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Best Constant in Weighted Sobolev Inequality with Weights Being Powers of Distance from the Origin

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We study the best constant in Sobolev inequality with weights being powers of distance from the origin in $\mathbb{R}^n$. In this variational problem, the invariance of $\mathbb{R}^n$ by the group of dilatations creates some possible loss of compactness. As a result we will see that the existence of extremals and the value of best constant essentially depends upon the relation among parameters in the inequality.

Keywords: Weighted Sobolev inequality; concentration compactness.

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1 INTRODUCTION

We begin with recalling the famous theorem due to Giorgio Talenti [11]:

THEOREM 1.1 *Let u be any real (or complex) valued function in $C_0^1(\mathbb{R}^n)$. Moreover, let p be any number such that: $1 < p < n$. Then :*

$$\int_{\mathbf{R}^n} |\nabla u|^p \, dx \geq S(p,q,n) \left\{ \int_{\mathbb{R}^n} |u|^q \, dx \right\}^{p/q}, \tag{1.1}$$

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where: $|\nabla u|$ *is the length of the gradient* ∇u *of* u, $q = np/(n-p)$ *and*

$$S(p,q,n) = 2^{p/n}\pi^{p/2}n\left(\frac{n-p}{p-1}\right)^{p-1}\left(\frac{p-1}{p}\right)^{p/n} \left\{\frac{\Gamma(n/p)\cdot\Gamma\big(n(1-1/p)\big)}{\Gamma(n/2)\cdot\Gamma(n)}\right\}^{p/n}. \tag{1.2}$$

The equality sign holds in (1.1) if u *has the form:*

$$u(x) = [a + b|x|^{p/(p-1)}]^{1-n/p}, \tag{1.3}$$

where $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ *and* a, b *are positive constants.*

The main purpose of the present paper is to study the best constant in the imbedding theorems for the weighted Sobolev spaces with weight functions being powers of $|x|$. Namely, we are interested in the best constant $S(p,q,\alpha,\beta,n)$ in the following inequality:

$$\int_{\mathbf{R}^n} |\nabla u|^p |x|^{p\alpha}\,dx \geq S(p,q,\alpha,\beta,n)\left\{\int_{\mathbf{R}^n} |u|^q |x|^{q\beta}\,dx\right\}^{p/q}, \tag{1.4}$$

where u is any function in $C_0^1(\mathbf{R}^n)$ and

$$0 < \frac{1}{p} - \frac{1}{q} = \frac{1-\alpha+\beta}{n}, \qquad -\frac{n}{q} < \beta \leq \alpha, \qquad 1 < p < \frac{n}{1-\alpha+\beta}. \tag{1.5}$$

For the proof of this inequality and related informations, see [9; Theorem 1 in §2] and [6; Theorem 1 in §3]. If $\alpha = 0$ and $\beta \leq 0$, then the best constant is already obtained in [11] and [4]. The equality sign in this case also holds if u has the similar form in Theorem 1.1. Therefore we are interested in (1.4) when α is a positive number. In this variational problem, the invariance of $\mathbb{R}^n$ by the group of dilatations creates some possible loss of compactness. As a result we show that the existence of extremal functions essentially depend upon the parameters (p,q,α,β,n). For example, there is no extremals if $\alpha = \beta$ and $p = 2$. Moreover if we restrict ourselves to the case when $p = 2$, we can make clear the behavior of the best constant $S(2,q,\alpha,\beta,n)$ rather precisely as a function of (q,α,β,n) under the condition (1.5).

It seems to be worth mentioning that the equality sign in (1.4) can not be achieved by any function with compact support. To see this we assume that there exists an extremal u having the support in a ball $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, namely, the infimum is attained by u. Here we may assume u is nonnegative. Moreover it has to satisfy the Euler Lagrange equation in distribution sense;

$$
\begin{aligned}
&-\operatorname{div}(|x|^{\alpha p}|\nabla u|^{p-2}\nabla u) = \lambda |x|^{\beta q} u^{q-1}, \qquad \text{in } B_r \\
&u|_{\partial B_r} = 0, \qquad u > 0 \qquad \text{in } B_r.
\end{aligned} \tag{1.6}
$$

Here $\lambda > 0$ is a Lagrange multiplier. Then it follows from the next lemma that u has to vanish almost everwhere in B_r.

LEMMA 1.2 *(Pohozaev identity) Let p, q, n, α and β satisfy $1 < p$, $0 \le 1/p - 1/q \le (1-\alpha+\beta)/n$, $(1-\alpha+\beta)p < n$ and $\beta > -n/q$. Assume that $u \in W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ satisfy the equation (1.6) with Dirichlet boundary condition in distribution sense. Then it holds that*

$$
\begin{aligned}
\lambda[1-\alpha+\beta-n(1/p-1/q)] &\int_{B_r} |x|^{\beta q} u^q \, dx \\
&= (1-1/p)\int_{\partial B_r} |x|^{\alpha p}(x,\nu)|\nabla u|^p \, dS,
\end{aligned} \tag{1.7}
$$

where ν is the unit outer normal to ∂B_r and S is the $(n-1)$-dimensional Lebesgue measure, and $W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ is defined by (2.2) and (2.3).

When $1/p - 1/q > (1-\alpha+\beta)/n$ it follows immediately from (1.7) that $u = 0$. When $1/p - 1/q = (1-\alpha+\beta)/n$, we deduce from (1.7) that $\frac{\partial u}{\partial \nu} = 0$ on ∂B_r, and then by (1.6)

$$
0 = -\int_{B_r} \operatorname{div}(|x|^{\alpha p}|\nabla u|^{p-2}\nabla u)\, dx = \lambda \int_{B_r} |x|^{\beta q} u^{q-1}\, dx, \tag{1.8}
$$

thus $u = 0$.

Proof of Lemma 1.2 By a standard argument of regularization, we see that u is smooth. Then the equality is established by the computation of div P and an integration by parts for

$$
P = |x|^{\alpha p}|\nabla u|^{p-2}(\nabla u, x)\nabla u. \tag{1.9}
$$

For the precise see [4; Prop. 13], [5] and [10].

2 WEIGHTED SOBOLEV SPACES AND INEQUALITIES

In this section we shall modify the classical Sobolev spaces so that we can treat the variational problems in the subsequent sections. To this end we recall the weighted inequality of Sobolev type.

LEMMA 2.1 *Let p satisfy $1 \leq p < +\infty$ and let n satisfy $n \geq 2$.Suppose $(1-\alpha+\beta)p < n,\ \ 0 \leq 1/p - 1/q = (1-\alpha+\beta)/n$ and $-n/q < \beta \leq \alpha$, then there is a positive number C such that for any $u \in C_0^\infty(\mathbb{R}^n)$,*

$$\left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx\right)^{1/p}. \tag{2.1}$$

Here, $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots \frac{\partial u}{\partial x_n})$ *and* $|\nabla u| = \left(\sum_{k=1}^n |\frac{\partial u}{\partial x_k}|^2\right)^{1/2}$.

The proof of this is seen in many places, for example in Maz'ja's book [9; Theorem 1 and its corollaries in §2]. This result is also obtained as a corollary to the more general imbedding theorem in the author's paper [6; Theorem 1 in §3]. This lemma naturally leads us to define the following spaces: Let $1 \leq p < +\infty$ and α, β be real numbers $> -n/p$. Let $L^p_\alpha(\mathbb{R}^n)$ denote the space of Lebesgue measurable functions, defined on $\mathbb{R}^n$, for which

$$||u;\, L^p_\alpha(\mathbb{R}^n)|| = \left(\int_{\mathbb{R}^n} |u|^p |x|^{\alpha p}\, dx\right)^{1/p} < +\infty. \tag{2.2}$$

$W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ is defined by

$$W^{1,p}_{\alpha,\beta}(\mathbb{R}^n) = \{u \in L^{q(p)}_\beta(\mathbb{R}^n) : |\nabla u| \in L^p_\alpha(\mathbb{R}^n)\}, \tag{2.3}$$

where

$$\frac{1}{p} - \frac{1}{q(p)} = \frac{1-\alpha+\beta}{n} \quad \text{or} \quad q(p) = \frac{np}{n-(1-\alpha+\beta)p}. \tag{2.4}$$

We equip $W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ with the norm

$$||u;\, W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)|| = ||u;\, L^{q(p)}_\beta(\mathbb{R}^n)|| + |||\nabla u|;\, L^p_\alpha(\mathbb{R}^n)||. \tag{2.5}$$

We also set

$$R^{1,p}_{\alpha,\beta}(\mathbb{R}^n) = \{u \in W^{1,p}_{\alpha,\beta}(\mathbb{R}^n) : u \text{ is a radial function }\},$$
$$||u;\, R^{1,p}_{\alpha,\beta}(\mathbb{R}^n)|| = ||u;\, W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)||. \tag{2.6}$$

Under these notations we prepare a compactness proposition for the imbedding and restriction operators $W^{1,p}_{\alpha,\beta}(\mathbb{R}^n) \to L^r_\gamma(B)$ for any ball B.

PROPOSITION 2.2 *Let p satisfy $1 \leq p < +\infty$ and let n satisfy $n > 2$. By B we denote an arbitrary ball in $\mathbb{R}^n$.*
(1) Assume that $(1-\alpha+\beta)p < n, 0 \leq 1/p - 1/r < (1-\alpha+\beta)/n$ and $-n/q < \beta \leq \alpha$, then the following restrictions of the mapping are compact;

$$W^{1,p}_{\alpha,\beta}(\mathbb{R}^n) \to L^r_\beta(B), \qquad p \leq r < q(p) = np/[n - p(1-\alpha+\beta)]. \quad (2.7)$$

(2) Assume that $(1-\alpha+\beta)p < n, 0 \leq 1/p - 1/r < (1-\alpha+\beta)/n$ and $-n/q < \beta$, then the following imbedding mappings are compact:

$$R^{1,p}_\alpha(\mathbb{R}^n) \to L^r_\beta(B), \qquad p \leq r < q(p) = np/[n - p(1-\alpha+\beta)]. \quad (2.8)$$

In the assertion (2) of this proposition, r may exceed the so-called Sobolev exponent provided $\beta > \alpha$, because elements in $R^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ are essentially dependent upon one variable. And the proof is elementary by the use of the polar coordinate system. For the detail see [4; Lemma 10] for instance.

3 MAIN RESULTS

We shall study the following variational problems. Assume that p, q, n, α and β satisfy

$$\begin{cases} 1 < p < +\infty, \quad (1-\alpha+\beta)p < n, \quad n \geq 2, \\ 0 < 1/p - 1/q = (1-\alpha+\beta)/n \end{cases} \quad (3.1)$$

and

$$-n/q < \beta \leq \alpha. \quad (3.2)$$

Under these assumptions we set

$$S(p,q,\alpha,\beta,n) = \inf\left[\int_{\mathbb{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx : u \in W^{1,p}_{\alpha,\beta}(\mathbb{R}^n), \|u : L^q_\beta\| = 1\right]. \quad (P)$$

In the following problem we assume instead of the inequality (3.2)

$$-n/q < \beta. \quad (3.3)$$

$$S_R(p,q,\alpha,\beta,n) = \inf\left[\int_{\mathbb{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx : u \in R^{1,p}_{\alpha,\beta}(\mathbb{R}^n), \|u : L^q_\beta\| = 1\right]. \quad (P_R)$$

In the problem (P_R), if we make a change of variables defined by

$$|x| = r^{1/h}, \qquad h = \frac{(1-\alpha+\beta)(n-p+p\alpha)}{n-p(1-\alpha+\beta)}, \tag{3.4}$$

we get an equivalent variational problem (P'_R) for $v \in C^1(\mathbb{R}_+)$:

$$S_R(p,q,\alpha,\beta,n) = C(p,q,k)\times$$
$$\inf\left[\int_0^{+\infty} |v'(r)|^p r^{n/(1-\alpha+\beta)-1}\,dr : \int_0^{+\infty} |v(r)|^q r^{n/(1-\alpha+\beta)-1}\,dr = 1\right], \tag{P'_R}$$

where

$$C(p,q,h) = |S^{n-1}|^{1-p/q} \cdot h^{p-1+p/q} \tag{3.5}$$

and $|S^{n-1}|$ is the area of $n-1$-dimensional unit sphere. This problem was solved by Talenti using the notion of Hilbert invariant integral. Namely it follows from Lemma 2 in [11] that the infimum is achieved by functions of the form

$$v(r) = [a + b|x|^{\frac{hp}{p-1}}]^{1-\frac{n}{p(1-\alpha+\beta)}}. \tag{3.6}$$

Then with somewhat more calculations we see

Lemma 3.1 *Assume that (3.1) and (3.3). Then we have*

$$S_R(p,q,\alpha,\beta,n) =$$
$$\pi^{\frac{p\gamma}{2}} \cdot n \cdot \left(\frac{n-\gamma p}{p-1}\right)^{p-1} \cdot \left(\frac{n-p+p\alpha}{n-\gamma p}\right)^{p-\frac{p\gamma}{n}} \cdot \left(\frac{2(p-1)}{\gamma p}\right)^{\frac{p\gamma}{n}}$$
$$\times \left\{\frac{\Gamma(n/\gamma p)\Gamma(n(p-1)/\gamma p)}{\Gamma(n/2)\Gamma(n/\gamma)}\right\}^{\frac{p\gamma}{n}}, \tag{3.7}$$

where $\gamma = 1-\alpha+\beta$. In particular if $\alpha = \beta$, then we have

$$S_R(p,q,\alpha,\alpha,n) = S(p,q,n)\cdot\left(\frac{n-p+p\alpha}{n-p}\right)^{p-\frac{p}{n}}. \tag{3.8}$$

Therefore we immediately get

LEMMA 3.2
(1) Assume that $1/p - 1/q = 1/n$, $1 < p < n$ *and* $n > 2$. *Then we have*

$$\begin{cases} S(p,q,n) < S_R(p,q,\alpha,\alpha,n), & \text{if } \alpha > 0 \\ S(p,q,n) > S_R(p,q,\alpha,\alpha,n), & \text{if } \alpha < 0 \end{cases} \tag{3.9}$$

(2) Assume that (3.1) and (3.3). Then we have

$$S_R(p,q,\alpha,\beta,n) = k^{1-p-p/q} S_R(p,q,0,\beta-\alpha,n), \tag{3.10}$$

where $\quad k = \frac{n-p}{n-p+\alpha p}.$

From this lemma it seems that if $\alpha \le 0$, $S_R(p,q,\alpha,\beta,n)$ is also the best constant for the problem (P), and in the subsequent argument this proves to be true. The following lemma is partially proved by Talenti and Egnell (in the case that $\alpha = 0, \beta \le 0$).

LEMMA 3.3 *Assume that* p, q, α, β, n *satisfy (3.1) and (3.2). Assume that* $\beta \le \alpha \le 0$, *then*

$$S(p,q,\alpha,\beta,n) = S_R(p,q,\alpha,\beta,n). \tag{3.11}$$

Proof of Lemma 3.3 By a polar coordinate system, we rewrite (P) to obtain

$$\inf\left[\int_{S^{n-1}}\int_0^\infty \left(|\partial_r u|^2 + \frac{|\Lambda u|^2}{r^2}\right)^{p/2} r^{\alpha p+n-1}\, dr dS_\omega : \right.$$
$$\left. \int_{S^{n-1}}\int_0^\infty |u|^q r^{\beta q+n-1}\, dr dS_\omega = 1\right], \tag{P'}$$

where S_ω is a $n-1$-dimensional Lebesgue measure and Λ is the Laplace Beltrami operator on the unit sphere S^{n-1}. Making a change of variables defined by

$$r = \rho^k, \qquad k = \frac{n-p}{n-p+\alpha p}, \tag{3.12}$$

we have

$$\inf\left[k^{1-p-p/q}\int_{S^{n-1}}\int_0^\infty \left(|\partial_\rho v|^2 + k^2\frac{|\Lambda v|^2}{\rho^2}\right)^{p/2} \rho^{n-1}\, d\rho dS_\omega : \right.$$
$$\left. u \in W^{1,p}_{\alpha,\beta}(\mathbb{R}^n),\quad \int_{S^{n-1}}\int_0^\infty |v|^q \rho^{(n-p)q/p-1}\, d\rho dS_\omega = 1\right], \tag{P'}$$

where $v(\rho \cdot \omega) = k^{-1/q} u(\rho^k \cdot \omega)$. Since $\alpha \leq 0$ by the assumption, we see $k \geq 1$. Therefore we see

$$S(p, q, \alpha, \beta, n) \geq k^{1-p-p/q} \cdot S(p, q, 0, \beta - \alpha, n), \tag{3.13}$$

where we used $(n - p)q/p - 1 = q(\beta - \alpha) + n - 1$. Since $\beta - \alpha \leq 0$, the assertion (4.4) in Lemma 4.1 and the spherically symmetric decreasing rearrangement of v leads us to

$$S(p, q, 0, \beta - \alpha, n) = S_R(p, q, 0, \beta - \alpha, n). \tag{3.14}$$

Therefore the assertion follows from Lemma 3.2.

Now we are in a position to state our main result.

THEOREM 3.4
(1) Assume that $0 < \alpha = \beta$, $1/2 - 1/q = 1/n$, $n > 2$. *Then it holds that*

$$S(2, q, \alpha, \alpha, n) = S(2, q, 0, 0, n) = S(2, q, n). \tag{3.15}$$

Moreover there exists no extremal function which attains the infimum in $W^{1,2}_{\alpha,\alpha}(\mathbb{R}^n)$.
(2) Assume that $\alpha > 0, \alpha > \beta, 0 < 1/p - 1/q = (1 - \alpha + \beta)/n, n \geq 2$ *and* $1 < p < \frac{n}{1-\alpha+\beta}$. *Then the infimum* $S(p, q, \alpha, \beta, n)$ *is attained by an extremal function* u *in* $W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ *and this* u *satisfies in distribution sense the equation:*

$$-\operatorname{div}(|x|^{p\alpha} |\nabla u|^{p-2} \nabla u) = I \cdot |x|^{\beta q} |u|^{q-2} u, \tag{3.16}$$

where I *is a Lagrange multiplier.*

Remark In the assertion (1), the best constant $S(p, q, \alpha, \alpha, n)$ is not known unless $p = 2$. Because the proof in this paper essentially use the linearlity of the Euler Lagrange equation. But at least we see that $S(p, q, \alpha, \alpha, n) \leq S(p, q, n)$ in the proof of the assertion (1). And the best constant in assertion (2) is also unknown for the present. We also note that if we replace the weight function $|x|$ by $|x_n|$, we can show a similar result.

4 PROOF OF THE ASSERTION 1

First we prove the assertion 1. Let Ω be a domain of $\mathbb{R}^n$. For a nonnegative function $f \in C^0(\Omega)$ with having a compact support, we denote by $\mathbf{S}(f)$

the spherically symmetric decreasing rearrangement of f (the Schwarz symmetrization of f). That is:

$$\mathbf{S}(f)(x) = \sup\{t : \mu(t) > |S^{n-1}| \cdot |x|^n\}, \quad \mu(t) = |\{x : f(x) > t\}|. \tag{4.1}$$

We prepare the following lemmas. The first one is well-known (for the proof see [11; Lemma 1] for instance).

LEMMA 4.1 *Let $\mathbf{S}(f)$ be the spherically symmetric decreasing rearrangement of a nonnegative function $f \in C^0(\Omega)$ with a compact support. Let $g \in C^0((0,\infty))$ be a nonnegative decreasing function. Then, for every exponent $p \geq 1$, the followings hold:*

$$\int_{\mathbb{R}^n} \mathbf{S}(f)^p \, dx = \int_\Omega f^p \, dx, \tag{4.2}$$

$$\int_{\mathbb{R}^n} |\nabla \mathbf{S}(f)|^p \, dx \leq \int_\Omega |\nabla f|^p \, dx \tag{4.3}$$

$$\int_{\mathbb{R}^n} \mathbf{S}(f)^p g(|x|) \, dx \geq \int_\Omega f^p \cdot g(|x|) \, dx. \tag{4.4}$$

The next one is a variant of the Hardy-Sobolev inequality.

LEMMA 4.2 *Assume that $f \in C^2(\Omega)$, $u \in C_0^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ $(n > 2)$. Let us set $v(x) = \mathbf{S}(|f \cdot u|)(x)$. Then it holds that*

$$\int_{\mathbb{R}^n} |\nabla v|^2 \, dx + \frac{1}{2} \int_\Omega u^2 [\Delta(f^2) - 2|\nabla f|^2] \, dx \leq \int_\Omega |\nabla u|^2 f^2 \, dx. \tag{4.5}$$

Admitting this in the present we shall establish the assertion (1) in Theorem 3.4.

Proof of the assertion (1) By the use of these lemmas for $f = |x|^\alpha$, $\Omega = \mathbb{R}^n$, we see that

$$S(2,q,n)\left(\int_{\mathbb{R}^n} |v|^q \, dx\right)^{2/q} + \alpha(\alpha+n-2)\int_{\mathbb{R}^n} u^2 |x|^{2(\alpha-1)} \, dx$$
$$\leq \int_{\mathbb{R}^n} |\nabla u|^2 |x|^{2\alpha} \, dx. \tag{4.6}$$

Here $1/2 - 1/q = 1/n$, $\quad n > 2$. Hence if there exists an extremal function $u \in W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$, then we have

$$S(2,q,n) + \alpha(\alpha+n-2)\int_{\mathbb{R}^n} u^2 |x|^{2(\alpha-1)} \, dx \leq S(2,q,\alpha,\alpha,n). \tag{4.7}$$

This implies $S(2,q,\alpha,\alpha,n) \geq S(2,q,n)$, and obviously the equality sign holds in (4.7) only if $u \equiv 0$. Therefore it suffices to see the opposite inequality. Let us set, for $y \in \mathbb{R}^n \setminus \{0\}$,

$$S(y) = \inf\{\int_{\mathbb{R}^n} |\nabla u|^2 |y|^{2\alpha}\, dx \;:\; \int_{\mathbb{R}^n} |u|^q |y|^{\alpha q}\, dx = 1, u \in C_0^\infty(\mathbb{R}^n)\}. \tag{4.8}$$

One then checks easily that if we replace u by $\varepsilon^{-n/q} u(\cdot - y/\varepsilon)$, $\varepsilon > 0$, $q = 2n/(n-2)$ and let ε tend to 0, we have $S(2,q,\alpha,\alpha,n) \leq S(y)$. On the other hand, it holds

$$S(y) = \inf\{\int_{\mathbb{R}^n} |\nabla u|^2\, dx \;:\; \int_{\mathbb{R}^n} |u|^q\, dx = 1\} = S(2,q,n). \tag{4.9}$$

So that we see $S(2,q,\alpha,\alpha,2) = S(2,q,n)$.

Proof of Lemma 4.2 First we have

$$\int_\Omega |\nabla(f\cdot u)|^2\, dx = \int_\Omega [|\nabla u|^2 f^2 + u^2|\nabla f|^2]\, dx + \frac{1}{2}\int_{\mathbb{R}^n} \nabla u^2 \cdot \nabla f^2\, dx$$
$$= \int_\Omega |\nabla u|^2 f^2 - \frac{1}{2}\int_\Omega u^2[\Delta(f^2) - 2|\nabla f|^2]\, dx. \tag{4.10}$$

Then, from Lemma 1 we can show the desired result. Here we note that this proof still works if we put either $f(x) = |x|^\alpha$, $(\alpha \geq 0, n \geq 2)$ or $f(x) = |x_n|^\alpha$, $(\alpha \geq 1/2, n \geq 2)$.

5 PROOF OF THE ASSERTION 2

Let us set for $u \in W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$

$$\begin{cases} J(u) = \int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx, \\ E(u) = \int_{\mathbb{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx \end{cases} \tag{5.1}$$

Here $0 \leq (1-\alpha+\beta)/n = 1/p - 1/q, \alpha \geq \beta > -n/q,\ p(1-\alpha+\beta) < n$. We also set for $0 < \lambda \leq 1$

$$S^\lambda = \inf[E(u) : J(u) = \lambda, u \in W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)\,] \tag{5.2}$$

Assume that $\{u_j\} \subset W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ is a minimizing sequence such that

$$\lim_{j\to+\infty} E(u_j) = S \equiv S(p,q,\alpha,\beta,n) \qquad J(u_j) = 1 \quad (j = 1,2,3,\ldots). \tag{5.3}$$

In order to prove the existence of the extremal function in $W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$, first we show the tightness of the sequence considered. Let us also set

$$\rho_j = |\nabla u_j|^p |x|^{\alpha p} + |u_j|^q |x|^{\beta q},$$
$$Q_j(R) = \int_{B_R(0)} \rho_j \, dx \qquad (j = 1, 2, 3, \dots). \tag{5.4}$$

For $\delta = 1 - \alpha - n/p$ and $\varepsilon > 0$, we set $u^\varepsilon = \varepsilon^\delta u(x/\varepsilon)$. Then we see

$$J(u) = J(u^\varepsilon) \qquad \text{and} \qquad E(u) = E(u^\varepsilon). \tag{5.5}$$

Hence we may assume from the first

$$Q_j(1) = \frac{1}{2}, \qquad j = 1, 2, 3, \dots \tag{5.6}$$

Then we see

LEMMA 5.1 *For an arbitrary $\varepsilon > 0$, there exists some positive number R such that we have*

$$\int_{\mathbb{R}^n \setminus B_R(0)} \rho_j \, dx < \varepsilon, \qquad (j = 1, 2, 3, \dots) \tag{5.7}$$

Proof of Lemma 5.1 First we note that for some positive number L

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} \rho_j \, dx = L \geq 1 + S. \tag{5.8}$$

According to the argument in [8; Theorem 1 in part 1], we just have to show that dichotomy cannot occur. To this end we assume that dichotomy occurs. Then, extracting subsequence from $\{u_j\}$ if necessary, we see that for an arbitrary $\varepsilon > 0$, there exist positive numbers $A \in (0, L)$, R and a sequence of positive numbers $\{R_j\}$ such that:

$$\left| A - \int_{B_R(0)} \rho_j \, dx \right| < \varepsilon, \ \int_{B_{R_j}(0) \setminus B_R(0)} \rho_j \, dx < \varepsilon, \ \text{and} \ \lim_{j \to \infty} R_j = \infty. \tag{5.9}$$

Let f and g be nonnegative smooth functions such that

$$f = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 2, \end{cases} \qquad g = \begin{cases} 1, & |x| > 1 \\ 0, & |x| < 1/2. \end{cases} \tag{5.10}$$

Let us set

$$\begin{cases} u^1(x) = f(x/R) \cdot u, \\ u^2(x) = g(x/R_j) \cdot u. \end{cases} \tag{5.11}$$

Then, for any $\varepsilon' > 0$ there is a positive integer $N(\varepsilon)$ such that

$$\left| \int_{\mathbb{R}^n} [|\nabla f(x/R)|^p |u_j|^p + |\nabla g(x/R_j)|^p |u_j|^p] |x|^{p\alpha}\, dx < \varepsilon' \right. \tag{5.12}$$

for $j > N(\varepsilon')$. In fact we see

$$\begin{aligned} &\int_{R \le |x| \le 2R} |u_j|^p |x|^{p\alpha}\, dx \\ \le &\left(\int_{R \le |x| \le 2R} |u_j|^q |x|^{\beta q}\, dx \right)^{p/q} \cdot \left(\int_{|x| < 2R} |x|^{\frac{n(\alpha-\beta)}{1-\alpha+\beta}}\, dx \right)^{p(1-\alpha+\beta)/n} \\ &\le CR^p \varepsilon. \end{aligned} \tag{5.13}$$

And in a similar way,

$$\int_{R_j/2 \le |x| \le R_j} |u_j|^p |x|^{p\alpha}\, dx \le CR^p \varepsilon. \tag{5.14}$$

Here C is a positive number independent of each R and u_j. Therefore we see (5.12). On the other hand, we may assume that for some numbers $s, t \in [0, 1]$

$$\begin{cases} \int |u_j^1|^q |x|^{\beta q}\, dx \to s, \\ \int |u_j^2|^q |x|^{\beta q}\, dx \to t, \\ |1 - (s + t)| \le \varepsilon. \end{cases} \tag{5.15}$$

From Sobolev inequality, there is a positive number c such that

$$\int |\nabla u_j^i|^p |x|^{p\alpha}\, dx \ge c. \tag{5.16}$$

When ε tends to 0, we may assume that $s = s(\varepsilon)$ also converges to some number $\bar{s} \in [0, 1]$. In case that $\bar{s} = 0$ or 1, then we see $S \ge c + S - \varepsilon$ for any $\varepsilon > 0$, and this contradicts to Sobolev inequality. Therefore, $\bar{s} \in (0, 1)$. Since $t = t(\varepsilon) \to 1 - \bar{s}$, we have

$$S \ge S^{\bar{s}} + S^{1-\bar{s}} = [\bar{s}^{p/q} + (1 - \bar{s})^{p/q}] S > S, \tag{5.17}$$

and this is a contradiction.

After all we see that under the condition (5.6) the minimizing sequence $\{u_j\}_{j=1}^{\infty} \subset W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ and $\{\rho_j\}_{j=1}^{\infty}$ are tight in $L^q_\beta(\mathbb{R}^n)$ and a space of all bounded measures on $\mathbb{R}^n$ respectively. To see the existence of extremals, we need an apparent variant of the concentration compactness lemma due to Lions in [7] and [8]. For the sake of self-containedness we state it here. The proof is omitted. Let $\{u_j\}$ be a minimizing sequence satisfying (5.3).

Concentration Compactness Lemma 5.2 *Let $\{u_j\}$ be a bounded sequence in $W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ converging weakly to some u and such that $|\nabla u_j|^p |x|^{p\alpha}$ converges weakly to μ and $|u_j|^q |x|^{\beta q}$ converges tightly to ν where μ and ν are bounded nonnegative measures on $\mathbb{R}^n$. Then we have* (1) *There exist some at most countable set J and two families $\{x_j\}_{j\in J}$ of distinct points in $\mathbb{R}^n$, $\{\nu_j\}_{j\in J}$ in* $(0, \infty)$ *such that:*

$$\nu = |u|^q |x|^{\beta q} + \sum_J \nu_j \delta_{x_j}, \qquad \mu \geq |\nabla u|^p |x|^{\alpha p} + \sum_J \mu_j \delta_{x_j}, \tag{5.18}$$

for some $\mu_j > 0$. Moreover it holds that

$$\nu_j^{p/q} \leq \frac{\mu_j}{S}. \tag{5.19}$$

(2) If $v \in W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ and $|\nabla(u_j + v)|^p |x|^{\alpha p}$ converges weakly to some measure $\overline{\mu}$, then $\overline{\mu} - \mu \in L^1(\mathbb{R}^n)$.

(3) If $u \equiv 0$ and $\mu(\mathbb{R}^n) \leq S\nu(\mathbb{R}^n)^{p/q}$, then J is a singleton and we have

$$\nu = \gamma\delta_{x_0} = \frac{1}{S\gamma^{p/q}}\mu, \tag{5.20}$$

for some $\gamma > 0$ and $x_0 \in \mathbb{R}^n$.

End of Proof of Theorem. From this lemma we may assume that there is a weak limit $u \in W^{1,p}_{\alpha,\beta}(\mathbb{R}^n)$ of the minimizing sequence $\{u_j\}$. Therefore it suffices to show that u_j converges strongly to $u \neq 0$ identically under the condition $\alpha > \beta \geq 0$. From the assumption we see $\mu(\mathbb{R}^n) = S$ and $\nu(\mathbb{R}^n) = 1$ (tightness). Here we note that the lack of compactness can occur only at the origin. Because the weight function vanishes only there. More precisely, if D is bounded open subset of $\mathbb{R}^n$ having a positive distance to the origin, then the imbedding operator $W^{1,p}_{\alpha,\beta}(\mathbb{R}^n) \longrightarrow L^q_\beta(D)$ is compact under the conditions $1/p - 1/q = (1 - \alpha + \beta)/n$, $\alpha > \beta > -n/q$ and

$1 < p < n/(1-\alpha+\beta)$. Now if $u \equiv 0$, then from Lemma 5.2 and the above remark we see $\nu = \frac{1}{S}\mu = \delta_0$. But

$$\frac{1}{2} = Q_j(1) \geq \int_{B_1(0)} |u_j|^q |x|^{\beta q}\, dx \longrightarrow 1. \tag{5.21}$$

This is a contradiction. Next we see u_j converges strongly to u. Let us set $a = \int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx$ and assume that $0 \leq a < 1$. Then from the lemma we have

$$\nu_0 = 1 - a, \quad \mu_0 \geq S\nu_0^{p/q}, \quad \int |\nabla u|^p |x|^{p\alpha}\, dx \leq S - \mu_0. \tag{5.22}$$

Hence we see

$$\int |\nabla u|^p |x|^{p\alpha}\, dx \leq S - \mu_0 \leq S(1-\nu_0^{p/q}) < S(1-\nu_0)^{p/q} = Sa^{p/q}. \tag{5.23}$$

On the other hand, it holds

$$\int |\nabla u|^p |x|^{p\alpha}\, dx \geq S^a = Sa^{p/q}. \tag{5.24}$$

So we reach a contradiction.

Appendix

In this section we calculate the best constant $S(2, q(\alpha), \alpha, \beta(\alpha), n)$, where α, $\beta(\alpha)$, $q(\alpha)$ and n satisfy the relations;

$$2\alpha = \beta q(\alpha), n \geq 2, q(\alpha) = \frac{2(n+2\alpha)}{n+2\alpha-2}, \quad \beta(\alpha) = \frac{n+2\alpha-2}{n+2\alpha}\alpha. \tag{A.1}$$

PROPOSITION A.1 *In addition to these assumptions we assume that 2α is a positive integer. Then it holds that*

$$\begin{aligned} &S(2, q(\alpha), \alpha, \beta(\alpha), n) = S_R(2, q(\alpha), \alpha, \beta(\alpha), n) \\ &= S(2, 2n/(n-2), n+2\alpha) \cdot \pi^{-2\alpha/(n+2\alpha)} \cdot \left(\frac{\Gamma((n+2\alpha)/2)}{\Gamma(n/2)}\right)^{2/(n+2\alpha)}. \end{aligned} \tag{A.2}$$

Proof We abbreviate $S(2, q(\alpha), \alpha, \beta(\alpha), n)$ to S. Let us set $V = [0, \infty)^n$. Then

$$S \geq 2^{\frac{2n}{n+2\alpha}} \inf \left[\int_V |\nabla u|^2 |x|^{2\alpha} \, dx : \int_V |u|^{q(\alpha)} |x|^{2\alpha} \, dx = 1 \right]. \tag{A.3}$$

Note that

$$\int_V |\nabla u|^2 |x|^{2\alpha} \, dx = \sum_{|\sigma|=\alpha} c_\sigma \int_V |\nabla u|^2 x_1^{2\sigma_1} x_2^{2\sigma_2} \cdots x_n^{2\sigma_n} \, dx, \tag{A.4}$$

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $c_\sigma = \frac{\alpha!}{\sigma_1! \sigma_2! \cdots \sigma_n!}$. Then we see

$$\begin{aligned} S \geq 2^{\frac{2n}{n+2\alpha}} & \inf_{\sum \varepsilon_\sigma = 1, \varepsilon_\sigma \geq 0} \sum_{|\sigma|=\alpha} c_\sigma (\varepsilon_\sigma / C_\sigma)^{2/q(\alpha)} \times \\ & \inf \left[\int_V |\nabla v_\sigma|^2 x_1^{2\sigma_1} \cdots x_n^{2\sigma_n} \, dx : \quad \int_V |v_\sigma|^{q(\alpha)} x_1^{2\sigma_1} \cdots x_n^{2\sigma_n} \, dx = 1 \right]. \end{aligned} \tag{A.5}$$

We needs more notations.

$$\begin{cases} z = (z^1, z^2, \ldots, z^n) \\ z^j = (z_1^j, z_2^j, \ldots, z_{2\sigma_j+1}^j) \in \mathbb{R}^{2\sigma_j+1} \\ k_\sigma = |S^{2\sigma_1}| \cdot |S^{2\sigma_2}| \cdots |S^{2\sigma_n}| \\ V_\sigma(z) = v(|z^1|, |z^2|, \ldots, |z^n|) \end{cases} \tag{A.6}$$

Under these notations

$$\begin{aligned} S \geq 2^{\frac{2n}{n+2\alpha}} & \inf_{\sum \varepsilon_\sigma = 1, \varepsilon_\sigma \geq 0} \sum_{|\sigma|=\alpha} c_\sigma (\varepsilon_\sigma / C_\sigma)^{2/q(\alpha)} \times \\ & k_\sigma^{-1} \inf \left[\int_{\mathbb{R}^{n+2\alpha}} |\nabla V_\sigma(z)|^2 \, dx : \quad \int_{\mathbb{R}^{n+2\alpha}} |V_\sigma(z)|^{q(\alpha)} \, dx = k_\sigma \right]. \\ = 2^{\frac{2n}{n+2\alpha}} & \inf_{\sum \varepsilon_\sigma = 1, \varepsilon_\sigma \geq 0} \sum_{|\sigma|=\alpha} \frac{c_\sigma}{k_\sigma} \left[\frac{\varepsilon_\sigma \cdot k_\sigma}{C_\sigma} \right]^{2/q(\alpha)} \times \\ & \inf \left[\int_{\mathbb{R}^{n+2\alpha}} |\nabla V_\sigma|^2 \, dx : \quad \int_{\mathbb{R}^{n+2\alpha}} |V_\sigma|^{q(\alpha)} \, dx = 1 \right] \\ = 2^{\frac{2n}{n+2\alpha}} & S(2, q(\alpha), n + 2\alpha) \sum_{|\sigma|=\alpha} \frac{c_\sigma}{k_\sigma} \left[\frac{\varepsilon_\sigma \cdot k_\sigma}{C_\sigma} \right]^{2/q(\alpha)} \end{aligned} \tag{A.7}$$

Here we prepare elementary lemmas.

LEMMA A.2 *Under the same notations, it holds that*

$$\inf_{\sum \varepsilon_\sigma=1, \varepsilon_\sigma \geq 0} \sum_{|\sigma|=\alpha} \frac{c_\sigma}{k_\sigma} \left[\frac{\varepsilon_\sigma \cdot k_\sigma}{C_\sigma} \right]^{2/q(\alpha)} = \left(\sum_{|\sigma|=\alpha} \frac{c_\sigma}{k_\sigma} \right)^{1-2/q(\alpha)}. \tag{A.8}$$

LEMMA A.3

$$\sum_{|\sigma|=\alpha} \frac{c_\sigma}{k_\sigma} = \frac{|S^{n-1}|}{2^n |S^{n+2\alpha-1}|} = \frac{1}{2^n \pi^\alpha} \frac{\Gamma\left(\frac{n+2\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \tag{A.9}$$

Proof of Lemma A.2 Let us set

$$F(\{\varepsilon_\sigma\}_{|\sigma|=\alpha}, \mu) = \sum_{|\sigma|=\alpha} \frac{c_\sigma}{k_\sigma} \left[\frac{\varepsilon_\sigma \cdot k_\sigma}{C_\sigma} \right]^{2/q(\alpha)} - \mu \left(\sum_{|\sigma|=\alpha} \varepsilon_\sigma - 1 \right) \tag{A.10}$$

Applying the method of a Lagrange multiplier to $F(\{\varepsilon_\sigma\}_{|\sigma|=\alpha}, \mu)$ under the restriction $\sum_{|\sigma|=\alpha} \varepsilon_\sigma = 1, \varepsilon_\sigma \geq 0$, the infimum in Lemma is attained when

$$\mu^{\frac{q(\alpha)}{2-q(\alpha)}} = \left(\frac{2}{q(\alpha)} \right)^{\frac{q(\alpha)}{2-q(\alpha)}} \cdot \frac{1}{\sum_{|\sigma|=\alpha} c_\sigma / k_\sigma}, \quad \varepsilon_\sigma = \frac{c_\sigma}{k_\sigma} \left(\sum_{|\sigma|=\alpha} \frac{c_\sigma}{k_\sigma} \right)^{-1}. \tag{A.11}$$

Therefore the assertion is now obvious.

Proof of Lemma A.3 In place of ε_σ, we set

$$\begin{aligned} \varepsilon_\sigma^{n,\alpha} &= 2^n \pi^\alpha \frac{c_\sigma}{k_\sigma} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+2\alpha}{2}\right)} = \frac{\alpha!}{2^{2\alpha}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+2\alpha}{2}\right)} \cdot \Pi_{j=1}^n \frac{(2\sigma_j)!}{(\sigma_j!)^2} \\ &= \frac{\alpha!}{n(n+2)\cdots(n+2\alpha-2) \cdot 2^\alpha} \Pi_{j=1}^n \binom{2\sigma_j}{\sigma_j}, \end{aligned} \tag{A.12}$$

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, $|\sigma| = \sigma_1 + \sigma_2 + \cdots + \sigma_n$. Then we show that $\sum_{|\sigma|=\alpha} \varepsilon_\sigma^{n,\alpha} = 1$. To this end let us set

$$\begin{cases} \varepsilon_\sigma^{n,\alpha} = l(n, \alpha) \cdot D_{n,\sigma}, \\ D_{n,\sigma} = \Pi_{j=1}^n \binom{2\sigma_j}{\sigma_j}, \\ l(n, \alpha) = \frac{\alpha!}{2^\alpha n(n+2)\cdots(n+2\alpha-2)} \text{ and } l(n, 0) = 1. \end{cases} \tag{A.13}$$

We shall show the assertion inductively. So we assume that

$$\sum_{|\sigma|=\sigma_1+\cdots+\sigma_k=\beta} \varepsilon_{\sigma}^{k,\beta} = 1, \tag{A.14}$$

for any nonnegative integers k and β satisfying $k \leq n-1$ and $\beta \leq \alpha$, and we consider the case $k = n$. By the hypothesis of the induction, we see

$$\sum_{|\sigma|=\alpha} D_{n,\sigma} = \sum_{k=0}^{\alpha} \binom{2k}{k} \sum_{|\sigma|=\alpha-k} D_{n-1,\sigma} = \sum_{k=0}^{\alpha} \binom{2k}{k} \cdot l(n-1, \alpha-k)^{-1}. \tag{A.15}$$

Then it suffices to show that $\sum_{k=0}^{\alpha} \binom{2k}{k} \cdot l(n-1, \alpha-k)^{-1} = l(n,\alpha)^{-1}$, namely

LEMMA A.4 *For $n, \alpha \geq 0$, we set*

$$P(n,\alpha) = \sum_{k=0}^{\alpha} \binom{2k}{k} \frac{(n-1)(n+1)\cdots(n-3+2(\alpha-k))}{2^k(\alpha-k)!}. \tag{A.16}$$

Then

$$P(n,\alpha) = \frac{n(n+2)\cdots(n+2\alpha-2)}{\alpha!}. \tag{A.17}$$

Proof of Lemma A.4 Again we make use of the induction on the values of $\alpha + n$. We assume that (A.18) holds when $\alpha + n \leq m$. Now we assume that $\alpha + n = m + 1$ First we see that for any nonnegative integers α and n

$$P(n,\alpha+1) = 2P(n,\alpha) + P(n-2,\alpha+1). \tag{A.18}$$

Then the desired equality (A.18) easily follows from these relations.

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