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Research Article

Fixed Point Theory of Weak Contractions in Partially Ordered Metric Spaces

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We prove two new fixed point theorems in the framework of partially ordered metric spaces. Our results generalize and improve many recent fixed point theorems in the literature.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, D a subset of X, and $f : D \to X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that, for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \tag{1}$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. In nonlinear analysis, the study of fixed points of given mappings satisfying certain contractive conditions in various abstract spaces has been investigated deeply. The Banach contraction principle [1] is one of the initial and crucial results in this direction. Also, this principle has many generalizations. For instance, Alber and Guerre-Delabriere in [2] suggested a generalization of the Banach contraction mapping principle by introducing the concept of weak contraction in Hilbert spaces. In [2], the authors also proved that the result of Eslamian and Abkar [3] is equivalent to the result of Dutta and Choudhury [4]. Later, weakly contractive mappings and mappings satisfying other weak contractive inequalities have been discussed in several works, some of which are noted in [4–16].

In 2008, Dutta and Choudhury proved the following theorem.

Theorem 1 (see [4]). *Let (X, d) be a complete metric space, and let $f : X \to X$ be such that*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad \text{for each } x, y \in X, \tag{2}$$

where $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and nondecreasing, and $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then f has a fixed point in X.

Recently, Eslamian and Abkar [3] proved the following theorem.

Theorem 2 (see [3]). *Let (X, d) be a complete metric space, and let $f : X \to X$ be such that*

$$\psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \quad \text{for each } x, y \in X, \tag{3}$$

where $\psi, \alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0, \qquad \psi(t) = 0 \quad \text{if and only if } t = 0, \quad \alpha(0) = \beta(0) = 0. \tag{4}$$

Then f has a fixed point in X.

In the recent, fixed point theory has developed rapidly in partially ordered metric spaces (e.g., [17–22]).

In 2012, Choudhury and Kundu [23] proved the following fixed point theorem as a generalization of Theorem 2.

Theorem 3 (see [23]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space and let $f : X \to X$ be a nondecreasing mapping such that*

$$\psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \quad \text{for each } x, y \in X \text{ such that } x \sqsubseteq y, \tag{5}$$

where $\psi, \alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0,$$
$$\psi(t) = 0 \quad \text{if and only if } t = 0, \quad \alpha(0) = \beta(0) = 0. \tag{6}$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{7}$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X.

In this paper, we prove two new fixed point theorems in the framework of partially ordered metric spaces. Our results generalize and improve many recent fixed point theorems in the literature.

2. Fixed Point Results (I)

We start with the following definition.

Definition 4. Let $(X, \sqsubseteq)$ be a partially ordered set and $f : X \to X$. Then f is said to be monotone nondecreasing if, for $x, y \in X$,

$$x \sqsubseteq y \Longrightarrow fx \sqsubseteq fy. \tag{8}$$

Let $(X, \sqsubseteq)$ be a partially ordered set. $x, y \in X$ are said to be comparable if either $x \sqsubseteq y$ or $y \sqsubseteq x$ holds.

In the section, we denote by Ψ the class of functions $\psi : {\mathbb{R}^+}^3 \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_1) ψ is an increasing, continuous function in each coordinate;

(ψ_2) for all $t \in \mathbb{R}^+$, $\psi(t, t, t) \leq t$, $\psi(0, 0, t) \leq t$ and $\psi(t, 0, 0) \leq t$.

Next, we denote by Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(ϕ_1) ϕ is a continuous, nondecreasing function;

(ϕ_2) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;

(ϕ_3) ϕ is subadditive; that is, $\phi(t_1 + t_2) \leq \phi(t_1) + \phi(t_2)$ for all $t_1, t_2 > 0$.

And we denote the following sets of functions:

$$\Theta = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{such that } \varphi \text{ is continuous}\},$$
$$\Xi = \{\xi : \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{such that } \xi \text{ is lower continuous}\}. \tag{9}$$

Let X be a nonempty set, and let $(X, \sqsubseteq)$ be a partially ordered set endowed with a metric d. Then, the triple $(X, \sqsubseteq, d)$ is called a partially ordered complete metric space.

We now state the main fixed point theorem for $(\varphi, \psi, \phi, \xi)$-contractions in partially ordered metric spaces, as follows.

Theorem 5. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be monotone nondecreasing, and*

$$\varphi(d(fx, fy)) \leq \psi(\phi(d(x, y)), \phi(d(x, fx)), \phi(d(y, fy))) - \xi(\max\{d(x, y), d(x, fx), d(y, fy)\}), \tag{10}$$

for all comparable $x, y \in X$, where $\varphi \in \Theta$, $\psi \in \Psi$, $\phi \in \Phi$, and $\xi \in \Xi$, and

$$\varphi(t) - \phi(t) + \xi(t) > 0 \quad \forall t > 0,$$
$$\varphi(t) = 0 \quad \text{if and only if } t = 0, \quad \phi(0) = \xi(0) = 0. \tag{11}$$

Suppose that either

(a) *f is continuous or*

(b) *if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{12}$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f has a fixed point in X.

Proof. Since f is nondecreasing, by induction, we construct the sequence $\{x_n\}$ recursively as

$$x_n = f^n x_0 = fx_{n-1} \quad \forall n \in \mathbb{N}. \tag{13}$$

Thus, we also conclude that

$$x_0 \sqsubset x_1 = fx_0 \sqsubseteq x_2 = fx_1 \sqsubseteq \cdots \sqsubseteq x_n = fx_{n-1} \sqsubseteq \cdots. \tag{14}$$

If any two consecutive terms in (14) are equal, then the f has a fixed point, and hence the proof is completed. So we may assume that

$$d(x_{n-1}, x_n) \neq 0, \quad \forall n \in \mathbb{N}. \tag{15}$$

Now, we claim that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. If not, we assume that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$ for some $n \in \mathbb{N}$;

substituting $x = x_n$ and $y = x_{n+1}$ in (10) and using the definition of the function ψ, we have

$$\begin{aligned}
&\psi(\phi(d(x_n, x_{n+1})), \phi(d(x_n, fx_n)), \phi(d(x_{n+1}, fx_{n+1}))) \\
&\quad = \psi(\phi(d(x_n, x_{n+1})), \phi(d(x_n, x_{n+1})), \phi(d(x_{n+1}, x_{n+2}))) \\
&\quad \leq \phi(d(x_{n+1}, x_{n+2})), \\
&\xi(\max\{d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\}) \\
&\quad = \xi(\max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \\
&\quad = \xi(d(x_{n+1}, x_{n+2})),
\end{aligned} \tag{16}$$

and hence

$$\begin{aligned}
\varphi(d(x_{n+1}, x_{n+2})) &= \varphi(d(fx_n, fx_{n+1})) \\
&\leq \phi(d(x_{n+1}, x_{n+2})) - \xi(d(x_{n+1}, x_{n+2})).
\end{aligned} \tag{17}$$

Since $\varphi(t) - \phi(t) + \xi(t) > 0$ for all $t > 0$, we have that $d(x_{n+1}, x_{n+2}) = 0$, which contradicts (15). Therefore, we conclude that

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \quad \forall n \in \mathbb{N}. \tag{18}$$

From the previous argument, we also have that for each $n \in \mathbb{N}$

$$\varphi(d(x_n, x_{n+1})) \leq \phi(d(x_{n-1}, x_n)) - \xi(d(x_{n-1}, x_n)). \tag{19}$$

It follows in (18) that the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing; it must converge to some $\eta \geq 0$. Taking limit as $n \to \infty$ in (19) and using the continuities of φ and ϕ and the lower semicontinuous of ξ, we get

$$\varphi(\eta) \leq \phi(\eta) - \xi(\eta), \tag{20}$$

which implies that $\eta = 0$. So we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{21}$$

□

We next claim that $\{x_n\}$ is a Cauchy sequence; that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$, then $d(x_p, x_q) < \varepsilon$.

Suppose, on the contrary, that there exists $\epsilon > 0$ such that, for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying

$$d(x_{q_n}, x_{p_n}) \geq \epsilon. \tag{22}$$

Further, corresponding to $q_n \geq n$, we can choose p_n in such a way that it the smallest integer with $p_n > q_n \geq n$ and $d(x_{q_n}, x_{p_n}) \geq \epsilon$. Therefore $d(x_{q_n}, x_{p_n-1}) < \epsilon$. Now we have that for all $n \in \mathbb{N}$

$$\begin{aligned}
\epsilon &\leq d(x_{p_n}, x_{q_n}) \\
&\leq d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n}) \\
&< d(x_{p_n}, x_{p_n-1}) + \epsilon.
\end{aligned} \tag{23}$$

By letting $n \to \infty$. we get that

$$\lim_{n\to\infty} d(x_{p_n}, x_{q_n}) = \epsilon. \tag{24}$$

On the other hand, we have

$$\begin{aligned}
&d(x_{p_n}, x_{q_n}) \\
&\quad \leq d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n-1}) + d(x_{q_n-1}, x_{q_n}), \\
&d(x_{p_n-1}, x_{q_n-1}) \\
&\quad \leq d(x_{p_n-1}, x_{p_n}) + d(x_{p_n}, x_{q_n}) + d(x_{q_n}, x_{q_n-1}).
\end{aligned} \tag{25}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d(x_{p_n-1}, x_{q_n-1}) = \epsilon. \tag{26}$$

By (14), we have that the elements x_{p_n} and x_{q_n} are comparable. Substituting $x = x_{p_n-1}$ and $y = x_{q_n-1}$ in (10), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\psi(\phi(d(x_{p_n-1}, x_{q_n-1})), \phi(d(x_{p_n-1}, fx_{p_n-1})), \\
&\quad \phi(d(x_{q_n-1}, fx_{q_n-1}))) \\
&\qquad \leq \psi(\phi(d(x_{p_n-1}, x_{q_n-1})), \phi(d(x_{p_n-1}, x_{p_n})), \\
&\qquad\qquad \phi(d(x_{q_n-1}, x_{q_n}))), \\
&M(x_{p_n-1}, x_{q_n-1}) \\
&\quad = \max\{d(x_{p_n-1}, x_{q_n-1}), d(x_{p_n-1}, fx_{p_n-1}), \\
&\qquad\qquad d(x_{q_n-1}, fx_{q_n-1})\} \\
&\quad = \max\{d(x_{p_n-1}, x_{q_n-1}), d(x_{p_n-1}, x_{p_n}), \\
&\qquad\qquad d(x_{q_n-1}, x_{q_n})\}.
\end{aligned} \tag{27}$$

By the previous argument and using inequality (10), we can conclude that

$$\begin{aligned}
\varphi(\epsilon) &\leq \psi(\phi(\epsilon), 0, 0) - \xi(\epsilon) \\
&\leq \phi(\epsilon) - \xi(\epsilon),
\end{aligned} \tag{28}$$

which implies that $\epsilon = 0$, a contradiction. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} x_n = \nu. \tag{29}$$

Suppose that (a) holds. Then

$$\nu = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\nu. \tag{30}$$

Thus, ν is a fixed point in X.

Suppose that (b) holds; that is, $x_n \sqsubseteq \nu$ for all $n \in \mathbb{N}$. Substituting $x = x_n$ and $y = \nu$ in (10), we have that

$$\begin{aligned}\varphi(d(x_{n+1}, f\nu)) &= \varphi(d(fx_n, f\nu)) \\ &\leq \psi(\phi(d(x_n,\nu)), \phi(d(x_n, fx_n)), \phi(d(\nu, f\nu))) \\ &\quad - \xi(\max\{d(x_n,\nu), d(x_n, fx_n), d(\nu, f\nu)\}).\end{aligned} \tag{31}$$

Taking limit as $n \to \infty$ in equality (31), we have

$$\begin{aligned}\varphi(d(\nu, f\nu)) &\leq \psi(\phi(0), \phi(o), \phi(d(\nu, f\nu))) - \xi(d(\nu, f\nu)) \\ &\leq \phi(d(\nu, f\nu)) - \xi(d(\nu, f\nu)),\end{aligned} \tag{32}$$

which implies that $d(\nu, f\nu) = 0$; that is $\nu = f\nu$. So we complete the proof.

If we let

$$\begin{aligned}&\psi(\phi(d(x,y)), \phi(d(x,fx)), \phi(d(y,fy))) \\ &\quad = \max\{\phi(d(x,y)), \phi(d(x,fx)), \phi(d(y,fy))\},\end{aligned} \tag{33}$$

it is easy to get the following theorem.

Theorem 6. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be monotone nondecreasing, and*

$$\begin{aligned}\varphi(d(fx, fy)) &\leq \max\{\phi(d(x,y)), \phi(d(x,fx)), \phi(d(y,fy))\} \\ &\quad - \xi(\max\{d(x,y), d(x,fx), d(y,fy)\}),\end{aligned} \tag{34}$$

for all comparable $x, y \in X$, where $\varphi \in \Theta$, $\phi \in \Phi$, and $\xi \in \Xi$, and

$$\begin{gathered}\varphi(t) - \phi(t) + \xi(t) > 0 \quad \forall t > 0, \\ \varphi(t) = 0 \quad \text{if and only if } t = 0, \quad \phi(0) = \xi(0) = 0.\end{gathered} \tag{35}$$

Suppose that either

(a) *f is continuous or*

(b) *if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{36}$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f has a fixed point in X.

3. Fixed Point Results (II)

In the section, we denote by Ψ the class of functions $\psi : \mathbb{R}^{+3} \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_1) ψ is an increasing and continuous function in each coordinate;

(ψ_2) for $t \in \mathbb{R}^+$, $\phi(t,t,t) \leq t$, $\phi(t,0,0) \leq t$, and $\phi(0,0,t) \leq t$.

Next, we denote by Θ the class of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(φ_1) φ is continuous and nondecreasing;

(φ_2) for $t > 0$, $\varphi(t) > 0$ and $\varphi(0) = 0$.

And we denote by Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions.

(ϕ_1) ϕ is continuous;

(ϕ_2) for $t > 0$, $\phi(t) > 0$ and $\phi(0) = 0$.

We now state the main fixed point theorem for the (ψ, φ, ϕ)-contractions in partially ordered metric spaces, as follows.

Theorem 7. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f : X \to X$ be monotone nondecreasing, and*

$$\begin{aligned}\varphi(d(fx, fy)) &\leq \psi(\varphi(d(x,y)), \varphi(d(x,fx)), \varphi(d(y,fy))) \\ &\quad - \phi(M(x,y)) + L \cdot m(x,y),\end{aligned} \tag{37}$$

for all comparable $x, y \in X$ and $\psi \in \Psi$, $\varphi \in \Theta$, $\phi \in \Phi$, where $L > 0$ and

$$\begin{aligned}M(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy)\}, \\ m(x,y) &= \min\{d(x,y), d(x,fx), d(y,fy), d(x,fy), \\ &\qquad d(y,fx)\}.\end{aligned} \tag{38}$$

Suppose that either

(a) *f is continuous or*

(b) *if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{39}$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f has a fixed point in X.

Proof. If $fx_0 = x_0$, then the proof is finished. Suppose that $x_0 \sqsubset fx_0$. Since f is nondecreasing, by induction, we construct the sequence $\{x_n\}$ recursively as

$$x_n = f^n x_0 = fx_{n-1} \quad \forall n \in \mathbb{N}. \tag{40}$$

Thus, we also conclude that

$$x_0 \sqsubset x_1 = fx_0 \sqsubseteq x_2 = fx_1 \sqsubseteq \cdots \sqsubseteq x_n = fx_{n-1} \sqsubseteq \cdots. \tag{41}$$

We now claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{42}$$

Put $x = x_{n-1}$ and $y = x_n$ in (37). Note that

$$\begin{aligned} m(x_{n-1}, x_n) &= \min\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), \\ &\qquad d(x_n, fx_n), d(x_{n-1}, fx_n), d(x_n, fx_{n-1})\} \\ &= \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\qquad d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ &= 0. \end{aligned} \tag{43}$$

So, we obtain that

$$\begin{aligned} &\varphi(d(x_n, x_{n+1})) \\ &\quad = \varphi(d(fx_{n-1}, fx_n)) \\ &\quad \le \psi(\varphi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, fx_{n-1})), \\ &\qquad \varphi(d(x_n, fx_n))) - \phi(M(x_{n-1}, x_n)) \\ &\quad \le \psi(\varphi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n)), \varphi(d(x_n, x_{n+1}))) \\ &\qquad - \phi(M(x_{n-1}, x_n)), \end{aligned} \tag{44}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{45}$$

We now claim that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{46}$$

If not, we assume that $d(x_{n-1}, x_n) \le d(x_n, x_{n+1})$; then $\varphi(d(x_{n-1}, x_n)) \le \varphi(d(x_n, x_{n+1}))$, since φ is non-decreasing. Using inequality (44) and the conditions of the function ψ, we have that, for each $n \in \mathbb{N}$,

$$\varphi(d(x_n, x_{n+1})) \le \varphi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})), \tag{47}$$

which implies that $\phi(d(x_n, x_{n+1})) = 0$, and hence $d(x_n, x_{n+1}) = 0$. This contradicts our initial assumption.

From the previous argument, we have that, for each $n \in \mathbb{N}$,

$$\begin{aligned} \varphi(d(x_n, x_{n+1})) &\le \varphi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)), \\ d(x_n, x_{n+1}) &< d(x_{n-1}, x_n). \end{aligned} \tag{48}$$

And since the sequence $\{d(x_n, x_{n+1})\}$ is decreasing, it must converge to some $\eta \ge 0$. Taking limit as $n \to \infty$ in (48) and by the continuity of φ and ϕ, we get

$$\varphi(\eta) \le \varphi(\eta) - \phi(\eta), \tag{49}$$

and so we conclude that $\phi(\eta) = 0$ and $\eta = 0$.

We next claim that $\{x_n\}$ is Cauchy; that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \ge n$, then $d(x_p, x_q) < \varepsilon$.

Suppose, on the contrary, that there exists $\epsilon > 0$ such that, for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \ge n$ satisfying

$$d(x_{q_n}, x_{p_n}) \ge \epsilon. \tag{50}$$

Further, corresponding to $q_n \ge n$, we can choose p_n in such a way that it the smallest integer with $p_n > q_n \ge n$ and $d(x_{q_n}, x_{p_n}) \ge \epsilon$. Therefore $d(x_{q_n}, x_{p_n-1}) < \epsilon$. By the rectangular inequality, we have

$$\begin{aligned} \epsilon &\le d(x_{p_n}, x_{q_n}) \\ &\le d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n}) \\ &< d(x_{p_n}, x_{p_n-1}) + \epsilon. \end{aligned} \tag{51}$$

Letting $n \to \infty$, then we get

$$\lim_{n \to \infty} d(x_{p_n}, x_{q_n}) = \epsilon. \tag{52}$$

On the other hand, we have

$$\begin{aligned} &d(x_{p_n}, x_{q_n}) \\ &\quad \le d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n-1}) + d(x_{q_n-1}, x_{q_n}), \\ &d(x_{p_n-1}, x_{q_n-1}) \\ &\quad \le d(x_{p_n-1}, x_{p_n}) + d(x_{p_n}, x_{q_n}) + d(x_{q_n}, x_{q_n-1}). \end{aligned} \tag{53}$$

By letting $n \to \infty$, we get that

$$\lim_{n \to \infty} d(x_{p_n-1}, x_{q_n-1}) = \epsilon. \tag{54}$$

Using inequalities (37), (52), and (54) and putting $x = x_{p_n-1}$ and $y = x_{q_n-1}$, we have that

$$\begin{aligned} &\varphi(d(x_{p_n}, x_{q_n})) \\ &\quad = \varphi(d(fx_{p_n-1}, fx_{q_n-1})) \\ &\quad \le \psi(\varphi(d(x_{p_n-1}, x_{q_n-1})), \varphi(d(x_{p_n-1}, fx_{p_n-1})), \\ &\qquad \varphi(d(x_{q_n-1}, fx_{q_n-1}))) \\ &\qquad - \phi(M(x_{p_n-1}, x_{q_n-1})) + L \cdot m(x_{p_n-1}, x_{q_n-1}) \\ &\quad = \psi(\varphi(d(x_{p_n-1}, x_{q_n-1})), \varphi(d(x_{p_n-1}, x_{p_n})), \\ &\qquad \varphi(d(x_{q_n-1}, x_{q_n}))) \\ &\qquad - \phi(M(x_{p_n-1}, x_{q_n-1})) + L \cdot m(x_{p_n-1}, x_{q_n-1}), \end{aligned} \tag{55}$$

where

$$M\left(x_{p_n-1},x_{q_n-1}\right) = \max\left\{d\left(x_{p_n-1},x_{q_n-1}\right),d\left(x_{p_n-1},x_{p_n}\right),d\left(x_{q_n-1},x_{q_n}\right)\right\},$$
$$m\left(x_{p_n-1},x_{q_n-1}\right) = \min\left\{d\left(x_{p_n-1},x_{q_n-1}\right),d\left(x_{p_n-1},x_{p_n}\right), d\left(x_{q_n-1},x_{q_n}\right),d\left(x_{p_n-1},x_{q_n}\right),d\left(x_{q_n-1},x_{p_n}\right)\right\}. \quad (56)$$

Letting $n \to \infty$, then we obtain that

$$\lim_{n\to\infty} M\left(x_{p_n-1},x_{q_n-1}\right) = \epsilon,$$
$$\lim_{n\to\infty} m\left(x_{p_n-1},x_{q_n-1}\right) = 0, \quad (57)$$
$$\varphi(\epsilon) \le \psi(\varphi(\epsilon),0,0) - \phi(\epsilon) \le \varphi(\epsilon) - \phi(\epsilon).$$

This implies that $\phi(\epsilon) = 0$, and hence $\epsilon = 0$. So we get a contraction. Therefore $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} x_n = \nu. \quad (58)$$

Suppose that (a) holds. Then

$$\nu = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\nu. \quad (59)$$

Thus, ν is a fixed point in X

Suppose that (b) holds; that is, $x_n \sqsubseteq \nu$ for all $n \in \mathbb{N}$. Substituting $x = x_n$ and $y = \nu$ in (37), we have that

$$\begin{aligned}\varphi(d(x_{n+1},f\nu)) &= \varphi(d(fx_n,f\nu)) \\ &\le \psi(\varphi(d(x_n,\nu)),\varphi(d(x_n,fx_n)),\varphi(d(\nu,f\nu))) \\ &\quad - \phi(M(x_n,\nu)) + L\cdot m(x_n,\nu),\end{aligned} \quad (60)$$

where

$$M(x_n,\nu) = \max\{d(x_n,\nu),d(x_n,fx_n),d(\nu,f\nu)\},$$
$$m(x_n,\nu) = \min\{d(x_n,\nu),d(x_n,fx_n),d(\nu,f\nu),d(x_n,f\nu) \; d(\nu,fx_n)\}. \quad (61)$$

Letting $n \to \infty$, then we obtain that

$$M(x_n,\nu) \longrightarrow d(\nu,f\nu), \qquad m(x_n,\nu) \longrightarrow 0,$$
$$\begin{aligned}\varphi(d(\nu,f\nu)) &\le \psi(\varphi(0),\varphi(0),\varphi(d(\nu,f\nu))) - \phi(d(\nu,f\nu)) \\ &\le \varphi(d(\nu,f\nu)) - \phi(d(\nu,f\nu)),\end{aligned} \quad (62)$$

which implies that $d(\nu,f\nu) = 0$; that is, $\nu = f\nu$. So we complete the proof. □

If we let

$$\psi(\varphi(d(x,y)),\varphi(d(x,fx)),\varphi(d(y,fy))) = \max\{\varphi(d(x,y)),\varphi(d(x,fx)),\varphi(d(y,fy))\}, \quad (63)$$

it is easy to get the following theorem.

Theorem 8. *Let $(X,\sqsubseteq,d)$ be a partially ordered complete metric space, and let $f : X \to X$ be monotone nondecreasing, and*

$$\varphi(d(fx,fy)) \le \max\{\varphi(d(x,y)),\varphi(d(x,fx)),\varphi(d(y,fy))\} - \phi(M(x,y)) + L\cdot m(x,y), \quad (64)$$

for all comparable $x, y \in X$ and $\varphi \in \Theta$, $\phi \in \Phi$, where $L > 0$ and

$$M(x,y) = \max\{d(x,y),d(x,fx),d(y,fy)\},$$
$$m(x,y) = \min\{d(x,y),d(x,fx),d(y,fy),d(x,fy), d(y,fx)\}. \quad (65)$$

Suppose that either

(a) *f is continuous or*

(b) *if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \quad (66)$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f has a fixed point in X.

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