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Inequalities for Laguerre Functions

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The main published inequality for Laguerre functions $L_\nu^\mu(z)$ seems to be for Laguerre polynomials $L_n^0(x)$ only; it is [**2**: 10.18(3)]:

$$|L_n(x)| \leq e^{x/2} \qquad \text{for} \quad x > 0.$$

This paper presents several inequalities for Laguerre polynomials $L_n^\mu(x)$ and Laguerre functions $L_\nu^\mu(x)$, most of which do not seem to be in the existing literature. The corresponding inequalities for confluent hypergeometric functions are noted.

For our work on expansions in series of Laguerre functions, M.N. Hunter and I needed an inequality for $L_\nu^\mu(x)$ when $\operatorname{re}\mu > -\frac{1}{2}$ and $\operatorname{re}\nu$ is large. The only extensions of the above inequality that we could obtain had multiples of e^x on the right hand side instead of $e^{x/2}$. This paper goes on to show that this is inevitable for non-integral ν, in that $\left|L_\nu^\mu(x)\right|$ can exceed a multiple of $e^{\lambda x}$ for every fixed $\lambda < 1$ if x is sufficiently large.

Keywords: Laguerre polynomial; Laguerre function; confluent hypergeometric function; fractional integration.

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Laguerre functions are the entire functions of z defined by

$$L_\nu^\mu(z) = \binom{\mu+\nu}{\nu} \Phi(-\nu, 1+\mu; z), \tag{1}$$

where Φ is the confluent hypergeometric function

$$\Phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!},$$

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$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},$$

and

$$\binom{\mu+\nu}{\nu} = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}.$$

Laguerre polynomials are the functions $L_n^\mu(z)$, n being any non-negative integer; they are polynomials in z because $(-n)_r = 0$ if $r > n$.

Inequalities The main (almost only) published inequality for Laguerre functions is for Laguerre polynomials; it is

$$\left|L_n^\mu(x)\right| \le \binom{\mu+n}{\mu} e^{x/2} \qquad \text{for} \quad \mu \ge 0 \quad \text{and} \quad x \ge 0. \tag{2}$$

[**2**: 10.18(3), (14) and (15)]; see also [**3**: Theorems 7.6.2 and 4].

To extend this to complex μ we start with the case $\mu = 0$, namely

$$|L_n(x)| \le e^{x/2} \qquad \text{for} \quad x > 0, \tag{3}$$

and apply the fractional integration formula [**2**: 10.12(30) with $\alpha = 0$]

$$\frac{x^\mu L_n^\mu(x)}{\Gamma(\mu+n+1)} = \int_0^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)} \frac{L_n(t)}{\Gamma(n+1)}\, \mathrm{d}\, t$$

for $x > 0$ and $\operatorname{re} \mu > 0$. With (3) this gives, writing $\mu' = \operatorname{re} \mu$,

$$\begin{aligned} \left|\frac{x^\mu L_n^\mu(x)}{\Gamma(\mu+n+1)}\right| &\le \int_0^x \frac{(x-t)^{\mu'-1}}{|\Gamma(\mu)|} \frac{e^{t/2}}{\Gamma(n+1)}\, \mathrm{d}\, t \\ &= \frac{e^{x/2}}{|\Gamma(\mu)|\,\Gamma(n+1)} \int_0^x (x-t)^{\mu'-1}\, e^{(t-x)/2}\, \mathrm{d}\, t \\ &= \frac{e^{x/2}}{|\Gamma(\mu)|\,\Gamma(n+1)} \int_0^x s^{\mu'-1} e^{-s/2}\, \mathrm{d}\, s. \end{aligned} \tag{4}$$

Increasing $e^{-s/2}$ up to 1,

$$\begin{aligned} x^{\mu'} \left|L_n^\mu(x)\right| &< \left|\frac{\Gamma(\mu+n+1)}{\Gamma(\mu)\Gamma(n+1)}\right| e^{x/2} \frac{x^{\mu'}}{\mu'}, \\ \left|L_n^\mu(x)\right| &< \left|\frac{\mu}{\mu'}\binom{\mu+n}{n}\right| e^{x/2}. \end{aligned} \tag{5}$$

Alternatively, putting $s = 2t$ in (4), the right side of (4) becomes

$$\frac{2^{\mu'} e^{x/2}}{|\Gamma(\mu)|\,\Gamma(n+1)} \int_0^{x/2} t^{\mu'-1} e^{-t}\ \mathrm{d}t < \frac{2^{\mu'} e^{x/2} \Gamma(\mu')}{|\Gamma(\mu)|\,\Gamma(n+1)},$$

so that

$$\left|L_n^{\mu}(x)\right| < \left|\frac{\Gamma(\mu+n+1)}{\Gamma(n+1)}\right| \left|\frac{\Gamma(\mu')}{\Gamma(\mu)}\right| \frac{e^{x/2}}{(x/2)^{\mu'}}. \tag{6}$$

The above arguments establish

THEOREM 1 *If $x > 0$, $re\ \mu > 0$ and n is a non-negative integer, then*

$$\left|L_n^{\mu}(x)\right| < \left|\frac{\mu}{re\ \mu}\binom{\mu+n}{n}\right| e^{x/2}. \tag{7}$$

and

$$\left|L_n^{\mu}(x)\right| < \left|\frac{\Gamma(\mu+n+1)}{\Gamma(n+1)}\right| \left|\frac{\Gamma(re\ \mu)}{\Gamma(\mu)}\right| \frac{e^{x/2}}{(x/2)^{\mathrm{re}\,\mu}}. \tag{8}$$

COROLLARY 1 *If $x > 0$, $re\ c > 1$ and n is a non-negative integer, then*

$$|\Phi(-n, c; x)| < \left|\frac{c-1}{re\ c-1}\right| e^{x/2} \tag{9}$$

and

$$|\Phi(-n, c; x)| < |(c-1)\Gamma(re\ c-1)| \frac{e^{x/2}}{(x/2)^{\mathrm{re}\,c-1}}. \tag{10}$$

For some work on expansions in series of Laguerre functions, M.N. Hunter and I needed estimates of $L_\nu^{\mu}(x)$ for re $\mu > -\frac{1}{2}$ and ν complex. Theorem 1 is clearly not adequate for this.

THEOREM 2 *If $x > 0$, $re\ \mu > -\frac{1}{2}$ and $re\ (\mu+\nu) > -1$, then*

$$\left|L_\nu^{\mu}(x)\right| \le \frac{\Gamma(re\ (\nu+\mu+1))}{|\Gamma(\nu+1)|\,\Gamma(re\ \mu+1)} \frac{\Gamma(re\ \mu+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|} e^{x}.$$

Proof By Poisson's Integral for Bessel functions, since re $\mu > -\frac{1}{2}$,

$$J_\mu(z) = \frac{\left(\frac{1}{2}z\right)^\mu}{\Gamma(\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\int_0^\pi e^{iz\cos\theta}\sin^{2\mu}\theta \quad \mathrm{d}\,\theta,$$

$$\left|J_\mu(x)\right| < \frac{\left(\frac{1}{2}x\right)^{\mu'}}{\Gamma(\frac{1}{2})\left|\Gamma(\mu+\frac{1}{2})\right|}\int_0^\pi \sin^{2\mu'}\theta \quad \mathrm{d}\,\theta = \frac{\left(\frac{1}{2}x\right)^{\mu'}}{\Gamma(\mu'+1)}\frac{\Gamma(\mu'+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|}, \tag{11}$$

where $\mu' =$ re μ again. By [**1**: 6.11(17)], Whittaker's confluent hypergeometric function $M_{\lambda,\frac{1}{2}\mu}(x)$ is expressible, if re $\left(\lambda+\frac{1}{2}(\mu+1)\right) > 0$, by

$$M_{\lambda,\frac{1}{2}\mu}(x) = \frac{\Gamma(\mu+1)}{\Gamma\left(\lambda+\frac{1}{2}(\mu+1)\right)}e^{\frac{1}{2}x}x^{\frac{1}{2}}\int_0^\infty e^{-t}t^{\lambda-\frac{1}{2}}J_\mu\left(2(xt)^{\frac{1}{2}}\right) \,\mathrm{d}\,t.$$

By (1) and [**1**: 6.9(3)], with $\lambda = \nu + \frac{1}{2}(\mu+1)$,

$$\begin{aligned} L_\nu^\mu(x) &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}\Phi(-\nu,\mu+1;x) \\ &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}e^{\frac{1}{2}x}x^{-\frac{1}{2}(\mu+1)}M_{\lambda,\frac{1}{2}\mu}(x) \\ &= \frac{e^x x^{-\frac{1}{2}\mu}}{\Gamma(\nu+1)}\int_0^\infty e^{-t}t^{\frac{1}{2}\mu+\nu}J_\mu\left(2(xt)^{\frac{1}{2}}\right) \,\mathrm{d}\,t, \end{aligned} \tag{12}$$

whence, using (11),

$$\left|L_\nu^\mu(x)\right| < \frac{e^x}{|\Gamma(\nu+1)|\,\Gamma(\mu'+1)}\frac{\Gamma(\mu'+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|}\int_0^\infty e^{-t}t^{\mu'+\nu'} \,\mathrm{d}\,t.$$

The result follows from this, the integral being a gamma function. □

COROLLARY 2 *If* $x > 0$, *re* $c >$ *re* a *and re* $c > \frac{1}{2}$, *then*

$$|\Phi(a,c;x)| \le \frac{\Gamma(\text{re}\,c - \text{re}\,a)}{|\Gamma(1-a)|\,\Gamma(\text{re}\,c)}\frac{\Gamma(\text{re}\,c-\frac{1}{2})}{\left|\Gamma(c-\frac{1}{2})\right|}e^x;$$

and if also a and c are real, then

$$|\Phi(a,c;x)| \le \frac{\Gamma(c-a)}{|\Gamma(1-a)|\,\Gamma(c)}e^x.$$

Contrast There is a marked contrast between the right hand sides of the inequalities (7) and (8) in Theorem 1 and that in Theorem 2. The former are $O(e^{x/2})$ as $x \to \infty$, while the latter is $O(e^x)$. In Theorem 3 it will be shown that this contrast is not illusory.

THEOREM 3 *If* $\operatorname{re}\mu > -1$, $\operatorname{re}\nu > 0$, $\mu + \nu + 1$ *is real and positive,* $0 < \delta < 1$ *and* $0 < \lambda < 1$, *then for all sufficiently large positive* x

$$\left|L_\nu^\mu(x)\right| > \Gamma(\mu+\nu+1)\frac{|\sin \nu\pi|}{\pi}(1-\delta)e^{\lambda x}.$$

Proof If ν is an integer there is nothing to prove. Suppose that ν is not an integer and that $x > 0$. Write

$$a = -\nu, \qquad b = \mu+\nu+1, \qquad c = \mu+1,$$

so that

$$c - a = b > 0 \tag{13}$$

Then

$$\begin{aligned} L_\nu^\mu(x) &= \frac{\Gamma(b)}{\Gamma(1-a)\Gamma(c)}\Phi(a,c;x) \\ &= \frac{\Gamma(b)}{\Gamma(1-a)\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{x^n}{n!} \\ &= \frac{\Gamma(b)}{\Gamma(1-a)\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}. \end{aligned} \tag{14}$$

By [**1**: 1.18(4)] and (13),

$$\frac{\Gamma(a+n)}{\Gamma(c+n)} = n^{-b}\{1 + O(1/n)\} \qquad \text{as } n \to \infty,$$

so there is a positive integer M such that

$$\left|\frac{\Gamma(a+n)}{\Gamma(c+n)} - n^{-b}\right| < n^{-b}\cdot\tfrac{1}{2}\delta \qquad \text{for all } n \geq M. \tag{15}$$

Also there is an integer $N \geq M$ such that $n^{-b} > \lambda^n$ for all $n \geq N$.

$$\begin{aligned} \left|\sum_{n=N}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!} - \sum_{n=N}^{\infty} n^{-b}\frac{x^n}{n!}\right| &\leq \sum_{n=N}^{\infty}\left|\frac{\Gamma(a+n)}{\Gamma(c+n)} - n^{-b}\right|\frac{x^n}{n!} \\ &< \tfrac{1}{2}\delta\sum_{n=N}^{\infty} n^{-b}\frac{x^n}{n!} \end{aligned} \tag{16}$$

by (15). By the triangle inequality,

$$\left|\sum_{n=N}^{\infty} n^{-b}\frac{x^n}{n!}\right| \le \left|\sum_{n=N}^{\infty} n^{-b}\frac{x^n}{n!} - \sum_{n=N}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}\right| + \left|\sum_{n=N}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}\right|;$$

whence, using (16),

$$\begin{aligned}\left|\sum_{n=N}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}\right| &> \sum_{n=N}^{\infty} n^{-b}\frac{x^n}{n!} - \tfrac{1}{2}\delta\sum_{n=N}^{\infty} n^{-b}\frac{x^n}{n!}\\ &= \left(1-\tfrac{1}{2}\delta\right)\sum_{n=N}^{\infty} n^{-b}\frac{x^n}{n!}\\ &> \left(1-\tfrac{1}{2}\delta\right)\sum_{n=N}^{\infty} \lambda^n\frac{x^n}{n!}\\ &= \left(1-\tfrac{1}{2}\delta\right)\left(e^{\lambda x} - \sum_{n=0}^{N-1}\frac{(\lambda x)^n}{n!}\right)\\ &= \left(1-\tfrac{1}{2}\delta\right)e^{\lambda x} - P(x) \qquad (17)\end{aligned}$$

where P is a polynomial of degree less than N.

By the triangle inequality again,

$$\left|\sum_{n=N}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}\right| \le \left|\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}\right| + \left|-\sum_{n=0}^{N-1}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}\right|.$$

With (17) this gives

$$\left|\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}\right| > \left(1-\tfrac{1}{2}\delta\right)e^{\lambda x} - P(x) - |Q(x)|, \qquad (18)$$

where Q is a polynomial of degree less than N. Now (18) persists if the coefficients in P and Q are replaced by their absolute values; let $\frac{1}{2}\delta \cdot R(x)$ be the polynomial so replacing $P(x) + |Q(x)|$. Thus

$$\begin{aligned}\left|\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{x^n}{n!}\right| &> (1-\delta)e^{\lambda x} + \tfrac{1}{2}\delta\left(e^{\lambda x} - R(x)\right),\\ &> (1-\delta)e^{\lambda x}\end{aligned}$$

for x sufficiently large. With (14) this gives

$$\begin{aligned}\left|L_\nu^\mu(x)\right| &> \frac{\Gamma(b)}{|\Gamma(a)\Gamma(1-a)|}(1-\delta)e^{\lambda x}\\ &= \frac{\Gamma(b)\,|\sin a\pi|}{\pi}(1-\delta)e^{\lambda x},\end{aligned}$$

and using (13) the theorem follows. □

COROLLARY 3 *If re* $a < 0 <$ *re* c, $c - a$ *is real,* $0 < \delta < 1$ *and* $0 < \lambda < 1$, *then*

$$|\Phi(a, c; x)| > \left|\frac{\Gamma(c)}{\Gamma(a)}\right| (1 - \delta)e^{\lambda x}$$

for all sufficiently large positive x.

Conclusion

Theorem 3 shows that, in general, e^x in Theorem 2 cannot be replaced by any lower order exponential.

References

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