Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 634927, 15 pages
doi:10.1155/2012/634927

Research Article

Hybrid Algorithms of Nonexpansive Semigroups for Variational Inequalities

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Received 19 March 2012; Accepted 30 April 2012

Academic Editor: Giuseppe Marino

Two hybrid algorithms for the variational inequalities over the common fixed points set of nonexpansive semigroups are presented. Strong convergence results of these two hybrid algorithms have been obtained in Hilbert spaces. The results improve and extend some corresponding results in the literature.

1. Introduction

Let H be a real Hilbert space and C a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for every $x, y \in C$. A family $S = \{T(\tau) : 0 < \tau < \infty\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$,

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$,

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C, s \geq 0$,

(iv) for all $x \in C, s \mapsto T(s)x$ is continuous.

We denote by $\mathrm{Fix}(S)$ the set of all common fixed points of S, that is, $\mathrm{Fix}(S) = \bigcap_{0\leq\tau<\infty} \mathrm{Fix}(T(\tau))$. It is known that $\mathrm{Fix}(S)$ is closed and convex.

Approximation of fixed points of nonexpansive mappings has been considered extensively by many authors, see, for instance, [1–18]. Nonlinear ergodic theorem for nonexpansive semigroups have been researched by some authors, see, for example, [19–23]. Our main purpose in the present paper is devoted to finding the common fixed points of nonexpansive semigroups.

Let $F : C \to C$ be a nonlinear operator. The variational inequality problem is formulated as finding a point $x^* \in C$ such that

$$\mathrm{VI}(F, C) : \langle Fx^*, v - x^* \rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

Now it is well known that VI problem is an interesting problem and it covers as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance. Several numerical methods including the projection and its variant forms have been developed for solving the variational inequalities and related problems, see [24–41].

It is clear that the $\mathrm{VI}(F, C)$ is equivalent to the fixed point equation

$$x^* = P_C\left[x^* - \mu F(x^*)\right], \tag{1.3}$$

where P_C is the projection of H onto the closed convex set C and $\mu > 0$ is an arbitrarily fixed constant. So, fixed point methods can be implemented to find a solution of the $\mathrm{VI}(F, C)$ provided F satisfies some conditions and $\mu > 0$ is chosen appropriately. The fixed point formulation (1.3) involves the projection P_C, which may not be easy to compute, due to the complexity of the convex set C. In order to reduce the complexity probably caused by the projection P_C, Yamada [24] (see also [42]) recently introduced a hybrid steepest-descent method for solving the $\mathrm{VI}(F, C)$.

Assume that F is an η-strongly monotone and κ-Lipschitzian mapping with $\kappa > 0$, $\eta > 0$ on C. An equally important problem is how to find an approximate solution of the $\mathrm{VI}(F, C)$ if any. A great deal of effort has been done in this problem; see [43, 44].

Take a fixed number μ such that $0 < \mu < 2\eta/\kappa^2$. Assume that a sequence $\{\lambda_n\}$ of real numbers in $(0, 1)$ satisfies the following conditions:

(C1) $\lim_{n\to\infty}\lambda_n = 0$,

(C2) $\sum_{n=0}^{\infty} \lambda_n = \infty$,

(C3) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_n^2 = 0$.

Starting with an arbitrary initial guess $x_0 \in H$, one can generate a sequence $\{x_n\}$ by the following algorithm:

$$x_{n+1} = Tx_n - \lambda_{n+1}\mu F(Tx_n), \quad n \geq 0. \tag{1.4}$$

Yamada [24] proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the $\mathrm{VI}(F, C)$. Xu and Kim [30] proved the strong convergence of $\{x_n\}$

to the unique solution of the VI(F, C) if $\{\lambda_n\}$ satisfies conditions (C1), (C2), and (C4): $\lim_{n\to\infty}\lambda_n/\lambda_{n+1} = 1$, or equivalently, $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1} = 1$. Recently, Yao et al. [25] presented the following hybrid algorithm:

$$\begin{aligned} y_n &= x_n - \lambda_n F(x_n), \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n W_n y_n, \quad n \geq 0, \end{aligned} \tag{1.5}$$

where F is a κ-Lipschitzian and η-strongly monotone operator on H and W_n is a W-mapping. It is shown that the sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.5) converge strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, which solves the following variational inequality:

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in \bigcap_{n=1}^{\infty} F(T_n). \tag{1.6}$$

Very recently, Wang [26] proved that the sequence $\{y_n\}$ generated by the iterative algorithm (1.5) converges to a common fixed point of an infinite family of nonexpansive mappings under some weaker assumptions.

Motivated and inspired by the above works, in this paper, we introduce two hybrid algorithms for finding a common fixed point of a nonexpansive semigroup $\{T(\tau)\}_{\tau\geq 0}$ in Hilbert spaces. We prove that the presented algorithms converge strongly to a common fixed point x^* of $\{T(\tau)\}_{\tau\geq 0}$. Such common fixed point x^* is the unique solution of some variational inequality in Hilbert spaces.

2. Preliminaries

In this section, we will collect some basic concepts and several lemmas that will be used in the next section.

Suppose that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For the sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ means that $\{x_n\}$ converges strongly to x. We denote by $\omega_w(x_n)$ the weak ω-limit set of $\{x_n\}$, that is

$$\omega_w(x_n) = \left\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\right\}. \tag{2.1}$$

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $F : C \to C$ is called κ-Lipschitzian if there exists a positive constant κ such that

$$\|Fx - Fy\| \leq \kappa\|x - y\|, \quad \forall x, y \in C. \tag{2.2}$$

F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C. \tag{2.3}$$

The following equalities are well known:

$$\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2,$$
$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.4}$$

for all $x, y \in H$ and $\lambda \in [0,1]$ (see [45]).

In the sequel, we will make use of the following well-known lemmas.

Lemma 2.1 (see [46]). *Let C be a nonempty bounded closed convex subset of H and let $S = \{T(s) \mid 0 \le s < \infty\}$ be a nonexpansive semigroup on C. Then, for any $h \ge 0$,*

$$\lim_{t\to\infty}\sup_{x\in C}\left\|\frac{1}{t}\int_0^t T(s)x\,ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x\,ds\right)\right\| = 0. \tag{2.5}$$

Lemma 2.2 (see [47]). *Assume that $T : H \to H$ is a nonexpansive mapping. If T has a fixed point, then $I-T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in H weakly converging to some $x \in H$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here, I is the identity operator of H.*

Lemma 2.3 (see [27]). *Let $\{\gamma_n\}$ be a real sequence satisfying $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$. Assume that $\{x_n\}$ and $\{z_n\}$ are bounded sequences in Banach space E, which satisfy the following condition: $x_{n+1} = (1-\gamma_n)x_n + \gamma_n z_n$. If $\limsup_{n\to\infty}(\|z_{n+1}-z_n\|) - \|x_{n+1}-x_n\|) \le 0$, then $\lim_{n\to\infty}\|x_n - z_n\| = 0$.*

Lemma 2.4 (see [48]). *Let F be a κ-Lipschitzian and η-strongly monotone operator on a Hilbert space H with $0 < \eta \le \kappa$ and $0 < t < \eta/\kappa^2$. Then, $S = (I - tF) : H \to H$ is a contraction with contraction coefficient $\tau_t = \sqrt{1 - t(2\eta - t\kappa^2)}$.*

Lemma 2.5 (see [49]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \le (1-\lambda_n)a_n + \lambda_n\delta_n + \gamma_n, \quad n \ge 0, \tag{2.6}$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\lambda_n \subset [0,1]$ *and* $\sum_{n=0}^{\infty}\lambda_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=0}^{\infty}\lambda_n\delta_n < \infty$,

(iii) $\gamma_n \ge 0\ (n \ge 0)$, $\sum_{n=0}^{\infty}\gamma_n < \infty$.

Then, $\lim_{n\to\infty}a_n = 0$.

3. Main Results

In this section we will show our main results.

Theorem 3.1. *Let H be a real Hilbert space. Let $S = \{T(\tau) \mid 0 \le \tau < \infty\} : H \to H$ be a nonexpansive semigroup such that $\mathrm{Fix}(S) \ne \emptyset$. Let F be a κ-Lipschitzian and η-strongly monotone* *operator on H with $0 < \eta \le \kappa$. Let $\{\gamma_t\}_{0<t<1}$ be a continuous net of positive real numbers such that $\lim_{t\to 0+}\gamma_t = +\infty$. Putting $\tau_t = \sqrt{1 - t(2\eta - t\kappa^2)}$, for each $t \in (0, \eta/\kappa^2)$, let the net $\{x_t\}$ be defined by the following implicit scheme:*

$$x_t = \frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)[(I - tF)x_t]d\tau. \tag{3.1}$$

Then, as $t \to 0^+$, the net $\{x_t\}$ converges strongly to a fixed point x^* of S, which is the unique solution of the following variational inequality:

$$\langle Fx^*, x^* - u\rangle \le 0, \quad \forall u \in \mathrm{Fix}(S). \tag{3.2}$$

Proof. First, we note that the net $\{x_t\}$ defined by (3.1) is well defined. We define a mapping

$$P_t x := \frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)[(I - tF)x]d\tau, \quad t \in \left(0, \frac{\eta}{\kappa^2}\right), \ x \in H. \tag{3.3}$$

It follows that

$$\begin{aligned} \|P_t x - P_t y\| &\le \frac{1}{\gamma_t}\int_0^{\gamma_t} \|T(\tau)[(I - tF)x] - T(\tau)[(I - tF)y]\|d\tau \\ &\le \|(I - tF)x - (I - tF)y\|. \end{aligned} \tag{3.4}$$

Obviously, P_t is a contraction. Indeed, from Lemma 2.4, we have

$$\|P_t x - P_t y\| \le \|(I - tF)x - (I - tF)y\| \le \|x - y\|, \tag{3.5}$$

for all $x, y \in C$. So it has a unique fixed point. Therefore, the net $\{x_t\}$ defined by (3.1) is well defined.

We prove that $\{x_t\}$ is bounded. Taking $u \in \mathrm{Fix}(S)$ and using Lemma 2.4, we have

$$\begin{aligned} \|x_t - u\| &= \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)[(I - tF)x_t]d\tau - u\right\| \\ &= \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)[(I - tF)x_t]d\tau - \frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)u d\tau\right\| \\ &\le \frac{1}{\gamma_t}\int_0^{\gamma_t} \|T(\tau)[(I - tF)x_t]d\tau - T(\tau)u\|d\tau \\ &\le \|(I - tF)x_t - u\| \\ &\le \|(I - tF)x_t - (I - tF)u - tFu\| \\ &\le \tau_t\|x_t - u\| + t\|Fu\|. \end{aligned} \tag{3.6}$$

It follows that

$$\|x_t - u\| \le \frac{t}{1-\tau_t}\|Fu\|. \tag{3.7}$$

Observe that

$$\lim_{t\to 0^+}\frac{t}{1-\tau_t} = \frac{1}{\eta}. \tag{3.8}$$

Thus, (3.7) and (3.8) imply that the net $\{x_t\}$ is bounded for small enough t. Without loss of generality, we may assume that the net $\{x_t\}$ is bounded for all $t \in (0, \eta/\kappa^2)$. Consequently, we deduce that $\{Fx_t\}$ is also bounded.

On the other hand, from (3.1), we have

$$\begin{aligned}
\|x_t - T(\tau)x_t\| &\le \left\| T(\tau)x_t - T(\tau)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right)\right\| \\
&\quad + \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau - x_t\right\| + \left\| T(\tau)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right\| \\
&\le 2\left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau - x_t\right\| \\
&\quad + \left\| T(\tau)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right\| \\
&= 2\left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau - \frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)[(I-tF)]x_t d\tau\right\| \\
&\quad + \left\| T(\tau)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right\| \\
&\le 2\frac{1}{\gamma_t}\int_0^{\gamma_t}\|T(\tau)x_t - T(\tau)[(I-tF)x_t]\|d\tau \\
&\quad + \left\| T(\tau)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right\| \\
&\le 2t\|Fx_t\| + \left\| T(\tau)\left(\frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right) - \frac{1}{\gamma_t}\int_0^{\gamma_t} T(\tau)x_t d\tau\right\|.
\end{aligned} \tag{3.9}$$

This together with Lemma 2.1 implies that

$$\lim_{t\to 0+}\|x_t - T(\tau)x_t\| = 0. \tag{3.10}$$

Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $\tilde{x} \in C$. Noticing (3.10), we can use Lemma 2.2 to get $\tilde{x} \in \mathrm{Fix}(S)$.

Again, from (3.1), we have

$$\begin{aligned}\|x_t-u\|^2&=\left\|\frac{1}{\gamma_t}\int_0^{\gamma_t}T(\tau)[(I-tF)x_t]d\tau-u\right\|^2\\&\le\left\|\frac{1}{\gamma_t}\int_0^{\gamma_t}[T(\tau)[(I-tF)x_t]-T(\tau)u]d\tau\right\|^2\\&\le\|(I-tF)x_t-(I-tF)u-tFu\|^2\\&\le\tau_t^2\|x_t-u\|^2+t^2\|Fu\|^2+2t\langle(I-tF)u-(I-tF)x_t,Fu\rangle\\&\le\tau_t\|x_t-u\|^2+t^2\|Fu\|^2+2t\langle u-x_t,Fu\rangle+2t^2\langle Fx_t-Fu,Fu\rangle\\&\le\tau_t\|x_t-u\|^2+t^2\|Fu\|^2+2t\langle u-x_t,Fu\rangle+2\kappa t^2\|x_t-u\|\|Fu\|.\end{aligned}\tag{3.11}$$

Therefore,

$$\|x_t-u\|^2\le\frac{t^2}{1-\tau_t}\|Fu\|^2+\frac{2t}{1-\tau_t}\langle u-x_t,Fu\rangle+\frac{2t^2\kappa}{1-\tau_t}\|x_t-u\|\|Fu\|.\tag{3.12}$$

It follows that

$$\|x_n-\tilde{x}\|^2\le\frac{t_n^2}{1-\tau_{t_n}}\|F\tilde{x}\|^2+\frac{2t_n}{1-\tau_{t_n}}\langle\tilde{x}-x_n,F\tilde{x}\rangle+\frac{2t_n^2\kappa}{1-\tau_{t_n}}\|x_n-\tilde{x}\|\|F\tilde{x}\|.\tag{3.13}$$

Thus, $x_n\rightharpoonup\tilde{x}$ implies that $x_n\to\tilde{x}$.

Again, from (3.12), we obtain

$$\|x_n-u\|^2\le\frac{t_n^2}{1-\tau_{t_n}}\|Fu\|^2+\frac{2t_n}{1-\tau_{t_n}}\langle u-x_n,Fu\rangle+\frac{2t_n^2\kappa}{1-\tau_{t_n}}\|x_n-u\|\|Fu\|.\tag{3.14}$$

It is clear that $\lim_{n\to\infty}(t_n^2/(1-\tau_{t_n}))=0$, $\lim_{n\to\infty}(2t_n/(1-\tau_{t_n}))=2/\eta$, and $\lim_{n\to\infty}(2t_n^2\kappa/(1-\tau_{t_n}))=0$. We deduce immediately from (3.14) that

$$\langle Fu,\tilde{x}-u\rangle\le 0,\tag{3.15}$$

which is equivalent to its dual variational inequality

$$\langle F\tilde{x},\tilde{x}-u\rangle\le 0.\tag{3.16}$$

That is, $\tilde{x}\in\mathrm{Fix}(S)$ is a solution of the variational inequality (3.2).

Suppose that $x^*\in\mathrm{Fix}(S)$ and $\tilde{x}\in\mathrm{Fix}(S)$ both are solutions to the variational inequality (3.2); then

$$\begin{aligned}&\langle Fx^*,x^*-\tilde{x}\rangle\le 0,\\&\langle F\tilde{x},\tilde{x}-x^*\rangle\le 0.\end{aligned}\tag{3.17}$$

Adding up (3.17) and the last inequality yields

$$\langle Fx^*-F\tilde{x},x^*-\tilde{x}\rangle\le 0.\tag{3.18}$$

The strong monotonicity of F implies that $x^* = \tilde{x}$ and the uniqueness is proved. Later, we use $x^* \in \mathrm{Fix}(S)$ to denote the unique solution of (3.2).

Therefore, $\tilde{x} = x^*$ by uniqueness. In a nutshell, we have shown that each cluster point of $\{x_t\}(t \to 0)$ equals x^*. Hence $x_t \to x^*$ as $t \to 0$. This completes the proof. □

Next we introduce an explicit algorithm for finding a solution of the variational inequality (3.2).

Algorithm 3.2. For given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= x_n - \lambda_n F(x_n), \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(\tau)y_n d\tau, \quad n \geq 0, \end{aligned} \tag{3.19}$$

where $\{\lambda_n\}$ and $\{t_n\}$ are sequences in $(0,\infty)$ and $\{\alpha_n\}$ is a sequence in $[0,1]$.

Theorem 3.3. *Let H be a real Hilbert space. Let F be a κ-Lipschitzian and η-strongly monotone operator on H with $0 < \eta \leq \kappa$. Let $S = \{T(\tau) \mid 0 \leq \tau < \infty\} : H \to H$ be a nonexpansive semigroup with* $\mathrm{Fix}(S) \neq \emptyset$. *Assume that*

(i) $\limsup_{n\to\infty}\lambda_n < \eta/\kappa^2$ *and* $\sum_{n=1}^{\infty}\lambda_n = \infty$,

(ii) $\lim_{n\to\infty}t_n = \infty$ *and* $\lim_{n\to\infty}(t_{n+1}/t_n) = 1$,

(iii) $0 < \gamma \leq \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$, *for some* $\gamma \in (0,1)$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.19) *converge strongly to $x^* \in \mathrm{Fix}(S)$ if and only if $\lambda_n F(x_n) \to 0$, where x^* solves the variational inequality* (3.2).

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that $\lambda_n F(x_n) \to 0$. First, we show that x_n is bounded. In fact, letting $u \in \mathrm{Fix}(S)$, we have

$$\begin{aligned} \|x_{n+1} - u\| &= \left\|(1-\alpha_n)y_n + \frac{\alpha_n}{t_n}\int_0^{t_n} T(\tau)y_n d\tau - u\right\| \\ &= \left\|(1-\alpha_n)(y_n - u) + \alpha_n\left(\frac{1}{t_n}\int_0^{t_n} T(\tau)y_n d\tau - u\right)\right\| \\ &\leq (1-\alpha_n)\|y_n - u\| + \alpha_n\left\|\frac{1}{t_n}\int_0^{t_n} T(\tau)y_n d\tau - u\right\| \\ &\leq (1-\alpha_n)\|y_n - u\| + \alpha_n\frac{1}{t_n}\int_0^{t_n}\|T(\tau)y_n - T(\tau)u\| d\tau \\ &\leq (1-\alpha_n)\|y_n - u\| + \alpha_n\|y_n - u\| \\ &= \|y_n - u\|. \end{aligned} \tag{3.20}$$

From condition (i), without loss of generality, we can assume that $\lambda_n \le a < \eta/\kappa^2$ for all n. By (3.19) and Lemma 2.4, we have

$$\begin{aligned} \|y_n - u\| &= \|x_n - \lambda_n F(x_n) - u\| \\ &= \|(I - \lambda_n F)x_n - (I - \lambda_n F)u - \lambda_n F u\| \\ &\le \tau_{\lambda_n}\|x_n - u\| + \lambda_n \|Fu\|, \end{aligned} \tag{3.21}$$

where $\tau_{\lambda_n} = \sqrt{1 - \lambda_n(2\eta - \lambda_n\kappa^2)} \in (0,1)$.

Then, from (3.20) and (3.21), we obtain

$$\begin{aligned} \|x_{n+1} - u\| &\le \tau_{\lambda_n}\|x_n - u\| + \lambda_n\|Fu\| \\ &= [1 - (1 - \tau_{\lambda_n})]\|x_n - u\| + (1 - \tau_{\lambda_n})\frac{\lambda_n}{1 - \tau_{\lambda_n}}\|Fu\| \\ &\le \max\left\{\|x_n - u\|, \frac{\lambda_n\|Fu\|}{1 - \tau_{\lambda_n}}\right\}. \end{aligned} \tag{3.22}$$

Since $\lim_{n\to\infty}(\lambda_n/(1 - \tau_{\lambda_n})) = 1/\eta$, we have by induction

$$\|x_{n+1} - u\| \le \max\{\|x_0 - u\|, M_1\|Fu\|\}, \tag{3.23}$$

where $M_1 = \sup_n\{\lambda_n/(1 - \tau_{\lambda_n})\} < \infty$. Hence, $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{T(\tau)y_n\}$, and $\{Fx_n\}$ are all bounded.

Define $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z_n$ for all n. Observe that

$$\begin{aligned} \|z_{n+1} - z_n\| &= \left\| \frac{x_{n+2} - (1 - \alpha_{n+1})x_{n+1}}{\alpha_{n+1}} - \frac{x_{n+1} - (1 - \alpha_n)x_n}{\alpha_n} \right\| \\ &= \left\| \frac{(1 - \alpha_{n+1})y_{n+1} + \alpha_{n+1}(1/t_{n+1})\int_0^{t_{n+1}} T(\tau)y_{n+1}d\tau - (1 - \alpha_{n+1})x_{n+1}}{\alpha_{n+1}} \right. \\ &\quad \left. - \frac{(1 - \alpha_n)y_n + \alpha_n(1/t_n)\int_0^{t_n} T(\tau)y_n d\tau - (1 - \alpha_n)x_n}{\alpha_n} \right\| \\ &= \left\| \frac{\alpha_{n+1}(1/t_{n+1})\int_0^{t_{n+1}} T(\tau)y_{n+1}d\tau - (1 - \alpha_{n+1})\lambda_{n+1}F(x_{n+1})}{\alpha_{n+1}} \right. \\ &\quad \left. - \frac{\alpha_n(1/t_n)\int_0^{t_n} T(\tau)y_n d\tau - (1 - \alpha_n)\lambda_n F(x_n)}{\alpha_n} \right\| \\ &\le \frac{1 - \alpha_{n+1}}{\alpha_{n+1}}\|\lambda_{n+1}F(x_{n+1})\| + \frac{1 - \alpha_n}{\alpha_n}\|\lambda_n F(x_n)\| \\ &\quad + \left\| \frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(\tau)y_{n+1}d\tau - \frac{1}{t_n}\int_0^{t_n} T(\tau)y_n d\tau \right\|. \end{aligned} \tag{3.24}$$

Next, we estimate $\|(1/t_{n+1})\int_0^{t_{n+1}} T(\tau)y_{n+1}d\tau - (1/t_n)\int_0^{t_n} T(\tau)y_n d\tau\|$. As a matter of fact, we have

$$\begin{aligned}
&\left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(\tau)y_{n+1}d\tau - \frac{1}{t_n}\int_0^{t_n} T(\tau)y_n d\tau\right\| \\
&\quad\le \left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(\tau)y_{n+1}d\tau - \frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(\tau)y_n d\tau\right\| + \left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(\tau)y_n d\tau - \frac{1}{t_n}\int_0^{t_n} T(\tau)y_n d\tau\right\| \\
&\quad\le \|y_{n+1}-y_n\| + \left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(\tau)y_n d\tau - \frac{1}{t_n}\int_0^{t_n} T(\tau)y_n d\tau\right\| \\
&\quad\le \|y_{n+1}-y_n\| + \left|\frac{1}{t_{n+1}} - \frac{1}{t_n}\right|\left\|\int_0^{t_n} T(\tau)y_n d\tau\right\| + \frac{1}{t_{n+1}}\left\|\int_{t_n}^{t_{n+1}} T(\tau)y_n d\tau\right\| \\
&\quad\le \|x_{n+1} - \lambda_{n+1}F(x_{n+1}) - x_n + \lambda_n F(x_n)\| + \left|\frac{t_n}{t_{n+1}} - 1\right| M_2 \\
&\quad\le \|x_{n+1}-x_n\| + \|\lambda_{n+1}F(x_{n+1})\| + \|\lambda_n F(x_n)\| + M_2\left|\frac{t_n}{t_{n+1}} - 1\right|,
\end{aligned} \tag{3.25}$$

where $M_2 = \sup_n\{2\|T(\tau)y_n\|\} < \infty$. From (3.24) and (3.25), we have

$$\begin{aligned}
\|z_{n+1}-z_n\| &\le \frac{1-\gamma}{\gamma}\|\lambda_{n+1}F(x_{n+1})\| + \frac{1-\gamma}{\gamma}\|\lambda_n F(x_n)\| + \|x_{n+1}-x_n\| + \|\lambda_{n+1}F(x_{n+1})\| \\
&\quad + \|\lambda_n F(x_n)\| + M_2\left|\frac{t_n}{t_{n+1}} - 1\right| \\
&\le \frac{1}{\gamma}\|\lambda_{n+1}F(x_{n+1})\| + \frac{1}{\gamma}\|\lambda_n F(x_n)\| + \|x_{n+1}-x_n\| + M_2\left|\frac{t_n}{t_{n+1}} - 1\right|.
\end{aligned} \tag{3.26}$$

Namely,

$$\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le \frac{1}{\gamma}\|\lambda_{n+1}F(x_{n+1})\| + \frac{1}{\gamma}\|\lambda_n F(x_n)\| + M_2\left|\frac{t_n}{t_{n+1}} - 1\right|. \tag{3.27}$$

Since $\lambda_n F(x_n) \to 0$ and $(t_n/t_{n+1}) - 1 \to 0$, we get

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0. \tag{3.28}$$

Consequently, by Lemma 2.3, we deduce $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Therefore,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\alpha_n\|z_n - x_n\| = 0. \tag{3.29}$$

Next, we claim that $\lim_{n\to\infty}\|x_n - T(\tau)x_n\| = 0$. Observe that

$$\begin{aligned}
\|T(\tau)x_n - x_n\| &\le \left\|T(\tau)x_n - T(\tau)\left(\frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right)\right\| + \left\|\frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau - x_n\right\| \\
&\quad + \left\|T(\tau)\left(\frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right) - \frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right\| \\
&\le 2\left\|\frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau - x_n\right\| \\
&\quad + \left\|T(\tau)\left(\frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right) - \frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right\|.
\end{aligned} \tag{3.30}$$

Note that

$$\begin{aligned}
\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right\| \\
&\le \|x_n - x_{n+1}\| + (1-\alpha_n)\left\|y_n - \frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right\| \\
&\quad + \alpha_n\left\|\frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau - \frac{1}{t_n}\int_0^{t_n} T(\tau)y_n d\tau\right\| \\
&\le \|x_n - x_{n+1}\| + (1-\alpha_n)\|y_n - x_n\| \\
&\quad + (1-\alpha_n)\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right\| + \alpha_n\|y_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \|y_n - x_n\| \\
&\quad + (1-\alpha_n)\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right\|.
\end{aligned} \tag{3.31}$$

It follows that

$$\begin{aligned}
\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(\tau)x_n d\tau\right\| &\le \frac{1}{\alpha_n}\left(\|x_n - x_{n+1}\| + \|y_n - x_n\|\right) \\
&= \frac{1}{\alpha_n}\left(\|x_n - x_{n+1}\| + \|\lambda_n F(x_n)\|\right) \\
&\longrightarrow 0.
\end{aligned} \tag{3.32}$$

By Lemma 2.1, (3.30), and (3.32), we derive

$$\lim_{n\to\infty}\|T(\tau)x_n - x_n\| = 0. \tag{3.33}$$

Next, we show that $\limsup_{n\to\infty}\langle Fx^*, x^* - x_n\rangle \le 0$, where $x^* = \lim_{n\to\infty} x_{t_n}$ and x_{t_n} is defined by $x_{t_n} = (1/t_n)\int_0^{t_n} T(\tau)[(I - t_n F)x_{t_n}]d\tau$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to ω. It is clear that $T(\tau)x_{n_k} \rightharpoonup \omega$. From Lemma 2.2, we have $\omega \in \mathrm{Fix}(S)$. Hence, by Theorem 3.1, we have

$$\limsup_{n\to\infty}\langle Fx^*, x^* - x_n\rangle = \lim_{k\to\infty}\langle Fx^*, x^* - x_{n_k}\rangle = \langle Fx^*, x^* - \omega\rangle \le 0. \tag{3.34}$$

Finally, we prove that $\{x_n\}$ converges strongly to $x^* \in \mathrm{Fix}(S)$. From (3.19), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)\left\|y_n - x^*\right\|^2 + \alpha_n\left\|\frac{1}{t_n}\int_0^{t_n} T(\tau)y_n d\tau - x^*\right\|^2 \\
&\le (1-\alpha_n)\left\|y_n - x^*\right\|^2 + \alpha_n\frac{1}{t_n}\int_0^{t_n}\left\|T(\tau)y_n - T(\tau)x^*\right\|^2 d\tau \\
&\le \left\|y_n - x^*\right\|^2 \\
&= \|x_n - \lambda_n F(x_n) - x^*\|^2 \\
&= \|(I - \lambda_n F)x_n - (I - \lambda_n F)x^* - \lambda_n Fx^*\|^2 \\
&\le \tau_{\lambda_n}^2\|x_n - x^*\|^2 + \lambda_n^2\|F(x^*)\|^2 \\
&\quad + 2\lambda_n\langle (I - \lambda_n F)x^* - (I - \lambda_n F)x_n, F(x^*)\rangle \\
&\le \tau_{\lambda_n}\|x_n - x^*\|^2 + \lambda_n^2\|F(x^*)\|^2 + 2\lambda_n\langle x^* - x_n, Fx^*\rangle \\
&\quad + 2\lambda_n\langle \lambda_n Fx_n, Fx^*\rangle - 2\lambda_n^2\|Fx^*\|^2 \\
&\le [1 - (1 - \tau_{\lambda_n})]\|x_n - x^*\|^2 + 2\lambda_n\langle x^* - x_n, Fx^*\rangle \\
&\quad + 2\lambda_n\|\lambda_n F(x_n)\|\|Fx^*\| - \lambda_n^2\|Fx^*\|^2 \\
&\le [1 - (1 - \tau_{\lambda_n})]\|x_n - x^*\|^2 \\
&\quad + (1 - \tau_{\lambda_n})\left[\frac{2\lambda_n}{1 - \tau_{\lambda_n}}\langle x^* - x_n, Fx^*\rangle + \frac{2\lambda_n\|Fx^*\|}{1 - \tau_{\lambda_n}}\|\lambda_n Fx_n\|\right] \\
&= (1 - \delta_n)\|x_n - x^*\|^2 + \delta_n\sigma_n,
\end{aligned} \tag{3.35}$$

where $\delta_n = 1 - \tau_{\lambda_n}$ and $\sigma_n = (2\lambda_n/1 - \tau_{\lambda_n})\langle x^* - x_n, Fx^*\rangle + (2\lambda_n\|Fx^*\|/(1 - \tau_{\lambda_n}))\|\lambda_n Fx_n\|$. Obviously, we can see that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. Hence, all conditions of Lemma 2.5 are satisfied. Therefore, we immediately deduce that the sequence $\{x_n\}$ converges strongly to $x^* \in \mathrm{Fix}(S)$.

Observe that

$$\left\|y_n - x^*\right\| \le \left\|y_n - x_n\right\| + \|x_n - x^*\| \le \|\lambda_n F(x_n)\| + \|x_n - x^*\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.36}$$

Consequently, it is clear that $\{y_n\}$ converges strongly to $x^* \in \mathrm{Fix}(S)$. From $x^* = \lim_{t\to 0} x_t$ and Theorem 3.1, we get that x^* is the unique solution of the variational inequality

$$\langle Fx^*, x^* - u\rangle \leq 0, \quad \forall u \in \mathrm{Fix}(S). \tag{3.37}$$

This completes the proof. □

Acknowledgments

Y. Yao was supported in part by NSFC 11071279 and NSFC 71161001-G0105. Y.-C. Liou was partially supported by NSC 100-2221-E-230-012. R. Chen was supported in part by NSFC 11071279.

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