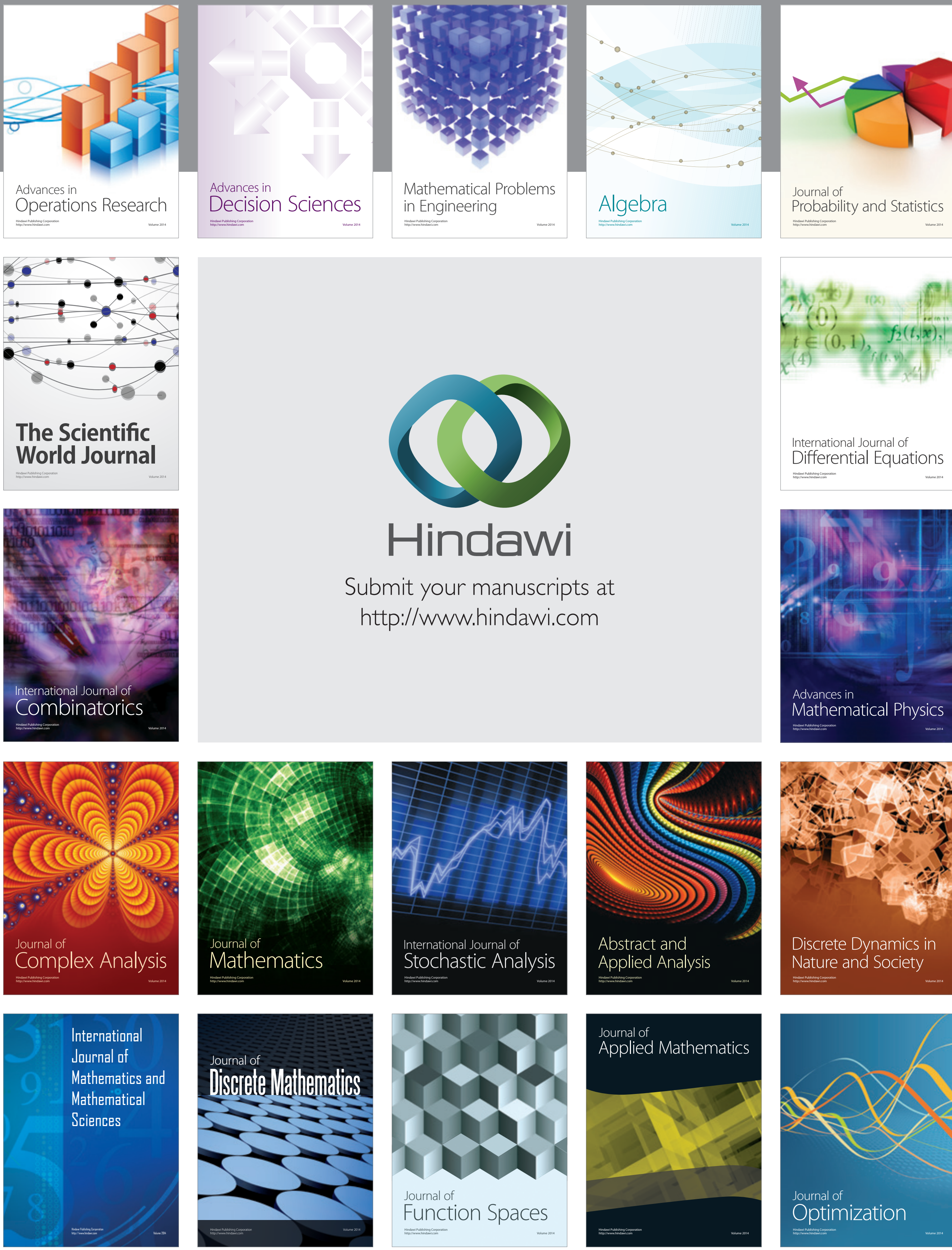
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