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Research Article

Generalizations of $\mathcal{N}$-Subalgebras in BCK/BCI-Algebras Based on Point $\mathcal{N}$-Structures

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The aim of this article is to obtain more general forms than the papers of (Jun et al. (2010); Jun et al. (in press)). The notions of $\mathcal{N}$-subalgebras of types $(\in, q_k)$, $(\in, \in \vee q_k)$, and $(q, \in \vee q_k)$ are introduced, and the concepts of q_k-support and $\in \vee q_k$-support are also introduced. Several related properties are investigated. Characterizations of $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ are discussed, and conditions for an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ to be an $\mathcal{N}$-subalgebra of type $(\in, \in)$ are considered.

1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalizations of the crisp set have been conducted on the unit interval $[0, 1]$, and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fits the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [1] introduced a new function which is called negative-valued function and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Jun et al. [2] considered closed ideals in BCH-algebras based on $\mathcal{N}$-structures. To obtain more general form of an $\mathcal{N}$-subalgebra in BCK/BCI-algebras, Jun et al. [3] defined the notions of $\mathcal{N}$-subalgebras of types $(\in, \in)$, $(\in, q)$, $(\in, \in \vee q)$, $(q, \in)$, (q, q), and $(q, \in \vee q)$ and investigated related properties. They also gave conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subalgebra of type $(q, \in \vee q)$. Jun et al. provided a characterization of an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$ (see [3, 4]).

In this paper, we try to have more general form of the papers [3, 4]. We introduce the notions of $\mathcal{N}$-subalgebras of types $(\in, q_k)$, $(\in, \in \vee q_k)$, and $(q, \in \vee q_k)$. We also introduce the concepts of q_k-support and $\in \vee q_k$-support and investigate several properties. We discuss characterizations of $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$. We consider conditions for an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ to be an $\mathcal{N}$-subalgebra of type $(\in, \in)$. The important achievement of the study of $\mathcal{N}$-subalgebras of types $(\in, q_k)$, $(\in, \in \vee q_k)$, and $(q, \in \vee q_k)$ is that the notions of $\mathcal{N}$-subalgebras of types $(\in, q)$, $(\in, \in \vee q)$, and $(q, \in \vee q)$ are a special case of $\mathcal{N}$-subalgebras of types $(\in, q_k)$, $(\in, \in \vee q_k)$, and $(q, \in \vee q_k)$, and thus so many results in the papers [3, 4] are corollaries of our results obtained in this paper.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. By a BCI-algebra, we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = y * x = 0 \Rightarrow x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK*-algebra. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X) \quad (x \leq y \Longleftrightarrow x * y = 0). \tag{2.1}$$

In a BCK/BCI-algebra X, the following hold:

(a1) $(\forall x \in X)(x * 0 = x)$,

(a2) $(\forall x, y, z \in X)((x * y) * z = (x * z) * y)$,

for all $x, y, z \in X$.

A nonempty subset S of a BCK/BCI-algebras X is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$. For our convenience, the empty set $\emptyset$ is regarded as a subalgebra of X.

We refer the reader to the books [5, 6] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases}$$
$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases} \tag{2.2}$$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure, we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. In what follows, let X denote a BCK/BCI-algebras and f an $\mathcal{N}$-function on X unless otherwise specified.

Definition 2.1 (see [1]). By a subalgebra of X based on $\mathcal{N}$-function f (briefly, $\mathcal{N}$-subalgebra of X), we mean an $\mathcal{N}$-structure (X, f) in which f satisfies the following assertion:

$$(\forall x, y \in X) \quad \left(f(x * y) \leq \bigvee \{ f(x), f(y) \} \right). \tag{2.3}$$

For any $\mathcal{N}$-structure (X, f) and $t \in [-1, 0)$, the set

$$C(f; t) := \{ x \in X \mid f(x) \leq t \} \tag{2.4}$$

is called a *closed t-support* of (X, f), and the set

$$O(f; t) := \{ x \in X \mid f(x) < t \} \tag{2.5}$$

is called an *open t-support* of (X, f).

Using the similar method to the transfer principle in fuzzy theory (see [7, 8]), Jun et al. [2] considered transfer principle in $\mathcal{N}$-structures as follows.

Theorem 2.2 (see [2]; $\mathcal{N}$-transfer principle). *An $\mathcal{N}$-structure (X, f) satisfies the property $\overline{\mathcal{P}}$ if and only if for all $\alpha \in [-1, 0]$,*

$$C(f; \alpha) \neq \emptyset \Longrightarrow C(f; \alpha) \text{ satisfies the property } \mathcal{P}. \tag{2.6}$$

Lemma 2.3 (see [1]). *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of X if and only if every open t-support of (X, f) is a subalgebra of X for all $t \in [-1, 0)$.*

3. General Form of $\mathcal{N}$-Subalgebras with Type $(\in, \in \vee q)$

In what follows, let t and k denote arbitrary elements of $[-1, 0)$ and $(-1, 0]$, respectively, unless otherwise specified.

Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f(y) = \begin{cases} 0 & \text{if } y \neq x, \\ t & \text{if } y = x. \end{cases} \tag{3.1}$$

In this case, f is denoted by x_t, and we call (X, x_t) a point $\mathcal{N}$-structure. For any $\mathcal{N}$-structure (X, g), we say that a point $\mathcal{N}$-structure (X, x_t) is an $\mathcal{N}_\in$-subset (resp., $\mathcal{N}_q$-subset) of (X, g) if $g(x) \leq t$ (resp., $g(x) + t + 1 < 0$). If a point $\mathcal{N}$-structure (X, x_t) is an $\mathcal{N}_\in$-subset of (X, g) or an $\mathcal{N}_q$-subset of (X, g), we say (X, x_t) is an $\mathcal{N}_{\in \vee q}$-subset of (X, g). We say that a point $\mathcal{N}$-structure

(X, x_t) is an $\mathcal{N}_{q_k}$-subset of (X, g) if $g(x) + t - k + 1 < 0$. Clearly, every $\mathcal{N}_{q_k}$-subset with $k = 0$ is an $\mathcal{N}_q$-subset. Note that if $k, r \in (-1, 0]$ with $k < r$, then every $\mathcal{N}_{q_k}$-subset is an $\mathcal{N}_{q_r}$-subset.

Definition 3.1. An $\mathcal{N}$-structure (X, f) is called an $\mathcal{N}$-subalgebra of type

(i) $(\in, \in)$ (resp., $(\in, q)$ and $(\in, \in \vee q)$) if whenever two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_\in$-subsets of (X, f) then the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_\in$-subset (resp., $\mathcal{N}_q$-subset and $\mathcal{N}_{\in\vee q}$-subset) of (X, f).

(ii) $(q, \in)$ (resp., (q, q) and $(q, \in \vee q)$) if whenever two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_q$-subsets of (X, f) then the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_\in$-subset (resp., $\mathcal{N}_q$-subset and $\mathcal{N}_{\in\vee q}$-subset) of (X, f).

Definition 3.2. An $\mathcal{N}$-structure (X, f) is called an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ (resp., $(q, \in \vee q_k)$) if whenever two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_\in$-subsets (resp., $\mathcal{N}_q$-subsets) of (X, f) then the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in\vee q_k}$-subset of (X, f).

Example 3.3. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(3.2)

Let (X, f) be an $\mathcal{N}$-structure in which f is defined by

$$f = \begin{pmatrix} 0 & a & b & c \\ -0.6 & -0.7 & -0.3 & -0.3 \end{pmatrix}. \tag{3.3}$$

It is routine to verify that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_{-0.2})$.

Note that if $k, r \in (-1, 0]$ with $k < r$, then every $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_r)$, but the converse is not true as seen in the following example.

Example 3.4. The $\mathcal{N}$-subalgebra (X, f) of type $(\in, \in \vee q_{-0.2})$ in Example 3.3 is not of type $(\in, \in \vee q_{-0.4})$ since $(X, a_{-0.65})$ and $(X, a_{-0.68})$ are $\mathcal{N}_\in$-subsets of (X, f), but

$$\left(X, (a * a)_{\bigvee\{-0.65, -0.68\}}\right) \tag{3.4}$$

is not an $\mathcal{N}_{\in\vee q_{-0.4}}$-subset of (X, f).

Theorem 3.5. *Every $\mathcal{N}$-subalgebra of type $(\in, \in)$ is of type $(\in, \in \vee q_k)$.*

Proof. Straightforward. □

Taking $k = 0$ in Theorem 3.5 induces the following corollary.

Corollary 3.6. *Every $\mathcal{N}$-subalgebra of type $(\in, \in)$ is of type $(\in, \in \vee q)$.*

The converse of Theorem 3.5 is not true as seen in the following example.

Example 3.7. Consider the $\mathcal{N}$-subalgebra (X, f) of type $(\in, \in \vee q_{-0.2})$ which is given in Example 3.3. Then (X, f) is not an $\mathcal{N}$-subalgebra of type $(\in, \in)$ since $(X, a_{-0.65})$ and $(X, a_{-0.68})$ are $\mathcal{N}_{\in}$-subsets of (X, f), but $(X, (a * a)_{\vee\{-0.65,-0.68\}})$ is not an $\mathcal{N}_{\in}$-subset of (X, f).

Definition 3.8. An $\mathcal{N}$-structure (X, f) is called an $\mathcal{N}$-subalgebra of type $(\in, q_k)$ if whenever two point $\mathcal{N}$-structure (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_{\in}$-subsets of (X, f) then the point $\mathcal{N}$-structure $(X, (x * y)_{\vee\{t_1,t_2\}})$ is an $\mathcal{N}_{q_k}$-subset of (X, f).

Theorem 3.9. *Every $\mathcal{N}$-subalgebra of type $(\in, q_k)$ is of type $(\in, \in \vee q_k)$.*

Proof. Straightforward. □

Taking $k = 0$ in Theorem 3.9 induces the following corollary.

Corollary 3.10. *Every $\mathcal{N}$-subalgebra of type $(\in, q)$ is of type $(\in, \in \vee q)$.*

The converse of Theorem 3.9 is not true as seen in the following example.

Example 3.11. Consider the $\mathcal{N}$-subalgebra (X, f) of type $(\in, \in \vee q_{-0.2})$ which is given in Example 3.3. Then $(X, a_{-0.65})$ and $(X, b_{-0.25})$ are $\mathcal{N}$-subsets of (X, f), but

$$\left(X, (a * b)_{\vee\{-0.65,-0.25\}}\right) = (X, c_{-0.2}) \tag{3.5}$$

is not an $\mathcal{N}_{q_k}$-subset of (X, f) for $k = -0.2$ since $f(c) - 0.25 - 0.2 + 1 > 0$.

We consider a characterization of an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$.

Theorem 3.12. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ if and only if it satisfies*

$$(\forall x, y \in X) \quad \left(f(x * y) \le \bigvee\left\{f(x), f(y), \frac{k-1}{2}\right\}\right). \tag{3.6}$$

Proof. Let (X, f) be an $\mathcal{N}$-structure of type $(\in, \in \vee q_k)$. Assume that (3.6) is not valid. Then there exists $a, b \in X$ such that

$$f(a * b) > \bigvee\left\{f(a), f(b), \frac{k-1}{2}\right\}. \tag{3.7}$$

If $\bigvee\{f(a), f(b)\} > (k-1)/2$, then $f(a * b) > \bigvee\{f(a), f(b)\}$. Hence

$$f(a * b) > t \ge \bigvee\{f(a), f(b)\} \tag{3.8}$$

for some $t \in [-1, 0)$. It follows that point $\mathcal{N}$-structures (X, a_t) and (X, b_t) are $\mathcal{N}_{\in}$-subsets of (X, f), but the point $\mathcal{N}$-structure $(X, (a * b)_t)$ is not an $\mathcal{N}_{\in}$-subset of (X, f). Moreover,

$$f(a * b) + t - k + 1 > 2t - k + 1 = 0, \tag{3.9}$$

and so $(X, (a * b)_t)$ is not an $\mathcal{N}_{q_k}$-subset of (X, f). Consequently, $(X, (a * b)_t)$ is not an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f). This is a contradiction. If $\bigvee\{f(a), f(b)\} \le (k-1)/2$, then $f(a) \le (k-1)/2, f(b) \le (k-1)/2$ and $f(a * b) > (k-1)/2$. Thus $(X, a_{(k-1)/2})$ and $(X, b_{(k-1)/2})$ are $\mathcal{N}_{\in}$-subsets of (X, f), but $(X, (a * b)_{(k-1)/2})$ is not an $\mathcal{N}_{\in}$-subset of (X, f). Also,

$$f(a * b) + \frac{k-1}{2} - k + 1 > \frac{k-1}{2} + \frac{k-1}{2} - k + 1 = 0, \tag{3.10}$$

that is, $(X, (a * b)_{(k-1)/2})$ is not an $\mathcal{N}_{q_k}$-subset of (X, f). Hence $(X, (a * b)_{(k-1)/2})$ is not an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f), a contradiction. Therefore (3.6) is valid.

Conversely, suppose that (3.6) is valid. Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_{\in}$-subsets of (X, f). Then

$$f(x * y) \le \bigvee\left\{f(x), f(y), \frac{k-1}{2}\right\} \le \bigvee\left\{t_1, t_2, \frac{k-1}{2}\right\}. \tag{3.11}$$

Assume that $t_1 \ge (k-1)/2$ or $t_2 \ge (k-1)/2$. Then $f(x * y) \le \bigvee\{t_1, t_2\}$, and so $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in}$-subset of (X, f). Now suppose that $t_1 < (k-1)/2$ and $t_2 < (k-1)/2$. Then $f(x * y) \le (k-1)/2$, and thus

$$f(x * y) + \bigvee\{t_1, t_2\} - k + 1 < \frac{k-1}{2} + \frac{k-1}{2} - k + 1 = 0, \tag{3.12}$$

that is, $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{q_k}$-subset of (X, f). Therefore $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f) and consequently (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$. □

Corollary 3.13 (see [3]). *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$ if and only if it satisfies*

$$(\forall x, y \in X) \quad \left(f(x * y) \le \bigvee\{f(x), f(y), -0.5\}\right). \tag{3.13}$$

Proof. It follows from taking $k = 0$ in Theorem 3.12. □

We provide conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subalgebra of type $(q, \in \vee q_k)$.

Theorem 3.14. *Let S be a subalgebra of X and let (X, f) be an $\mathcal{N}$-structure such that*

(a) $(\forall x \in X)(x \in S \Rightarrow f(x) \le (k-1)/2)$,

(b) $(\forall x \in X)(x \notin S \Rightarrow f(x) = 0)$.

Then (X, f) is an $\mathcal{N}$-subalgebra of type $(q, \in \vee q_k)$.

Proof. Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that two point $\mathcal{N}$-structures (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_q$-subsets of (X, f). Then $f(x) + t_1 + 1 < 0$ and $f(y) + t_2 + 1 < 0$. Thus $x * y \in S$ because if it is impossible, then $x \notin S$ or $y \notin S$. Thus $f(x) = 0$ or $f(y) = 0$, and so $t_1 < -1$ or $t_2 < -1$. This is a contradiction. Hence $f(x * y) \le (k-1)/2$. If $\bigvee\{t_1, t_2\} < (k-1)/2$, then $f(x * y) + \bigvee\{t_1, t_2\} - k + 1 < ((k-1)/2) + ((k-1)/2) - k + 1 = 0$ and so the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{q_k}$-subset of (X, f). If $\bigvee\{t_1, t_2\} \ge (k-1)/2$, then $f(x * y) \le (k-1)/2 \le \bigvee\{t_1, t_2\}$ and so the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_\in$-subset of (X, f). Therefore the point $\mathcal{N}$-structure $(X, (x * y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f). This shows that (X, f) is an $\mathcal{N}$-subalgebra of type $(q, \in \vee q_k)$. □

Taking $k = 0$ in Theorem 3.14, we have the following corollary.

Corollary 3.15 (see [3]). *Let S be a subalgebra of X and let (X, f) be an $\mathcal{N}$-structure such that*

(a) $(\forall x \in X)(x \in S \Rightarrow f(x) \le -0.5)$,

(b) $(\forall x \in X)(x \notin S \Rightarrow f(x) = 0)$.

Then (X, f) is an $\mathcal{N}$-subalgebra of type $(q, \in \vee_q)$.

Theorem 3.16. *Let (X, f) be an $\mathcal{N}$-subalgebra of type $(q_k, \in \vee q_k)$. If f is not constant on the open 0-support of (X, f), then $f(x) \le (k-1)/2$ for some $x \in X$. In particular, $f(0) \le (k-1)/2$.*

Proof. Assume that $f(x) > (k-1)/2$ for all $x \in X$. Since f is not constant on the open 0-support of (X, f), there exists $x \in O(f; 0)$ such that $t_x = f(x) \ne f(0) = t_0$. Then either $t_0 < t_x$ or $t_0 > t_x$. For the case $t_0 < t_x$, choose $r < (k-1)/2$ such that $t_0 + r - k + 1 < 0 < t_x + r - k + 1$. Then the point $\mathcal{N}$-structure $(X, 0_r)$ is an $\mathcal{N}_{q_k}$-subset of (X, f). Since (X, x_{-1}) is an $\mathcal{N}_{q_k}$-subset of (X, f). It follows from (a1) that the point $\mathcal{N}$-structure $(X, (x * 0)_{\bigvee\{r,-1\}}) = (X, x_r)$ is an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f). But, $f(x) > (k-1)/2 > r$ implies that the point $\mathcal{N}$-structure (X, x_r) is not an $\mathcal{N}_\in$-subset of (X, f). Also, $f(x) + r - k + 1 = t_x + r - k + 1 > 0$ implies that the point $\mathcal{N}$-structure (X, x_r) is not an $\mathcal{N}_{q_k}$-subset of (X, f). This is a contradiction. Assume that $t_0 > t_x$ and take $r < (k-1)/2$ such that $t_x + r - k + 1 < 0 < t_0 + r - k + 1$. Then (X, x_r) is an $\mathcal{N}_{q_k}$-subset of (X, f). Since

$$f(x * x) = f(0) = t_0 > -r + k - 1 > -\frac{k-1}{2} + k - 1 = \frac{k-1}{2} > r, \tag{3.14}$$

$(X, (x * x)_{\bigvee\{r,r\}})$ is not an $\mathcal{N}_\in$-subset of (X, f). Since

$$f(x * x) + \bigvee\{r, r\} - k + 1 = f(0) + r - k + 1 = t_0 + r - k + 1 > 0, \tag{3.15}$$

$(X, (x * x)_{\bigvee\{r,r\}})$ is not an $\mathcal{N}_{q_k}$-subset of (X, f). Hence $(X, (x * x)_{\bigvee\{r,r\}})$ is not an $\mathcal{N}_{\in \vee q_k}$-subset of (X, f), which is a contradiction. Therefore $f(x) \le (k-1)/2$ for some $x \in X$. We now prove that $f(0) \le (k-1)/2$. Assume that $f(0) = t_0 > (k-1)/2$. Note that there exists $x \in X$ such that $f(x) = t_x \le (k-1)/2$ and so $t_x < t_0$. Choose $t_1 < t_0$ such that $t_x + t_1 - k + 1 < 0 < t_0 + t_1 - k + 1$. Then $f(x) + t_1 - k + 1 = t_x + t_1 - k + 1 < 0$, and thus the point $\mathcal{N}$-structure (X, x_{t_1}) is an $\mathcal{N}_{q_k}$-subset of (X, f). Now we have

$$f(x * x) + \bigvee\{t_1, t_1\} - k + 1 = f(0) + t_1 - k + 1 = t_0 + t_1 - k + 1 > 0 \tag{3.16}$$

and $f(x*x) = f(0) = t_0 > t_1 = \bigvee\{t_1, t_1\}$. Hence $(X, (x*x)_{\bigvee\{t_1,t_1\}})$ is not an $\mathcal{N}_{\in\vee q_k}$-subset of (X, f). This is a contradiction, and therefore $f(0) \le (k-1)/2$. □

Corollary 3.17 (see [3]). *Let (X, f) be an $\mathcal{N}$-subalgebra of type $(q, \in\vee_q)$. If f is not constant on the open 0-support of (X, f), then $f(x) \le -0.5$ for some $x \in X$. In particular, $f(0) \le -0.5$.*

Theorem 3.18. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$ if and only if for every $t \in [(k-1)/2, 0]$ the nonempty closed t-support of (X, f) is a subalgebra of X.*

Proof. Assume that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$ and let $t \in [(k-1)/2, 0]$ be such that $C(f;t) \ne \emptyset$. Let $x, y \in C(f;t)$. Then $f(x) \le t$ and $f(y) \le t$. It follows from Theorem 3.12 that

$$f(x*y) \le \bigvee\left\{f(x), f(y), \frac{k-1}{2}\right\} \le \bigvee\left\{t, \frac{k-1}{2}\right\} = t \tag{3.17}$$

so that $x*y \in C(f;t)$. Therefore $C(f;t)$ is a subalgebra of X.

Conversely, let (X, f) be an $\mathcal{N}$-structure such that the nonempty closed t-support of (X, f) is a subalgebra of X for all $t \in [(k-1)/2, 0]$. If there exist $a, b \in X$ such that $f(a*b) > \bigvee\{f(a), f(b), (k-1)/2\}$, then we can take $s \in [-1, 0]$ such that

$$f(a*b) > s \ge \bigvee\left\{f(a), f(b), \frac{k-1}{2}\right\}. \tag{3.18}$$

Thus $a, b \in C(f;s)$ and $s \ge (k-1)/2$. Since $C(f,s)$ is a subalgebra of X, it follows that $a*b \in C(f;s)$ so that $f(a*b) \le s$. This is a contradiction, and therefore $f(x*y) \le \bigvee\{f(x), f(y), (k-1)/2\}$ for all $x, y \in X$. Using Theorem 3.12, we conclude that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$. □

Taking $k = 0$ in Theorem 3.18, we have the following corollary.

Corollary 3.19 (see [4]). *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee_q)$ if and only if for every $t \in [-0.5, 0]$ the nonempty closed t-support of (X, f) is a subalgebra of X.*

Theorem 3.20. *Let S be a subalgebra of X. For any $t \in [(k-1)/2, 0)$, there exists an $\mathcal{N}$-subalgebra (X, f) of type $(\in, \in\vee q_k)$ for which S is represented by the closed t-support of (X, f).*

Proof. Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f(x) = \begin{cases} t & \text{if } x \in S, \\ 0 & \text{if } x \notin S, \end{cases} \tag{3.19}$$

for all $x \in X$ where $t \in [(k-1)/2, 0)$. Assume that $f(a*b) > \bigvee\{f(a), f(b), (k-1)/2\}$ for some $a, b \in X$. Since the cardinality of the image of f is 2, we have $f(a*b) = 0$ and $\bigvee\{f(a), f(b), (k-1)/2\} = t$. Since $t \ge (k-1)/2$, it follows that $f(a) = t = f(b)$ so that $a, b \in S$. Since S is a subalgebra of X, we obtain $a*b \in S$ and so $f(a*b) = t < 0$. This is a contradiction. Therefore $f(x*y) \le \bigvee\{f(x), f(y), (k-1)/2\}$ for all $x, y \in X$. Using Theorem 3.12, we conclude that

(X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$. Obviously, S is represented by the closed t-support of (X, f). □

Corollary 3.21 (see [4]). *Let S be a subalgebra of X. For any $t \in [-0.5, 0)$, there exists an $\mathcal{N}$-subalgebra (X, f) of type $(\in, \in \vee_q)$ for which S is represented by the closed t-support of (X, f).*

Proof. It follows from taking $k = 0$ in Theorem 3.20. □

Note that every $\mathcal{N}$-subalgebra of type $(\in, \in)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$, but the converse is not true in general (see Example 3.7). Now, we give a condition for an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ to be an $\mathcal{N}$-subalgebra of type $(\in, \in)$.

Theorem 3.22. *Let (X, f) be an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$ such that $f(x) > (k-1)/2$ for all $x \in X$. Then (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.*

Proof. Let $x, y \in X$ and $t \in [-1, 0)$ be such that (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_{\in}$-subsets of (X, f). Then $f(x) \le t_1$ and $f(y) \le t_2$. It follows from Theorem 3.12 and the hypothesis that

$$f(x * y) \le \bigvee\left\{f(x), f(y), \frac{k-1}{2}\right\} = \bigvee\{f(x), f(y)\} \le \bigvee\{t_1, t_2\} \tag{3.20}$$

so that $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in}$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$. □

Corollary 3.23 (see [4]). *Let (X, f) be an $\mathcal{N}$-structure of type $(\in, \in \vee_q)$ such that $f(x) > -0.5$ for all $x \in X$. Then (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.*

Proof. It follows from taking $k = 0$ in Theorem 3.22. □

Theorem 3.24. *Let $\{(X, f_i) \mid i \in \Lambda\}$ be a family of $\mathcal{N}$-subalgebras of type $(\in, \in \vee q_k)$. Then $(X, \bigcup_{i \in \Lambda} f_i)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$, where $\bigcup_{i \in \Lambda} f_i$ is an $\mathcal{N}$-function on X given by $(\bigcup_{i \in \Lambda} f_i)(x) = \bigvee_{i \in \Lambda} f_i(x)$ for all $x \in X$.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that (X, x_{t_1}) and (X, y_{t_2}) are $\mathcal{N}_{\in}$-subsets of $(X, \bigcup_{i \in \Lambda} f_i)$. Assume that $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is not an $\mathcal{N}_{\in \vee q_k}$-subset of $(X, \bigcup_{i \in \Lambda} f_i)$. Then $(X, (x * y)_{\bigvee\{t_1, t_2\}})$ is neither an $\mathcal{N}_{\in}$-subset nor an $\mathcal{N}_{q_k}$-subset of $(X, \bigcup_{i \in \Lambda} f_i)$. Hence $(\bigcup_{i \in \Lambda} f_i)(x * y) > \bigvee\{t_1, t_2\}$ and

$$\left(\bigcup_{i \in \Lambda} f_i\right)(x * y) + \bigvee\{t_1, t_2\} - k + 1 \ge 0, \tag{3.21}$$

which imply that

$$\left(\bigcup_{i \in \Lambda} f_i\right)(x * y) > \frac{k-1}{2}. \tag{3.22}$$

Let $A_1 := \{i \in \Lambda \mid (X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{\in}$-subset of $(X, f_i)\}$ and $A_2 := \{i \in \Lambda \mid (X, (x * y)_{\bigvee\{t_1, t_2\}})$ is an $\mathcal{N}_{q_k}$-subset of $(X, f_i)\} \cap \{j \in \Lambda \mid (X, (x * y)_{\bigvee\{t_1, t_2\}})$ is not an $\mathcal{N}_{\in}$-subset of $(X, f_j)\}$. Then $\Lambda = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. If $A_2 = \emptyset$, then $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{\in}$-subset of (X, f_i) for all $i \in \Lambda$, that is, $f_i(x*y) \leq \bigvee\{t_1, t_2\}$ for all $i \in \Lambda$. Thus $(\bigcup_{i\in\Lambda} f_i)(x*y) \leq \bigvee\{t_1, t_2\}$. This is a contradiction. Hence $A_2 \neq \emptyset$, and so for every $i \in A_2$, we have $f_i(x*y) > \bigvee\{t_1, t_2\}$ and $f_i(x*y) + \bigvee\{t_1, t_2\} - k + 1 < 0$. It follows that $\bigvee\{t_1, t_2\} < (k-1)/2$. Since (X, x_{t_1}) is an $\mathcal{N}_{\in}$-subset of $(X, \bigcup_{i\in\Lambda} f_i)$, we have

$$f_i(x) \leq \left(\bigcup_{i\in\Lambda} f_i\right)(x) \leq t_1 \leq \bigvee\{t_1, t_2\} < \frac{k-1}{2} \tag{3.23}$$

for all $i \in \Lambda$. Similarly, $f_i(y) < (k-1)/2$ for all $i \in \Lambda$. Next suppose that $t := f_i(x*y) > (k-1)/2$. Taking $(k-1)/2 < r < t$, we know that (X, x_r) and (X, y_r) are $\mathcal{N}_{\in}$-subsets of (X, f_i), but $(X, (x*y)_{\bigvee\{r,r\}}) = (X, (x*y)_r)$ is not an $\mathcal{N}_{\in\vee q_k}$-subset of (X, f_i). This contradicts that (X, f_i) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$. Hence $f_i(x*y) \leq (k-1)/2$ for all $i \in \Lambda$, and so $(\bigcup_{i\in\Lambda} f_i)(x*y) \leq (k-1)/2$ which contradicts (3.22). Therefore $(X, (x*y)_{\bigvee\{t_1,t_2\}})$ is an $\mathcal{N}_{\in\vee q_k}$-subset of $(X, \bigcup_{i\in\Lambda} f_i)$ and consequently $(X, \bigcup_{i\in\Lambda} f_i)$ is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$. □

For any $\mathcal{N}$-structure (X, f) and $t \in [-1, 0)$, the q-support and the $\in\vee_q$-support of (X, f) related to t are defined to be the sets (see [4])

$$\mathcal{N}_q(f;t) := \{x \in X \mid (X, x_t) \text{ is an } \mathcal{N}_q\text{-subset of } (X, f)\}, \tag{3.24}$$

$$\mathcal{N}_{\in\vee q}(f;t) := \{x \in X \mid (X, x_t) \text{ is an } \mathcal{N}_{\in\vee q}\text{-subset of } (X, f)\}, \tag{3.25}$$

respectively. Note that the $\in\vee q$-support is the union of the closed support and the q-support, that is,

$$\mathcal{N}_{\in\vee q}(f;t) = C(f;t) \cup \mathcal{N}_q(f;t), \quad t \in [-1, 0). \tag{3.26}$$

The q_k-support and the $\in\vee q_k$-support of (X, f) related to t are defined to be the sets

$$\mathcal{N}_{q_k}(f;t) := \{x \in X \mid (X, x_t) \text{ is an } \mathcal{N}_{q_k}\text{-subset of } (X, f)\}, \tag{3.27}$$

$$\mathcal{N}_{\in\vee q_k}(f;t) := \{x \in X \mid (X, x_t) \text{ is an } \mathcal{N}_{\in\vee q_k}\text{-subset of } (X, f)\}, \tag{3.28}$$

respectively. Clearly, $\mathcal{N}_{\in\vee q_k}(f;t) = C(f;t) \cup \mathcal{N}_{q_k}(f;t)$ for all $t \in [-1, 0)$.

Theorem 3.25. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$ if and only if the $\in\vee q_k$-support of (X, f) related to t is a subalgebra of X for all $t \in [-1, 0)$.*

Proof. Suppose that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in\vee q_k)$. Let $x, y \in \mathcal{N}_{\in\vee q_k}(f;t)$ for $t \in [-1, 0)$. Then (X, x_t) and (X, y_t) are $\mathcal{N}_{\in\vee q_k}$-subsets of (X, f). Hence $f(x) \leq t$ or $f(x) + t - k + 1 < 0$, and $f(y) \leq t$ or $f(y) + t - k + 1 < 0$. Then we consider the following four cases:

(c1) $f(x) \leq t$ and $f(y) \leq t$,

(c2) $f(x) \leq t$ and $f(y) + t - k + 1 < 0$,

(c3) $f(x) + t - k + 1 < 0$ and $f(y) \leq t$,

(c4) $f(x) + t - k + 1 < 0$ and $f(y) + t - k + 1 < 0$.

Combining (3.6) and (c1), we have $f(x*y) \leq \bigvee\{t,(k-1)/2\}$. If $t \geq (k-1)/2$, then $f(x*y) \leq t$ and so $(X,(x*y)_t)$ is an $\mathcal{N}_{\in}$-subset of (X,f). Hence $x*y \in C(f;t) \subseteq \mathcal{N}_{\in\vee q_k}(f;t)$. If $t < (k-1)/2$, then $f(x*\ y) \leq (k-1)/2$ and so $f(x*\ y)+t-k+1 < ((k-1)/2)+((k-1)/2)-k+1 = 0$, that is, $(X,(x*y)_t)$ is an $\mathcal{N}_{q_k}$-subset of (X,f). Therefore $x*y \in \mathcal{N}_{q_k}(f;t) \subseteq \mathcal{N}_{\in\vee q_k}(f;t)$. For the case (c2), assume that $t < (k-1)/2$. Then

$$\begin{aligned} f(x*y) &\leq \bigvee\left\{f(x),f(y),\frac{k-1}{2}\right\} \\ &\leq \bigvee\left\{t,f(y),\frac{k-1}{2}\right\} = \bigvee\left\{f(y),\frac{k-1}{2}\right\} \\ &= \begin{cases} f(y) & \text{if } f(y) > \dfrac{k-1}{2}, \\ \dfrac{k-1}{2} & \text{if } f(y) \leq \dfrac{k-1}{2}, \end{cases} \\ &< k-1-t, \end{aligned} \tag{3.29}$$

and so $f(x*y)+t-k+1 < 0$. Thus $(X,(x*y)_t)$ is an $\mathcal{N}_{q_k}$-subset of (X,f). If $t \geq (k-1)/2$, then

$$\begin{aligned} f(x*y) &\leq \bigvee\left\{f(x),f(y),\frac{k-1}{2}\right\} \\ &\leq \bigvee\left\{t,f(y),\frac{k-1}{2}\right\} = \bigvee\{t,f(y)\} \\ &= \begin{cases} f(y) & \text{if } f(y) > t, \\ t & \text{if } f(y) \leq t, \end{cases} \end{aligned} \tag{3.30}$$

and thus $x*y \in \mathcal{N}_{q_k}(f;t)$ or $x*y \in C(f;t)$. Consequently, $x*y \in \mathcal{N}_{\in\vee q_k}(f;t)$. For the case (c3), it is similar to the case (c2). Finally, for the case (c4), if $t \geq (k-1)/2$, then $k-1-t \leq (k-1)/2 \leq t$. Hence

$$f(x*y) \leq \bigvee\left\{f(x),f(y),\frac{k-1}{2}\right\} \leq \bigvee\left\{k-1-t,\frac{k-1}{2}\right\} = \frac{k-1}{2} \leq t, \tag{3.31}$$

which implies that $x*y \in C(f;t)$. If $t < (k-1)/2$, then $t < (k-1)/2 < k-1-t$. Therefore

$$f(x*y) \leq \bigvee\left\{f(x),f(y),\frac{k-1}{2}\right\} \leq \bigvee\left\{k-1-t,\frac{k-1}{2}\right\} = k-1-t, \tag{3.32}$$

that is, $f(x*y)+t-k+1 < 0$, which means that $(X,(x*y)_t)$ is an $\mathcal{N}_{q_k}$-subset of (X,f). Consequently, the $\in\vee q_k$-support of (X,f) related to t is a subalgebra of X for all $t \in [-1,0)$.

Conversely, let (X, f) be an $\mathcal{N}$-structure for which the $\in \vee q_k$-support of (X, f) related to t is a subalgebra of X for all $t \in [-1, 0)$. Assume that there exist $a, b \in X$ such that $f(a*b) > \bigvee\{f(a), f(b), (k-1)/2\}$. Then

$$f(a*b) > s \geq \bigvee\left\{f(a), f(b), \frac{k-1}{2}\right\} \tag{3.33}$$

for some $s \in [(k-1)/2, 0)$. It follows that $a, b \in C(f; s) \subseteq \mathcal{N}_{\in \vee q_k}(f; s)$ but $a*b \notin C(f; s)$. Also, $f(a*b) + s - k + 1 > 2s - k + 1 \geq 0$, that is, $a*b \notin \mathcal{N}_{q_k}(f; s)$. Thus $a*b \notin \mathcal{N}_{\in \vee q_k}(f; s)$ which is a contradiction. Therefore

$$f(x*y) \leq \bigvee\left\{f(x), f(y), \frac{k-1}{2}\right\} \tag{3.34}$$

for all $x, y \in X$. Using Theorem 3.12, we conclude that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q_k)$. □

If we take $k = 0$ in Theorem 3.25, we have the following corollary.

Corollary 3.26 (see [4]). *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$ if and only if the $\in \vee q$-support of (X, f) related to t is a subalgebra of X for all $t \in [-1, 0)$.*

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