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# Small Values of Polynomials: Cartan, Pólya and Others

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Let $P(z)$ be a monic polynomial of degree $n$, and $\alpha, \varepsilon > 0$. A classic lemma of Cartan asserts that the lemniscate $E(P;\varepsilon) := \{z : |P(z)| \leq \varepsilon^n\}$ can be covered by balls $B_j$, $1 \leq j \leq n$, whose diameters $d(B_j)$ satisfy

$$\sum_{j=1}^{p} \left(d(B_j)\right)^{\alpha} \leq e(4\varepsilon)^{\alpha}.$$

For $\alpha = 2$, this shows that $E(P;\varepsilon)$ has an area at most $\pi e(2\varepsilon)^2$. Pólya showed in this case that the sharp estimate is $\pi\varepsilon^2$. We discuss some of the ramifications of these estimates, as well as some of their close cousins, for example when $P$ is normalized to have $L_p$ norm 1 on some circle, and Remez' inequality.

*Keywords:* Polynomials; small values; Cartan's lemma; Pólya; Remez; capacity.

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## 1 INTRODUCTION

On how large a set can a polynomial be small? This simple question and its cousins has fascinated mathematicians of the status of H. Cartan, G. Pólya and P. Erdös; its ramifications range from the theory of entire functions and potential theory to rational approximation and orthogonal polynomials. In this paper we shall discuss some of these results.

The first step is normalization of the polynomial. The obvious choices are normalizing $P$ to be monic, that is, to have leading coefficient 1:

$$P(z) = z^n + \dots \tag{1.1}$$

or to have some norm 1, for example, for some fixed $r > 0$, $0 < p \leq \infty$

$$\|P\|_{L_p(|z|=r)} = 1. \tag{1.2}$$

Let us begin with the former, namely monic polynomials. The classic 1928 result of Cartan [4], introduced with a view to studying minimum moduli of holomorphic functions, still reigns supreme. Let $P$ be monic of degree $n$, and $\varepsilon, \alpha > 0$. The lemma asserts that

$$E(P; \varepsilon) := \{z : |P(z)| \leq \varepsilon^n\} \subset \bigcup_{j=1}^{p} B_j \tag{1.3}$$

where $p \leq n$, and $B_1, B_2, \ldots B_p$ are balls with diameters $d(B_j)$ satisfying

$$\sum_{j=1}^{p} \left(d(B_j)\right)^\alpha \leq e(4\varepsilon)^\alpha. \tag{1.4}$$

What is remarkable is that this estimate does not depend on the degree $n$ of $P$. The naive approach of placing balls of radius $\varepsilon$ centred on each of the zeros of $P$ and using these to cover $E(P; \varepsilon)$ gives the much weaker estimate of $n(2\varepsilon)^\alpha$.

In particular, when $\alpha = 2$, and $meas_2$ denotes planar Lebesgue measure, (1.4) becomes

$$meas_2\big(E(P; \varepsilon)\big) \leq \pi e(2\varepsilon)^2.$$

In this special case, a result of Pólya from 1928 asserts that there is the sharp estimate

$$meas_2\big(E(P; \varepsilon)\big) \leq \pi \varepsilon^2$$

with equality iff $P(z) = (z - a)^n$, some $a \in \mathbb{C}$.

It would be a real challenge to determine for general $\alpha$, the sharp constant that should replace $e4^\alpha$ in (1.4). For $\alpha = 1$, the conjectured sharp constant is 4.

Like all great inequalities, Cartan's has inspired extensions. In Section 2, we shall state and prove an extension where $d(B_j)^\alpha$ is replaced by $h\big(d(B_j)\big)$ for suitable increasing functions $h$. This form is useful in studying thin sets that arise in potential theory and Padé approximation.

In Section 3, we shall prove Pólya's inequality. Because it's proof is so closely linked to Green's theorem, and the latter is tied to planar measure, Pólya's methods do not seem to have many generalizations. Or have they been missed?

The alternative normalization (1.2) is often more useful than the monic one. For example, in convergence theory of Padé approximation, one is called on to estimate the ratio

$$\|P\|_{L_\infty(|z|=r)}/\left|P(z)\right|$$

outside as small a set as possible. If one adopts the normalization (1.2) with $p = \infty$, then at least the numerator is taken care of, and then one wants to see how large can be the set on which $\left|P(z)\right|$ is small. Until quite recently, this has always been via capacitary estimates or Cartan's lemma: One splits, following Nuttall and others,

$$P(z) = c \prod_{|z_j|\le 2r} (z - z_j) \prod_{|z_j|>2r} \left(1 - \frac{z}{z_j}\right) =: cR(z)S(z).$$

Since for $|z| \le r$, $|z_j| > 2r$,

$$\frac{1}{2} < \left|1 - \frac{z}{z_j}\right| < \frac{3}{2}$$

and for $|z| \le r$, $|z_j| \le 2r$,

$$|z - z_j| \le 3r$$

we see that

$$\|P\|_{L_\infty(|z|=r)}/\left|P(z)\right| \le \left(3\max\{1, r\}\right)^n/\left|R(z)\right|.$$

As $R$ is monic, one can apply Cartan's lemma to deduce that

$$\|P\|_{L_\infty(|z|=r)}/\left|P(z)\right| \le \left(3\max\{1, r\}/\varepsilon\right)^n$$

for $|z| \le r$ outside a set that can be covered by balls $\{B_j\}$ admitting the estimate (1.4).

This procedure, of separating zeros into small and large ones, is useful in many contexts, and has been used by mathematicians of the status of Nuttall, Pommerenke, Goncar, Stahl and others but does not admit easy extension to several variables. Motivated by the latter, A. Cuyt, K. Driver and the author [5] were led to directly consider the normalization (1.2). This and the Walsh-Bernstein inequality, which bounds polynomials anywhere in the plane in terms of their maximum modulus on some given set that is not "too thin", very quickly give for the restricted lemniscate

$$E(P; r; \varepsilon) := \left\{z : |z| \le r, \left|P(z)\right| \le \varepsilon^n\right\}$$

the estimates

$$cap\big(E(P;r;\varepsilon)\big) \le 2r\varepsilon \tag{1.5}$$

and

$$meas_2\big(E(P;r;\varepsilon)\big) \le \pi(2r\varepsilon)^2 \tag{1.6}$$

and that these are sharp for each $r, n$. Here *cap* is logarithmic capacity (we shall define that in Section 2). We shall discuss this simple approach in Section 3, giving also some of its $L_p$ extensions.

In Section 4, we review some of the implications of Remez' inequality for small values of polynomials. In Section 5, we shall briefly review some of the multivariate extensions of Cartan's lemma. Here it is difficult to decide what is a monic polynomial! Moreover, the measures of thinness of sets become quite complicated, and the fact that the lemniscate is unbounded leads to difficulties.

## 2 THE CARTAN APPROACH

We begin by presenting Cartan's classical argument in a general form, the power of which lies in the arbitrary values assigned to the numbers $r_j$.

THEOREM 2.1 *Let $0 < r_1 < r_2 < \cdots < r_n$ and $P(z)$ be a monic polynomial of degree $n$. There exist positive integers $p \le n$, $\{\lambda_j\}_{j=1}^p$ and closed balls $\{B_j\}_{j=1}^p$ such that*

*(i)* $\lambda_1 + \lambda_2 + \cdots + \lambda_p = n$.

*(ii)* $d(B_j) = 4r_{\lambda_j}$, $1 \le j \le p$.

*(iii)*

$$\left\{z : \left|P(z)\right| \le \prod_{j=1}^{n} r_j\right\} \subset \bigcup_{j=1}^{p} B_j. \tag{2.1}$$

*Proof* We divide this into four steps:

**STEP 1:** We show that there exists $\lambda_1 \le n$ and a circle $C_1$ of radius $r_{\lambda_1}$ containing exactly $\lambda_1$ zeros of $P$, counting multiplicity.

For suppose such a circle does not exist. Then any circle $C$ of radius $r_1$ containing 1 zero of $P$ contains at least 2. The concentric circle of radius $r_2$ contains 2, so must contain 3 (otherwise we could choose $\lambda_1 = 2$ and $C_1$ to be this circle). Continuing in this way, we eventually find that the circle concentric with $C$ and radius $r_n$ must contain $n + 1$ zeros of $P$, which is impossible.

**Step 2:** We rank the zeros of $P$.

Choose the largest $\lambda_1$ with the property in Step 1, and let $C_1$ be the corresponding circle. Call the $\lambda_1$ zeros of $P$ inside $C_1$ zeros of rank $\lambda_1$. Next, applying the argument of Step 1 to the remaining $n - \lambda_2$ zeros of $P$, we obtain a largest positive integer $\lambda_2 \leq \lambda_1$ and a circle $C_2$ containing exactly $\lambda_2$ of the zeros of $P$ outside $C_1$. Call those zeros inside $C_2$ zeros of rank $\lambda_2$. Continuing in this way, we find $p \leq n$ largest integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ and corresponding circles $C_j$ of radius $r_{\lambda_j}$ containing exactly $\lambda_j$ zeros of $P$ outside $C_1 \cup C_2 \cup \cdots \cup C_{j-1}$. Moreover, as we eventually exhaust the zeros, $\lambda_1 + \lambda_2 + \cdots + \lambda_p = n$.

**Step 3:** We prove that if $S$ is a circle of radius $r_\lambda$ containing at least $\lambda$ zeros of $P$, then at least one of these zeros has rank at least $\lambda$.

First if $S$ contains more than $\lambda_1$ zeros, then at least one must lie in $C_1$ and so have rank $\lambda_1 \geq \lambda$. (If not, we would obtain a contradiction to the choice of $\lambda_1$ being as large as possible). Next suppose that $\lambda_j \geq \lambda > \lambda_{j+1}$, some $j$. If any of the zeros inside $S$ lies in $C_1, C_2, \ldots, C_j$ then these have rank $\geq \lambda_j \geq \lambda$, as required. If all the zeros lie outside these former $j$ circles, then the process of Step 1 yields a circle with $\geq \lambda > \lambda_{j+1}$ zeros contradicting the choice of $\lambda_{j+1}$ being as large as possible.

**Step 4:** Complete the proof

Let $B_j$ be the (closed) ball concentric with $C_j$ but twice the radius, so that $d(B_j) = 4r_{\lambda_j}$, $1 \leq j \leq p$. Fix $z \in \mathbb{C} \setminus \cup_{j=1}^{p} B_j$. We claim that a circle $S$, centre $z$, radius $r_\lambda$, can contain at most $\lambda - 1$ zeros of $P$. For if it contained at least $\lambda$, then by Step 3, at least one, $u$ say, would have, say, rank $\lambda_j \geq \lambda$, and so lie in $C_j$ and also in the concentric ball $B_j$ of twice the radius. Then the fact that $z \notin B_j$ and $u$ lies inside $C_j$ forces

$$|z - u| > dist(\mathbb{C} \setminus B_j, C_j) = r_{\lambda_j} \geq r_\lambda$$

contradicting our hypothesis that $S$ contains $u$.

Finally rearrange the zeros in order of increasing distance from $z$ as $z_1, z_2, \ldots, z_n$. Now the circle centre $z$, radius $r_j$ can contain at most $j-1$ zeros of $P$, and these could only be $z_1, z_2, \ldots, z_{j-1}$ so

$$|z - z_j| > r_j.$$

Thus

$$\left|P(z)\right| = \left|\prod_{j=1}^{n}(z - z_j)\right| > \prod_{j=1}^{n} r_j. \qquad \square$$

I wonder if a non-geometric proof will ever be found as an alternative to Cartan's beautiful one above. Now by choosing $\{r_j\}$ in various ways, we can obtain all sorts of estimates.

COROLLARY 2.2 *Let $P$ be a monic polynomial of degree $n$ and $\varepsilon, \alpha > 0$. Then*

$$E(P;\varepsilon) = \left\{z : \left|P(z)\right| \leq \varepsilon^n\right\} \subset \bigcup_{j=1}^{p} B_j$$

*where*

$$\sum_{j=1}^{p}\left(d(B_j)\right)^{\alpha} \leq e(4\varepsilon)^{\alpha}.$$

*Proof* We choose

$$r_j := \varepsilon j^{1/\alpha}(n!)^{-1/(\alpha n)}, \quad 1 \leq j \leq n.$$

Then

$$\prod_{j=1}^{n} r_j = \varepsilon^n$$

and with $\{\lambda_j\}$ as in Theorem 2.1,

$$\sum_{j=1}^{p}\left(d(B_j)\right)^{\alpha} = 4^{\alpha}\sum_{j=1}^{p}(r_{\lambda_j})^{\alpha} = (4\varepsilon)^{\alpha}(n!)^{-1/n}\sum_{j=1}^{p}\lambda_j$$
$$= (4\varepsilon)^{\alpha}\left(\frac{n^n}{n!}\right)^{1/n} \leq e(4\varepsilon)^{\alpha}$$

by the elementary inequality $n! \geq (n/e)^n$. $\square$

For some applications, one needs to replace $d(B_j)^\alpha$ by $h\big(d(B_j)\big)$ for some positive monotone increasing function $h(t)$ that has limit 0 at 0. Such functions are closely associated with Hausdorff $h$ — content or measure. Let $h : [0, \infty) \to [0, \infty)$ be a monotone increasing non-negative function with limit 0 at 0. The corresponding Hausdorff content, is defined for $E \subset \mathbb{C}$, by

$$h - c(E) := \inf\left\{\sum_{j=1}^{\infty} h\big(d(B_j)\big) : E \subset \bigcup_{j=1}^{\infty} B_j\right\}.$$

Note that only balls $B_j$ are considered for covering $E$. The Hausdorff $h$-measure is defined similarly: when taking the inf, one restricts each $B_j$ to have $d(B_j) \leq \delta$ and then lets $\delta \to 0+$. For our purposes, we note only that Hausdorff $h$-measure and $h$-content vanish on the same sets. The classic reference is Rogers [19]. In this language, if we let $h_\alpha(t) = t^\alpha$, the estimate of Corollary 2.2 may be written as

$$h_\alpha - c\big(E(P;\varepsilon)\big) \leq e(4\varepsilon)^\alpha.$$

In formulating our result for $h - c$, we need the generalized inverse of a continuous function $g : [0, 1] \to \mathbb{R}$, defined by

$$g^{[-1]}(s) := \min\{t : g(t) = s\}, s \in g\big([0, 1]\big).$$

THEOREM 2.3 *Let $h : [0, 1] \to [0, \infty)$ be strictly increasing in $[0, 1]$ with limit 0 at 0, and absolutely continuous in each closed subinterval of $(0, 1]$.*

*(I) Assume*

$$\int_0^1 \frac{h(t)}{t} dt < \infty. \tag{2.2}$$

*Let*

$$g(t) := \exp\left(-\frac{1}{h(t)}\int_0^t |\log u| h'(u)du\right),\ t \in (0, 1). \tag{2.3}$$

*Then for $n \geq 1$ and monic polynomials $P$ of degree $n$,*

$$h - c\big(E(P;\delta)\big) \leq h\big(g^{[-1]}(4\delta)\big),\ \delta \in \left[0, \frac{g(1)}{4}\right]. \tag{2.4}$$

*In particular, the bounds decays to 0 as $t \to 0+$, since*

$$\lim_{t\to 0+} g^{[-1]}(t) = 0. \tag{2.5}$$

*(II) Conversely, if there exists a monotone increasing function* $\chi : [0,1] \to [0,\infty)$ *with limit* 0 *at* 0, *such that for* $n \geq 1, \delta \in (0,1)$ *and monic polynomials P of degree n,*

$$h - c\big(E(P;\delta)\big) \leq \chi(\delta),\ \delta \in (0,1) \tag{2.6}$$

*while for some* $0 < \beta < 2, h(t)/t^{\beta}$ *is monotone decreasing near* 0, *then (2.2) holds.*

Under additional conditions, we can replace the above implicit estimate by something simpler:

COROLLARY 2.4 *Suppose there exists* $A > 1$ *such that*

$$\frac{1}{h(t)|\log t|}\int_0^t \frac{h(u)}{u}du \leq A - 1,\ t \in \left(0, \frac{1}{2}\right]. \tag{2.7}$$

*Then (2.4) may be replaced by*

$$h - c\big(E(P;\delta)\big) \leq h\big((4\delta)^{1/A}\big),\ \delta \in [0, 2^{A-2}]. \tag{2.8}$$

COROLLARY 2.5 *If* $h(t) = \left(\log \frac{1}{t}\right)^{-\gamma}, \gamma > 1$, *then (2.4) may be replaced by*

$$h - c\big(E(P;\delta)\big) \leq \left(\frac{\gamma - 1}{\gamma}\right)^{-\gamma} h(4\delta),\ \delta \in \left[0, \frac{1}{4}\right). \tag{2.9}$$

It is noteworthy that only the behaviour of $h$ in an arbitrarily small neighbourhood of 0 is important in applications, for which $\delta$ above is usually close to 0. Thus one may always modify $h$ away from 0 to ensure its definition throughout $(0, 1)$. We turn to the

*Proof of Theorem 2.3 (I)* Let $h^{[-1]}$ be the (ordinary) inverse of $h$, defined at least on $[0, h(1)]$ by $h\big(h^{[-1]}(u)\big) = u$. Fix $H \in [0,1]$ and set

$$r_j := \frac{1}{4}h^{[-1]}\left(\frac{j}{n}h(H)\right),\ 1 \leq j \leq n. \tag{2.10}$$

Then if $\{\lambda_j\}$ are as in Theorem 2.1,

$$\sum_{j=1}^{p} h\big(d(B_j)\big) = \sum_{j=1}^{p} h(4r_{\lambda_j}) = h(H)\sum_{j=1}^{p}\frac{\lambda_j}{n} = h(H).$$

Moreover,

$$\prod_{j=1}^{n} r_j = 4^{-n} \exp\left(\sum_{j=1}^{n} \log h^{[-1]}\left(\frac{j}{n}h(H)\right)\right) \geq 4^{-n} \exp(I)$$

where, by the monotonicity of $h$, $h^{[-1]}$,

$$I := \int_0^n \log h^{[-1]}\left(\frac{u}{n}h(H)\right) du.$$

The substitution $h(v) = \frac{u}{n}h(H)$ gives

$$I = \frac{n}{h(H)} \int_0^H (\log v) h'(v)\, dv = n \log g(H).$$

In summary,

$$\prod_{j=1}^{n} r_j \geq \left(\frac{g(H)}{4}\right)^n. \tag{2.11}$$

This works only if the integral defining $g$ is meaningful, which we now show: An integration by parts gives

$$\begin{aligned} \log g(t) &= \frac{1}{h(t)} \int_0^t (\log v) h'(v) \quad dv \\ &= \log t - \frac{1}{h(t)} \int_0^t \frac{h(v)}{v} dv. \end{aligned} \tag{2.12}$$

Here we have used the fact that $h(t) \log t$ has limit 0 at 0: this is an easy consequence of the convergence of the integral in (2.2). So we see that $g$ is well defined. Theorem 2.1 and (2.11) give

$$h - c\left(\left\{z : |P(z)| \leq \left(\frac{g(H)}{4}\right)^n\right\}\right) \leq h(H).$$

Now $g$ is continuous, $h' \geq 0$, so we see from (2.3) that $0 < g(t) \leq t$, $t \neq 0$ so (2.5) holds and $g([0, 1]) \supseteq [0, g(1)]$. Thus for each $\delta \in \left[0, \frac{g(1)}{4}\right]$, there exists $H \in [0, 1]$ such that $\delta = g(H)/4$. Moreover, we may choose $H = g^{[-1]}(4\delta)$.

□

*Proof of Corollary 2.4* From (2.12), we see that the bound given in (2.7) is equivalent to

$$-\log g(t) \le A|\log t|, \ t \in \left(0, \frac{1}{2}\right]$$
$$\Rightarrow g(t) \ge t^A \Rightarrow g^{[-1]}(u) \le u^{1/A}, \ u \in [0, 2^{-A}]. \qquad \square$$

*Proof of Corollary 2.5* A calculation shows that $g(t) = t^{\gamma/(\gamma-1)}$ and hence $g^{[-1]}(u) = u^{(\gamma-1)/\gamma}$. One then uses the specific form of $h$. $\square$

In the proof of Theorem 2.3(II) and also in the next section, we shall need the notion of (logarithmic) capacity. There are at least four equivalent definitions, but the simplest is the following: For compact $E \subset \mathbb{C}$,

$$cap(E) = \lim_{n\to\infty} \left( \min_{\substack{\deg(P)=n,\\ P \text{ monic}}} \|P\|_{L_\infty(E)} \right)^{1/n}.$$

For non-compact $F \subset \mathbb{C}$,

$$cap(F) = \sup\{cap(E) : E \subset F, E \text{ compact}\}.$$

For a proof that the above limit exists, and an introduction to *cap*, perhaps the best source is Chapter 16 of Hille [9]. Deeper treatments may be found in Carleson [3], Hayman and Kennedy [8] and Landkof [11]. What is particularly relevant for the purposes of this paper is that for monic polynomials $P$ and $\varepsilon > 0$,

$$cap\big(E(P;\varepsilon)\big) = \varepsilon.$$

(The inequality $cap\big(E(P;\varepsilon)\big) \le \varepsilon$ is easily proved from the definition of *cap*; the converse is a little more difficult, requiring the maximum modulus principle). This identity shows that *cap* is often the natural set function to measure small values of polynomials.

We turn to the

*Proof of Theorem 2.3(II)* Assume (2.6) holds. We shall assume the integral in (2.2) diverges and derive a contradiction. We use two results of S.J. Taylor [23]: Because of the assumed regularity condition on $h$, there exists a compact set $E$ of finite positive Hausdorff $h$-measure, and hence also with $0 < h - c(E) < \infty$. In addition, $E$ has so-called positive lower spherical density at each of its points. If the integral in (2.2) diverges, another theorem of Taylor ensures that $cap(E) = 0$. But then from the definition of *cap*, for arbitrarily small $\varepsilon > 0$, and correspondingly suitable $n$ and $P$ of degree $n$,

$$E \subset \{z : |P(z)| \leq \varepsilon^n\}$$

and hence (2.6) gives

$$h - c(E) \leq h - c\Big(\{z : |P(z)| \leq \varepsilon^n\}\Big) \leq \chi(\varepsilon).$$

Since $\chi$ has limit 0 at 0, we deduce that $h - c(E) = 0$, a contradiction. So the integral in (2.2) must converge. □

Theorem 2.3 and its corollaries are neater formulations of (presumably new) results in [14, 15]. One of their consequences are explicit estimates relating $cap$ and $h - c$. Since any compact set $E$ can be contained in a lemniscate $\{z : |P(z)| \leq (cap(E) + \varepsilon)^n\}$ for arbitrarily small $\varepsilon$, it follows that under the hypotheses of Theorem 2.3,

$$h - c(E) \leq h\Big(g^{[-1]}\big(4cap(E)\big)\Big) \tag{2.13}$$

provided $cap(E) < \frac{g(1)}{4}$. This extends to arbitrary sets $E$ with $cap(E) < \frac{g(1)}{4}$ and when $cap(E) \geq g(1)/4$, we may simply scale $E$ by multiplying all its elements by some small positive number. Similarly, Corollary 2.4 gives

$$h - c(E) \leq h\Big(\big[4cap(E)\big]^{1/A}\Big) \tag{2.14}$$

and Corollary 2.5 gives for $h(t) = \left(\log \frac{1}{t}\right)^{-\gamma}$, $\gamma > 1$,

$$h - c(E) \leq \left(\frac{\gamma - 1}{\gamma}\right)^{-\gamma} h\big(4cap(E)\big). \tag{2.15}$$

It is still possible to obtain non-trivial estimates when the integral in (2.2) diverges. One still chooses $r_j$ by (2.10), but instead estimates

$$\prod_{j=2}^{n} r_j \geq 4^{-(n-1)} \exp(I)$$

where

$$I = \int_1^n \log h^{[-1]}\left(\frac{u}{n} h(H)\right) du = \frac{n}{h(H)} \int_{h^{[-1]}(h(H)/n)}^{H} (\log v) h'(v)\, dv.$$

On integrating by parts, we see that the term coming from $r_1$ cancels, and we obtain

$$\prod_{j=1}^{n} r_j \geq 4^{-n} \exp\left(n\left(\log H - \frac{1}{h(H)} \int_{h^{[-1]}(h(H)/n)}^{H} \frac{h(v)}{v} dv\right)\right) =: \left(\frac{g_n(H)}{4}\right)^n$$

and hence

$$h - c\left(\left\{z : |P(z)| \leq \left(\frac{g_n(H)}{4}\right)^n\right\}\right) \leq h(H).$$

In specific cases such as $h(t) := \left(\log \frac{1}{t}\right)^{-\gamma}$ direct calculation of $g_n(H)$ leads to

$$h - c\left(\left\{z : |P(z)| \leq \delta^n\right\}\right) \leq \begin{cases} (1 + \log n)h(4\delta), & \gamma = 1 \\ n^{1-\gamma}(1-\gamma)^{-\gamma}h(4\delta), & 0 < \gamma < 1 \end{cases}. \tag{2.16}$$

While we have focused on polynomials, Cartan's lemma has some of its most powerful applications when applied to potentials [8, 11]. The arguments are similar to that of Theorem 2.3, but the formulation is different. Let us briefly indicate the extension to generalized polynomials [1, 6]. A generalized polynomial of degree $n$ is an expression

$$P(z) := \prod_{j=1}^{m} |z - z_j|^{\alpha_j} : \sum_{j=1}^{m} \alpha_j = n.$$

Here all $\alpha_j > 0$ but are not necessarily integers. (Even $n$ need not be an integer, as the reader will easily see). All the estimates of Theorem 2.3 and its corollaries go through for such $P$. Indeed, because of continuity, we can assume that all $\alpha_j$ are rational, and have form $k_j/N$ for some positive integers $k_j$ and some positive integer $N$ independent of $j$. Then if

$$Q(z) := \prod_{j=1}^{m} (z - z_j)^{k_j}$$

we see that $Q$ is monic of degree $nN$ and

$$E(P; \varepsilon) = \left\{z : |P(z)| \leq \delta^n\right\} = \left\{z : |Q(z)| \leq \delta^{nN}\right\}.$$

Since (2.4), (2.8), (2.9) do not depend on $n$, $N$, they remain valid for the more general form of $P$. However this should hardly be surprising as $E(P;\varepsilon)$ still has capacity $\varepsilon$, so we are simply reformulating special cases of (2.13) to (2.15).

## 3 THE PÓLYA APPROACH

We shall follow Goluzin [7] in proving Pólya's

THEOREM 3.1 *For monic polynomials $P$ of degree $n$,*

$$meas_2\big(E(P;\varepsilon)\big) \leq \pi\varepsilon^2 \tag{3.1}$$

*with equality iff $P(z) = (z-a)^n$.*

*Proof* We split this into several steps.

**STEP 1:** Describe the lemniscate $\Gamma := \{z : |P(z)| = \varepsilon^n\}$ as a union of contours $\Gamma_j$, $1 \leq j \leq m$.

Consider the map $\xi = P(z)$ and its inverse algebraic function $z = P^{[-1]}(\xi)$. If $\Gamma$ contains none of the points $z$ with $P'(z) = 0$, then $\Gamma$ consists of finitely many disjoint closed analytic Jordan curves, say $\Gamma_j$, $1 \leq j \leq m$. By the maximum principle, $|P(z)| < \varepsilon^n$ inside each $\Gamma_j$ and each $\Gamma_j$ encloses at least one zero of $P$. Then $m \leq n$.

**STEP 2:** Parametrize each $\Gamma_j$.

Let us suppose that $\Gamma_j$ contains zeros of total multiplicity $l_j$. As $z$ moves around $\Gamma_j$, $\xi = P(z)$ moves around $|\xi| = \varepsilon^n$ exactly $l_j$ times and $\xi = P(z)^{1/l_j}$ moves once around the circle $|\xi| = \varepsilon^{n/l_j}$. Hence one of the branches of $z = P^{[-1]}(\xi^{l_j})$ is analytic and single valued on $|\xi| = \varepsilon^{n/l_j}$, admitting there the Laurent series expansion

$$P^{[-1]}(\xi^{l_j}) = \sum_{k=-\infty}^{\infty} a_k^{(j)} \xi^k$$

so that for a suitable branch

$$P^{[-1]}(\xi) = \sum_{k=-\infty}^{\infty} a_k^{(j)} \xi^{k/l_j}. \tag{3.2}$$

Then setting $\xi = \varepsilon^n e^{i\theta}$, we obtain a parametrization of $\Gamma_j$,

$$\gamma_j(\theta) := \sum_{k=-\infty}^{\infty} a_k^{(j)} (\varepsilon^n e^{i\theta})^{k/l_j}, \theta \in [0, 2\pi l_j]. \tag{3.3}$$

**Step 3:** Calculate area enclosed by $\Gamma_j$ and hence the area enclosed by $\Gamma$.

A well known consequence of Green's theorem is a formula for the area $A_j$ enclosed by $\Gamma_j$:

$$A_j = \frac{1}{2}\int_{\Gamma_j} (xdy - ydx) = \frac{1}{2}\mathrm{Im}\left(\int_{\Gamma_j} \bar{z}\, dz\right)$$

where the second integral is a complex contour integral. Using our parametrization (3.3), we see that

$$A_j = \frac{1}{2}\mathrm{Im}\left(\int_0^{2l_j\pi} \overline{\gamma_j(\theta)}\gamma_j'(\theta)d\theta\right) = \pi \sum_{k=-\infty}^{\infty} \left|a_k^{(j)}\right|^2 k\varepsilon^{2kn/l_j}.$$

Adding over $j$, we see that the area $A$ enclosed by $\Gamma$ admits the identity

$$\frac{A}{\pi\varepsilon^2} = \sum_{j=1}^{m}\sum_{k=-\infty}^{\infty} \left|a_k^{(j)}\right|^2 k\varepsilon^{2kn/l_j-2}. \tag{3.4}$$

**Step 4:** Prove Pólya's inequality by letting $\varepsilon \to \infty$.

From (3.4), we see that for $k > 0$, the terms in the series have positive coefficients and positive powers of $\varepsilon$, increase with $\varepsilon$, while the terms for $k < 0$ have negative coefficients and negative powers of $\varepsilon$, also increase with $\varepsilon$. Thus $A/(\pi\varepsilon^2)$ increases with $\varepsilon$. But for large $\varepsilon$, $\Gamma$ consists of only one curve $\Gamma_1$ and $P^{[-1]}$ has on the curve $|\xi| = \varepsilon^n$ the expansion

$$P^{[-1]}(\xi) = \xi^{1/n} + a_0^{(1)} + a_{-1}^{(1)}\xi^{-1/n} + a_{-2}^{(1)}\xi^{-2/n} + \dots$$

corresponding to the inverse of the polynomial $\xi = P(z)$ about $\xi = \infty$. Integrating over $\Gamma_1$ gives as before (recall now $l_1 = n$ and $a_1^{(1)} = 1$ and $a_k^{(1)} = 0, k \geq 2$),

$$\frac{A}{\pi\varepsilon^2} = \sum_{k=-\infty}^{\infty} \left|a_k^{(1)}\right|^2 k\varepsilon^{2k-2} = 1 - \sum_{k=1}^{\infty} \left|a_{-k}^{(1)}\right|^2 k\varepsilon^{-2(k+1)}.$$

Thus for large $\varepsilon$, $A/(\pi\varepsilon^2) \leq 1$, and hence for all smaller $\varepsilon$ also. Moreover, we see that there is equality iff $a_{-k}^{(1)} = 0, k \geq 1$, that is,

$$P^{[-1]}(\xi) = \xi^{1/n} + a_0 \Leftrightarrow z = P(z)^{1/n} + a_0 \Leftrightarrow P(z) = (z - a_0)^n.$$

□

COROLLARY 3.2 *For bounded Borel sets $F$,*

$$meas_2 F \leq \pi(cap\ F)^2. \tag{3.5}$$

*Proof* If $F$ is compact, then given $\varepsilon > 0$, we can (by definition of $cap$) find $n$ and a monic polynomial $P$ of degree $n$ such that $F \subset E\big(P; cap(F) + \varepsilon\big)$. Then Theorem 3.1 gives

$$meas_2(F) \leq meas_2\big(E(P; cap\ F + \varepsilon)\big) \leq \pi(cap\ F + \varepsilon)^2.$$

Then (3.5) follows on letting $\varepsilon \to 0+$. The measurability and capacitability of general Borel sets can be used to deduce the general case [8]. □

We note one related result also due to Pólya, also proved about the same time: If $P$ is a monic polynomial of degree $n$, and $L$ is any line in the plane, then there is the sharp estimate

$$meas_1\big(E(P; \varepsilon) \cap L\big) \leq 2^{2-1/n}\varepsilon. \tag{3.6}$$

Here $meas_1$ denotes linear Lebesgue measure. The proof of this involves factorization of $P$ and successively moving the intervals of $E(P; \varepsilon) \cap L$ thereby showing that the measure is maximized when $E(P; \varepsilon)$ is a single interval and $P$ is essentially, a Chebyshev polynomial of degree $n$. This is similar to a Remez type argument, for those familiar with the latter.

It has been conjectured that if we replace $meas_1$ by one-dimensional Hausdorff measure, then (3.6) holds without having to project $E(P; \varepsilon)$ onto its intersection with the line $L$. If proved, this would show that for $\alpha = 1$, the sharp constant in Cartan's (1.4) is 4, not $4e$.

While Pólya's proof above is a beautiful application of Green's theorem, it seems very closely tied to planar measure. It would be nice to see the above argument modified to treat other measures, if this is at all possible.

## 4 POLYNOMIALS WITH $L_P$ NORMALIZATION

Despite their beauty, Cartan and Pólya's estimates are closely linked to one variable: Factorization of polynomials in higher dimensions is more complicated, and Green's theorem is very much a plane animal. Moreover, there are many notions of what constitutes a multivariate monic polynomial. So in the course of investigating convergence of multivariate

Padé approximants, A. Cuyt, K. Driver and the author were forced to consider alternative approaches, and the normalization (1.2) was undoubtedly the easiest to extend. To our surprise, we discovered sharp univariate inequalities with this normalization, and moreover the proofs are simple, though these do involve the notion of Green's functions.

In this section, we shall present some of these results for the univariate case, proved for $p = \infty$ in [5] and for $0 \leq p < \infty$ in [16]. Let us set

$$\|P\|_{L_p(|z|=r)} = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P(re^{i\theta})\right|^p d\theta\right)^{1/p}, & 0 < p < \infty \\ \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(re^{i\theta})\right| d\theta\right), & p = 0 \end{cases}.$$

For $p = \infty$, the norm is as usual. Moreover, let us define $\kappa_0 := 1$; $\kappa_\infty := 2$ and

$$\kappa_\lambda := 2\left[\frac{\Gamma(\frac{\lambda+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{\lambda}{2}+1)}\right]^{1/\lambda}, \quad 0 < \lambda < \infty.$$

It follows easily from Beta function identities that $\kappa_\lambda \leq 2$ [16]. Moreover, Stirling's formula gives $\kappa_\lambda = 2 + O\left(\frac{\log\lambda}{\lambda}\right)$, $\lambda \to \infty$.

Theorem 4.1 *Let $r, \varepsilon > 0$ and $0 \leq p \leq \infty$. Let $P$ be a polynomial of degree at most $n$, normalized by the condition*

$$\|P\|_{L_p(|z|=r)} = 1. \tag{4.1}$$

Let

$$E(P; r; \varepsilon) := \left\{z : |z| \leq r, \left|P(z)\right| \leq \varepsilon^n\right\}. \tag{4.2}$$

Then

$$cap\big(E(P; r; \varepsilon)\big) \leq r\varepsilon\kappa_{np}; \ \ meas_2\big(E(P; r; \varepsilon)\big) \leq \pi(r\varepsilon\kappa_{np})^2. \tag{4.3}$$

These are sharp for each $r, n$ in the sense that

$$\sup_{\substack{\varepsilon>0 \\ \deg(P)=n}} \frac{cap\big(E(P; r; \varepsilon)\big)}{\varepsilon} = r\kappa_{np}; \ \ \sup_{\substack{\varepsilon>0 \\ \deg(P)=n}} \frac{meas_2\big(E(P; r; \varepsilon)\big)}{\varepsilon^2} = \pi(r\kappa_{np})^2. \tag{4.4}$$

*Proof* Let us set $E := E(P; r; \varepsilon)$. The result is trivial if $cap(E) = 0$, so we assume that it is positive. We shall use some facts from potential theory. The most elementary treatment of these appears in [9], deeper treatments appear in [3, 8, 11]. The set $E$ has piecewise analytic boundary, and is regular with respect to the Dirichlet problem. As such it has a classical Green's function $g(z)$ with pole at $\infty$. This has the following properties: $g$ is harmonic in $\mathbb{C}\backslash E$; $g(z) = \log|z| + O(1), |z| \to \infty$; and $g$ has boundary value 0 on the boundary of $E$. Moreover, $g = 0$ in $E$. It is known that $g$ admits the representation

$$g(z) = \int_E \log|z - t| d\mu(t) + \log \frac{1}{cap(E)}$$

where $\mu$ is a probability measure with support on $E$, the so-called equilibrium measure of $E$. For polynomials $R$ of degree $m \leq n$, there is the Bernstein-Walsh inequality

$$\left|R(z)\right| \leq e^{ng(z)} \|R\|_{L_\infty(E)}, z \in \mathbb{C}. \tag{4.5}$$

The proof is simple: The function $F(z) := \log\left|R(z)\right| - mg(z) - \log\|R\|_{L_\infty(E)}$ is subharmonic in $\mathbb{C}$, with boundary value $\leq 0$ on $E$ and with a finite limit at $\infty$. The maximum principle for subharmonic functions gives $F(z) \leq 0$ in $\mathbb{C}\backslash E$, that is, (4.5) follows as $g \geq 0$. Likewise on $E$, the inequality is trivial as $g \geq 0$. Now in our case $R = P$ has maximum $\varepsilon^n$ on $E$, so our normalization (4.1) and (4.5) give

$$\begin{aligned} 1 = \|P\|_{L_p(|z|=r)} &\leq \varepsilon^n \|e^{ng(z)}\|_{L_p(|z|=r)} \\ &= \left(\frac{\varepsilon}{cap(E)}\right)^n \|e^{n\int \log|z-t|d\mu(t)}\|_{L_p(|z|=r)}. \end{aligned}$$

Let us suppose now $0 < p < \infty$. Using Jensen's inequality for integrals applied to the convex function $t \to e^t$, we can continue this as

$$\begin{aligned} &= \left(\frac{\varepsilon}{cap(E)}\right)^n \left[\frac{1}{2\pi}\int_0^{2\pi} e^{np\int \log|re^{i\theta}-t|d\mu(t)} d\theta\right]^{1/p} \\ &\leq \left(\frac{\varepsilon}{cap(E)}\right)^n \left[\frac{1}{2\pi}\int_0^{2\pi}\int |re^{i\theta} - t|^{np} d\mu(t)\, d\theta\right]^{1/p} \\ &= \left(\frac{\varepsilon}{cap(E)}\right)^n \left[\int\left[\frac{1}{2\pi}\int_0^{2\pi} |re^{i\theta} - t|^{np} d\theta\right] d\mu(t)\right]^{1/p} \\ &\leq \left(\frac{\varepsilon}{cap(E)}\right)^n \sup_{|t|\leq r}\left[\frac{1}{2\pi}\int_0^{2\pi} |re^{i\theta} - t|^{np} d\theta\right]^{1/p}. \end{aligned} \tag{4.6}$$

In the second last line, we used Fubini's theorem, and in the last line, we used the fact that $\mu$ has support in $E \subset \{z : |z| \le r\}$. It is clear that rotating $t$ does not change the value of the integral in the sup so we may take $t \in [0, r]$. Moreover, the fact that the $L_p$ norm of an analytic function on a circle centre 0, radius $t$, increases with $t$ [20, p. 337] gives that the sup is attained for $t = r$, so

$$1 \le \left(\frac{r\varepsilon}{cap(E)}\right)^n \left[\frac{1}{2\pi}\int_0^{2\pi} |e^{i\theta} - 1|^{np} d\theta\right]^{1/p}$$

$$= \left(\frac{2r\varepsilon}{cap(E)}\right)^n \left[\frac{1}{2\pi}\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right|^{np} d\theta\right]^{1/p}.$$

Expressing the last integral in terms of Beta functions gives

$$1 \le \left(\frac{r\varepsilon\kappa_{np}}{cap(E)}\right)^n$$

and hence we have the first inequality in (4.3). Pólya's inequality Theorem 3.1 then gives the second inequality in (4.3). The case $p = \infty$ is easier, as we see that the sup in (4.6) then becomes $(2r)^n$. The case $p = 0$ requires more care, see [16].

To prove the sharpness for $0 < p < \infty$, we let $0 < a < r$, and $P(z) := \left(\frac{z-a}{\lambda}\right)^n$, where $\lambda$ is chosen to give the normalization (4.1). It is easy to see that for small enough $\varepsilon$, $E(P; r; \varepsilon) = \{z : |z - a| \le \varepsilon\lambda\}$ and so

$$\frac{cap\big(E(P; r; \varepsilon)\big)}{\varepsilon} = \lambda; \quad \frac{meas_2\big(E(P; r; \varepsilon)\big)}{\varepsilon^2} = \pi\lambda^2.$$

The normalization (4.1) shows that

$$\lambda^n = \left[\frac{1}{2\pi}\int_0^{2\pi} |re^{i\theta} - a|^{np} d\theta\right]^{1/p} \to (r\kappa_{np})^n, \quad a \to r.$$

So $\lambda$ may be made arbitrarily close to $\kappa_{np}$. Then (4.4) follows. □

We note one generalization, proved in [16]:

THEOREM 4.2 *Let $\psi : (0, \infty) \to \mathbb{R}$ be strictly increasing and continuous with*

$$\psi(0) := \lim_{t\to 0+} \psi(t) < 1 < \lim_{t\to\infty} \psi(t) =: \psi(\infty).$$

*Assume, moreover, that $\psi(e^t)$ is convex in $(-\infty, \infty)$. Let $r, \varepsilon > 0$ and $0 < p < \infty$. Let $P$ be a polynomial of degree at most $n$, normalized by the condition*

$$\frac{1}{2\pi}\int_0^{2\pi} \psi\left(\left|P(re^{i\theta})\right|^p\right) d\theta = 1. \tag{4.7}$$

*Let $\kappa_{\lambda,\psi}$ be the root of the equation*

$$\frac{1}{2\pi}\int_0^{2\pi} \psi\left(\left[\frac{|1-e^{i\theta}|}{\kappa_{\lambda,\psi}}\right]^\lambda\right) d\theta = 1.$$

*Then the estimates (4.3) hold if we replace $\kappa_{np}$ by $\kappa_{np,\psi}$ and moreover these are sharp in the sense that (4.4) holds with $\kappa_{np}$ replaced by $\kappa_{np,\psi}$.*

The proof is very similar to that of Theorem 4.1, one just applies Jensen's inequality to $\psi(e^t)$ rather than $e^t$. See [16] for this and further extensions involving generalized polynomials and potentials.

## 5 REMEZ INEQUALITIES

Remez inequalities have been studied intensively by Tamas Erdelyi and his collaborators in recent years. They have been shown to be useful in proving Markov-Bernstein and Nikolskii inequalities, amongst others. The classical one involves the Chebyshev polynomial $T_n(x)$:

THEOREM 5.1 *Let $P$ be a polynomial of degree at most $n$, with real coefficients, and*

$$\delta := meas_1 \left\{x \in [-1, 1] : \left|P(x)\right| \leq 1\right\}. \tag{5.1}$$

*Then*

$$\|P\|_{L_\infty[-1,1]} \leq T_n\left(\frac{4}{\delta} - 1\right). \tag{5.2}$$

*There is equality iff $P(x) = \pm T_n\left(\frac{\pm 2x+2-\delta}{\delta}\right)$.*

The reader may find an elegant proof in the delightful book of Borwein and Erdelyi [1]: As with Pólya's (3.6), the proof involves shifting the intervals that comprise $\left\{x \in [-1, 1] : \left|P(x)\right| \leq 1\right\}$ until one has a single interval, at each stage increasing the measure.

For our purpose, the following corollary is of most interest: Recall that $T_n$ is strictly increasing in $[1, \infty)$ with $T_n(1) = 1$. Hence it has an inverse $T_n^{[-1]} : [1, \infty) \to [1, \infty)$.

COROLLARY 5.2 *Let $P$ be a polynomial of degree at most n, with real coefficients, normalized by*

$$\|P\|_{L_\infty[-1,1]} = 1. \tag{5.3}$$

*Let* $\varepsilon \in (0, 1]$. *Then*

$$meas_1 \left\{x \in [-1, 1] : \left|P(x)\right| \leq \varepsilon^n\right\} \leq \frac{4}{1 + T_n^{[-1]}(\varepsilon^{-n})} (\leq 2^{3-1/n}\varepsilon). \tag{5.4}$$

*Given* $\varepsilon \in (0, 1]$, *we have equality in (5.4) for suitable polynomials $P$ of degree n.*

Note the power of the corollary: It is sharp for each $\varepsilon$, not just as $\varepsilon \to 0+$.

*Proof of Corollary 5.2* For the given $P$, let $Q := \varepsilon^{-n} P$. Then

$$\left\{x \in [-1, 1] : \left|P(x)\right| \leq \varepsilon^n\right\} = \left\{x \in [-1, 1] : \left|Q(x)\right| \leq 1\right\} =: E.$$

By Remez's inequality (5.2),

$$\|Q\|_{L_\infty[-1,1]} \leq T_n \left(\frac{4}{meas_1 E} - 1\right)$$

and our normalization (5.3) gives

$$\varepsilon^{-n} \leq T_n \left(\frac{4}{meas_1 E} - 1\right).$$

Inverting this gives (5.4). The second inequality in (5.4) follows from the elementary inequality

$$T_n(x) \leq 2^{n-1}x^n, x \in [1, \infty) \Rightarrow T_n^{[-1]}(u) \geq 2^{-1+1/n}u^{1/n}, u \in [1, \infty). \tag{5.5}$$

The sharpness is a simple consequence of the sharpness of Theorem 5.1: Fix $\varepsilon$ and set

$$\delta := \frac{4}{1 + T_n^{[-1]}(\varepsilon^{-n})} \Rightarrow T_n \left(\frac{4}{\delta} - 1\right) = \varepsilon^{-n}.$$

Theorem 5.1 shows that $P(x) = \pm\varepsilon^n T_n \left(\frac{\pm 2x+2-\delta}{\delta}\right)$ gives equality in (5.4).□

We emphasise that Remez inequalities have been proved for generalized polynomials, potentials, in $L_p$ spaces, for Müntz polynomials, .... See [1, 6] for results and references. Since our emphasis is on ordinary polynomials, and regions in the plane, we restrict ourselves to the following result of Erdelyi, Li and Saff [6, Theorem 2.5] for the unit ball:

THEOREM 5.3 *Let $s \in \left[0, \frac{1}{4}\right]$ and $P$ be a polynomial of degree at most $n$ with*

$$meas_2\left(\left\{z : |z| \leq 1, \left|P(z)\right| \leq 1\right\}\right) \geq \pi - s.$$

*Then*

$$\|P\|_{L_\infty(|z|=1)} \leq e^{Cn\sqrt{s}}.$$

*Here $C$ is independent of $n$, $P$, $s$.*

What is fairly typical about this Remez extension is that it applies only when $meas_2\left(E(P;\,1;\,1)\right)$ is bounded away from 0, namely when it is $\geq \pi - \frac{1}{4}$. This is indicative of the rationale of Remez inequalities. Their greatest use is when $meas_2\left(E(P;\,1;\,1)\right)$ approaches its full measure $\pi$, while the inequalities of the previous sections are most useful when $meas_2\left(E(P;\,1;\,1)\right)$ approaches 0.

## 6 MULTIVARIATE POLYNOMIALS

The polynomial $P(z_1, z_2) := (z_1 z_2)^n$ illustrates many of the multivariate features. The lemniscate

$$E(P;\varepsilon) := \left\{(z_1, z_2) : \left|P(z_1, z_2)\right| \leq \varepsilon^n\right\} = \left\{(z_1, z_2) : |z_1 z_2| \leq \varepsilon\right\}$$

is unbounded and even has infinite (4 dimensional) Lebesgue measure. Moreover, is the degree of $P$, $n$ or $2n$? We shall define its degree as $n$. We shall say that the degree of a polynomial $P(z_1, z_2, \ldots z_k)$ of $k$ variables is $n$, if the highest power of each $z_j$ is at most $n$, with equality for at least one $j$. To take account of the unboundedness of $E(P;\varepsilon)$, we consider the restricted lemniscate

$$E(P;r;\varepsilon) := \left\{(z_1, z_2, \ldots z_k) : |z_j| \leq r \forall j, \left|P(z_1, z_2, \ldots z_k)\right| \leq \varepsilon^n\right\}.$$

A. Cuyt, K. Driver and the author [5] used Theorem 4.1 and induction on $k$ to prove:

THEOREM 6.1 *Let $r, \varepsilon > 0$. Let $P$ be a polynomial of degree at most $n$, normalized by*

$$\max\left\{\left|P(z_1, z_2, \dots z_k)\right| : |z_j| \le r \forall j\right\} = 1. \tag{6.1}$$

*Then if $meas_{2k}$ denotes Lebesgue measure in $\mathbb{C}^k (= \mathbb{R}^{2k})$,*

$$meas_{2k}\big(E(P; r; \varepsilon)\big) \le (16\pi r^2)^k \varepsilon^2 \max\left\{1, \log_2 \frac{2^{k-1}}{\varepsilon}\right\}^{k-1}. \tag{6.2}$$

While the constants are not sharp, the powers of $r, \varepsilon$ are, including the surprising factor $\log_2 \frac{1}{\varepsilon}$. For $k = 2$ and the polynomial $P(z_1, z_2) = (z_1 z_2 / r^2)^k$ a calculation shows that

$$meas_2\big(E(P; r; \varepsilon)\big) = (\pi r^2)^2 \varepsilon^2 \left[1 + 2 \log \frac{1}{\varepsilon}\right].$$

What about monic normalization? B. Paneah [17] generalized Cartan's lemma as follows. Given a multi-index $\underline{\alpha} = (\alpha_1, \alpha_2, \dots \alpha_k)$ we say that it is *leading* if

(I) $\partial^\alpha P := \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial z_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial z_k}\right)^{\alpha_k} P \neq 0$;

(II) $\forall 1 \le j \le k$ such that $\alpha_j \neq 0$, $\left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial z_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial z_j}\right)^{\alpha_j + 1} P \equiv 0$.

For example, $P(z_1, z_2, z_3) = z_1^3 + z_2^3 + z_3^2 + 3z_1^2 z_2^2 z_3$ has leading multi-indices $(3, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 2)$ but not $(2, 2, 1)$.

For $\underline{z} = (z_1, z_2, \dots z_k)$ and $1 \le j \le k$, we set $\hat{\underline{z}}_j := (z_1, z_2, \dots z_{j-1}, z_{j+1}, \dots z_k)$. One dimensional lines in $\mathbb{C}^k$ parallel to the $z_j$ axis have the form

$$\mathcal{C}_j(\underline{a}) = \left\{\underline{z} : \hat{\underline{z}}_j = \underline{a}\right\},$$

where $\underline{a} \in \mathbb{C}^{k-1}$. Paneah proved:

THEOREM 6.2 *Let $P$ be of degree $n$, and $\underline{\alpha}$ be a leading multi-index. Let $\delta_j \ge 0$ with equality iff $\alpha_j = 0$, $1 \le j \le k$. There exist subsets $\mathcal{M}_j$ of $\mathbb{C}^k$ such that*

$$\left|P(\underline{z})\right| \ge \left[\prod_{j=1}^{k} \left(\frac{\delta_j}{\alpha_j}\right)^{\alpha_j}\right] |\partial^\alpha P|, \underline{z} \in \mathbb{C}^k \setminus \bigcup_{j=1}^{k} \mathcal{M}_j.$$

*Moreover, $M_j$ intersects any line $C_j(\underline{a})$, $\underline{a} \in \mathbb{C}^{k-1}$, in at most $\alpha_j$ circles, with sum of diameters $\leq 4\delta_j$.*

Note that for $k = 1, \alpha_1 = n$ and we obtain $(\delta_1/n)^n n!$, precisely the quantity in the proof of Corollary 2.2.

What about measures other than Lebesgue measure? Notions of capacity are far more complicated in the multivariate case, and several basic questions remain unresolved. One of the main problems is the lack of an explicit formula for the Greens' function. See [2, 5, 10, 12, 13, 22] for partial results. Undoubtedly the greatest scope for work on small values of polynomials lies in the multivariate setting.

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