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Extension of Two Inequalities of Payne

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In this note isoperimetric bounds are derived for the maximum of the solution to the Poisson problem for a plane domain. This extends previous bounds of Payne valid for the torsion problem.

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1 INTRODUCTION

The boundary value problem

$$\begin{cases} \Delta\psi + 1 = 0 & \text{in } \Omega \subset \mathbb{R}^2 \\ \psi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

is usually called the torsion problem because of its mechanical interpretation. Another interpretation relates (1.1) to a laminar flow in a pipe of cross-section Ω. Then, ψ is proportional to the flow velocity. A third important possibility is a stationary heat flow problem with ψ measuring the temperature.

An important quantity in all these contexts is

$$S = \int_\Omega |\nabla\psi|^2\, dx = \int_\Omega \psi\, dx \quad (dx = \text{ area element})\ .$$

In the mechanical interpretation of (1.1) S is called the torsional rigidity. A second quantity of interest is

$$\psi_m = \max_\Omega\ \psi(x)\ .$$

Many bounds for ψ_m and S are known, see e.g. [1, 2, 4]. In particular Pólya and Szegö proved that

$$\psi_m \le \frac{A}{4\pi}\,, \tag{1.2}$$

with A denoting the area of Ω and furthermore that

$$S \le \frac{A^2}{8\pi}\,. \tag{1.3}$$

Later on Payne [3] proved the sharper inequality

$$\psi_m \le \left(\frac{S}{2\pi}\right)^{1/2} \tag{1.4}$$

and also gave the lower bound

$$4\pi \cdot \psi_m \ge A - (A^2 - 8\pi S)^{1/2}\,. \tag{1.5}$$

In all these inequalities the equality sign holds if Ω is a disk.

In this note the primary concern is to give an extension of Payne's inequalities (1.4), (1.5) to the Poisson problem in the plane, i.e. the boundary value problem

$$\begin{cases} \Delta u + p(x) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

where $p(x)$ is a smooth, strictly positive function satisfying

$$-\frac{\Delta(\log p)}{2p} \le K \tag{1.7}$$

for some constant K. If $K > 0$ an additional requirement is that

$$K \int_\Omega p\,dx < 4\pi\,. \tag{1.8}$$

Remark Problem (1.6) is equivalent to problem (1.1) for a domain on a surface of Gaussian curvature $K = -\frac{1}{2p}\,\Delta(\log p)$ (see [1, 4] for more details).

2 EXTENSION OF PAYNE'S INEQUALITIES

The analogue of inequalities (1.4) and (1.5) can be stated as

THEOREM *Suppose $p(x)$ satisfies (1.7) and (1.8) in the simply connected plane domain Ω and set*

$$A = \int_\Omega p\,dx, \quad S = \int_\Omega |\nabla u|^2\,dx = \int_\Omega u\,p\,dx\ .$$

Then one has for $u_m = \max_\Omega u$ the inequalities

$$u_m - \frac{1}{K}\,(1 - e^{-Ku_m}) \le \frac{K \cdot S}{4\pi} \tag{2.1}$$

and

$$u_m + \Big(\frac{A}{4\pi} - \frac{1}{K}\Big)(e^{Ku_m} - 1) \ge \frac{K \cdot S}{4\pi}\ . \tag{2.2}$$

Equality holds in (2.2), (2.3) if Ω is a disk and p is of the form

$$p = \frac{c}{(1 + \frac{cK}{4}\,r^2)^2},$$

$c =$ positive number

$r =$ distance from the center of the disk.

Remark For $K \to 0$ inequalities (2.1) and (2.2) reduce to the inequalities (1.4) and (1.5) of Payne as a Taylor expansion with respect to K shows.

Proof of the Theorem: Let Γ_t be the level-line where $u = t$ and Ω_t the domain enclosed by Γ_t. We set for $v \in (0, u_m)$

$$S(v) = \int_v^{u_m} \Big(\oint_{\Gamma_t} |\nabla u|\,ds\Big)\,dt\ . \tag{2.3}$$

Then

$$-\frac{dS}{dv} = \oint_{\Gamma_v} |\nabla u|\,ds = \int_{\Omega_v} p\,dx =: a(v) \tag{2.4}$$

where we have used Green's identity and defined the quantity $a(v)$ such that

$$a(0) = \int_\Omega p\,dx \equiv A \quad \text{and} \quad a(u_m) = 0\ .$$

Next we make use of the fact that (see [1], p. 53)

$$-\frac{da}{dv} = \oint_{\Gamma_v} p\,\frac{ds}{|\nabla u|} \qquad \text{a.e. in } (0, u_m)\ . \tag{2.5}$$

By Schwarz's inequality one has

$$\oint_{\Gamma_v} |\nabla u|\, ds \cdot \oint_{\Gamma_v} p\, \frac{ds}{|\nabla u|} \geq \left(\oint_{\Gamma_v} \sqrt{p}\, ds \right)^2 \tag{2.6}$$

At this point we can use Bol's inequality (see [1], p. 36) which states that if $p(x)$ satisfies (1.7) and (1.8) then

$$\left(\oint_{\Gamma_v} \sqrt{p}\, ds \right)^2 \geq a(v)(4\pi - K a(v)) \,. \tag{2.7}$$

Combining now (2.4), (2.6) and (2.7) we are led to the inequality

$$\frac{d^2 S}{dv^2} - K \cdot \frac{dS}{dv} \geq 4\pi \tag{2.8}$$

or in equivalent form as

$$\frac{d}{dv} \left(e^{-Kv} \cdot \frac{dS}{dv} \right) \geq 4\pi e^{-Kv} \,. \tag{2.9}$$

Integration of (2.9) from a value $v = v_0$ to $v = u_m$ gives after some rearrangement

$$-\frac{dS}{dv}\bigg|_{v_0} \geq \frac{4\pi}{K} \left(1 - e^{-K(u_m - v_0)}\right) , \tag{2.10}$$

since $-\frac{dS}{dv}\Big|_{u_m} = a(u_m) = 0$.

For $v_0 = 0$ (2.10) reads

$$A \geq \frac{4\pi}{K} \left(1 - e^{-K u_m}\right) \tag{2.11}$$

or equivalently

$$u_m \leq \frac{1}{K} \log \left(\frac{4\pi}{4\pi - KA} \right) \tag{2.12}$$

as noted by Bandle (see [1]).

If we now integrate (2.10) one more time from $v_0 = 0$ to $v = u_m$ we are led to

$$\begin{aligned} S(0) &= \int_0^{u_m} \left(\oint_{\Gamma_t} |\nabla u|\, ds \right) dt = \int_\Omega |\nabla u|^2 dx = S \\ &\geq \frac{4\pi}{K} \left[u_m - \frac{1}{K} \left(1 - e^{-K u_m}\right) \right] , \end{aligned} \tag{2.13}$$

which is inequality (2.1).

(For the second equality sign in (2.13), see e.g. [4], p. 190). Inequality (2.2) is obtained in a completely analogous manner: the first integration of (2.9) is now from $v = 0$ to $v = v_0$ and the second is from $v_0 = 0$ to $v = u_m$ as before.

3 REMARKS

(a) It was shown by Bandle (see [1]) that

$$S \leq \frac{4\pi}{K^2} \log \frac{4\pi}{4\pi - KA} - \frac{A}{K} .$$

If we write (2.13) as

$$S + \frac{4\pi}{K^2} (1 - e^{-Ku_m}) \geq \frac{4\pi}{K} u_m$$

and use (2.11) and (2.12), we see that the upper bound for u_m given in (2.1) is sharper than the bound (2.12), but it requires the knowledge of S or a close upper bound for S.

(b) There are other types of bounds that can be obtained from the differential inequality (2.8). For example if we write it in terms of $a(v)$ as

$$-\frac{da}{dv} \geq 4\pi - Ka(v)$$

and then change the independent variable and writing u in the place of v it becomes

$$-\frac{du}{da} \leq \frac{1}{4\pi - Ka} . \tag{3.2}$$

This inequality can be integrated in many ways. As an example we perform a double integration as follows:

$$\int_0^A \left[\int_s^A \left(-\frac{du}{da}\right) da\right]^n ds = \int_\Omega u^n \, p \, dx \leq \int_0^A \left[\int_s^A \frac{da}{4\pi - Ka}\right]^n ds . \tag{3.3}$$

Setting $f = \frac{1}{K} \log\left(\frac{4\pi}{4\pi - KA}\right) =$ upper bound for u_m one has e.g.

$$\int_\Omega u^2 \, p \, dx \leq \frac{2}{K} \left(2\pi f^2 + \frac{A}{K} - \frac{4\pi}{K} f\right) . \tag{3.4}$$

If instead of the double integral $\int_0^A \int_s^A ()$ we select $\int_0^A \int_0^s ()$ then we obtain

$$S \geq A \cdot u_m + \frac{1}{K} (4\pi - KA) \cdot f - \frac{A}{K} . \tag{3.5}$$

(c) A number of other types of bounds for problems (1.1) and (1.6) can be found in [1, 2, 4].

References

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