A REFINEMENT OF THE POINCARÉ INEQUALITY FOR KOLMOGOROV OPERATORS ON $\mathbb{R}^d$

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We give a refinement of the Poincaré inequality for Kolmogorov operators on $\mathbb{R}^d$. This refinement yields some regularity result of the corresponding semigroups.

1. Introduction

Let $\{P_t\}$ be the semigroup on $B_b(\mathbb{R}^d)$ associated with the Kolmogorov operator

$$L_0 = \frac{1}{2}\Delta + F(x) \cdot D. \tag{1.1}$$

Here we denote by $B_b(\mathbb{R}^d)$ the Banach space of all Borel and bounded functions, endowed with the supremum norm. We assume a suitable dissipative assumption on the function $F = (F_1, \dots, F_d)$ such that there exists a unique invariant probability measure ν on $\mathbb{R}^d$ associated with $\{P_t\}$. Let $H^1(\nu)$ and $H^2(\nu)$ be the Sobolev spaces with the norms

$$\|\varphi\|_{H^1(\nu)} = \left[\int_{\mathbb{R}^d} \left[|\varphi|^2 + |D\varphi|^2\right] d\nu\right]^{1/2}, \tag{1.2}$$

$$\|\varphi\|_{H^2(\nu)} = \left[\int_{\mathbb{R}^d} \left[|\varphi|^2 + |D\varphi|^2 + \left|D^2\varphi\right|^2\right] d\nu\right]^{1/2}, \tag{1.3}$$

respectively. It is well known that the Poincaré inequality with respect to ν is the following:

$$\int_{\mathbb{R}^d} (\varphi - \overline{\varphi})^2 d\nu \le \frac{1}{2\alpha} \int_{\mathbb{R}^d} |D\varphi|^2 d\nu, \quad \varphi \in H^1(\nu), \tag{1.4}$$

where $\alpha > 0$ is a constant determined by F, and $\overline{\varphi} = \int_{\mathbb{R}^d} \varphi \, d\nu$. The Poincaré inequality (1.4) is so important that it implies existence of a *spectral gap* or, equivalently, *exponential*

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convergence of equilibrium of the semigroup $\{P_t\}$ such that

$$\int_{\mathbb{R}^d} \left| P_t\varphi - \overline{\varphi} \right|^2 d\nu \le e^{-2\alpha t} \int_{\mathbb{R}^d} |\varphi|^2 d\nu, \quad t \ge 0,\ \varphi \in L^2(\nu) \tag{1.5}$$

(cf. [2, Proposition 3.12]).

The aim of this paper is to give a refinement of the Poincaré inequality (1.4) such that

$$\int_{\mathbb{R}^d} (\varphi - \overline{\varphi})^2 d\nu + \frac{1}{2\alpha} \int_0^\infty dt \int_{\mathbb{R}^d} \left| D^2 P_t\varphi \right|^2 d\nu \le \frac{1}{2\alpha} \int_{\mathbb{R}^d} |D\varphi|^2 d\nu, \quad \varphi \in H^1(\nu). \tag{1.6}$$

When $F(x) = -\alpha x$ in (1.1) (i.e., $\{P_t\}$ is the Ornstein-Uhlenbeck semigroup), inequality (1.6) is reduced to an *equality*. Furthermore, we will show that inequality (1.6) yields the regularity result such that $P_t\varphi - \overline{\varphi} \in L^2((0,\infty), dt; H^2(\nu))$ for $\varphi \in H^1(\nu)$. This regularity result corresponds to the well-known regularity result such that $P_t\varphi - \overline{\varphi} \in L^2((0,\infty), dt; H^1(\nu))$ for $\varphi \in L^2(\nu)$ (cf. (1.5) and (3.18)).

In the proof of the Poincaré inequality (1.4), the following inequality was used for $\varphi \in C_b^1(\mathbb{R}^d)$:

$$\left| DP_t\varphi(x) \right|^2 \le e^{-2\alpha t} P_t\left(|D\varphi|^2\right)(x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \tag{1.7}$$

(cf. [2, Proposition 2.8]). In our proof of inequality (1.6), we will also use (1.7). However, we will derive another differential inequality so as not to lose the term $|D^2 P_t\varphi(x)|^2$. For this purpose, it is crucial to assume that the Kolmogorov operator L_0 has the form of (1.1). It seems hard for the author to apply our proof directly to a more general Kolmogorov operator such as $(1/2)\mathrm{tr}[C(x)D^2] + F(x) \cdot D$.

The contents of this paper are as follows. In Section 2, we will state the main results. They will be proved in Section 3.

2. Main results

First of all, we recall the results about invariant probability measures on $\mathbb{R}^d$ (for details, see [2]). Following [1, Hypothesis 1.1], we make the following assumptions on $F = (F_1, \dots, F_d)$ of (1.1).

(A) $F \in C^4(\mathbb{R}^d; \mathbb{R}^d)$, and there exist

$$\begin{gathered} m \ge 0 \text{ such that } \sup_{x \in \mathbb{R}^d} \frac{\left| D^\beta F(x) \right|}{1 + |x|^{2m+1-\beta}} < +\infty, \quad \beta = 0,1,2,3,4, \\ \alpha > 0 \text{ such that } DF(x)y \cdot y \le -\alpha |y|^2, \quad x, y \in \mathbb{R}^d, \\ a, \gamma, c > 0 \text{ such that } \left(F(x+y) - F(x)\right) \cdot y \le -a|y|^{2m+2} + c\left(|x|^\gamma + 1\right), \quad x, y \in \mathbb{R}^d. \end{gathered} \tag{2.1}$$

By [1, Proposition 1.2.2], the stochastic differential equation

$$d\xi(t,x) = F(\xi(t,x))dt + dw(t), \qquad \xi(0,x) = x, \tag{2.2}$$

admits a unique strong solution $(\xi(t,x))$, where $(w(t))$ is a d-dimensional standard Brownian motion on a probability space. Then we can define the semigroup $\{P_t\}$ on $B_b(\mathbb{R}^d)$ by

$$P_t\varphi(x) = \mathbb{E}[\varphi(\xi(t,x))]. \tag{2.3}$$

By [2, Proposition 2.7], there exists a unique probability measure ν on $\mathbb{R}^d$ satisfying the following: for any uniformly continuous and bounded function χ on $\mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} P_t\chi\, d\nu = \int_{\mathbb{R}^d} \chi\, d\nu, \quad t \geq 0. \tag{2.4}$$

Such a probability measure ν on $\mathbb{R}^d$ is called the invariant probability measure for $\{P_t\}$. Using this invariant probability measure ν, we can extend $\{P_t\}$ to a strongly continuous semigroup of contractions on $L^p(\nu)$ for every $p \geq 1$. We also denote by $\{P_t\}$ this extended strongly continuous semigroup. The generator $(L, \mathrm{dom}_p(L))$ of $\{P_t\}$ in $L^p(\nu)$ is the closure of the Kolmogorov operator $(L_0, C_0^\infty(\mathbb{R}^d))$, where L_0 is the operator defined by (1.1), and $C_0^\infty(\mathbb{R}^d)$ is the space of C^∞-functions with compact supports. An important example of L is the Ornstein-Uhlenbeck operator corresponding to the case $F(x) = -\alpha x$.

Next, we define the Sobolev spaces $H^1(\nu)$ and $H^2(\nu)$. The operators $(D, C_0^\infty(\mathbb{R}^d))$ and $(D^2, C_0^\infty(\mathbb{R}^d))$ are closable in $L^p(\nu)$ for every $p \geq 1$. We also denote their closures by $(D, \mathrm{dom}_p(D))$ and $(D^2, \mathrm{dom}_p(D^2))$, respectively. Then, we can define the Sobolev spaces $H^1(\nu)$ and $H^2(\nu)$ by $H^1(\nu) = \mathrm{dom}_2(D)$ and $H^2(\nu) = \mathrm{dom}_2(D^2)$, respectively. They become Hilbert spaces with the norms defined by (1.2) and (1.3), respectively. Then, the Poincaré inequality (1.4) holds for the constant α of (2.1).

Now, we state the main results of this paper.

Theorem 2.1. *Assume (2.1). Then, for every $\varphi \in H^1(\nu)$,*

$$P_t\varphi \in H^2(\nu), \quad t\text{-a.e. on } (0,\infty), \tag{2.5}$$

$$P_t\varphi - \overline{\varphi} \in L^2((0,\infty), dt; H^2(\nu)), \tag{2.6}$$

$$\int_{\mathbb{R}^d} (\varphi - \overline{\varphi})^2 d\nu + \frac{1}{2\alpha}\int_0^\infty dt \int_{\mathbb{R}^d} \left| D^2 P_t\varphi \right|^2 d\nu \leq \frac{1}{2\alpha}\int_{\mathbb{R}^d} |D\varphi|^2 d\nu. \tag{2.7}$$

When $F(x) = -\alpha x$, inequality (2.7) is reduced to an equality. (2.8)

Results (2.5) and (2.6) give a regularity result of $P_t\varphi$ for $\varphi \in H^1(\nu)$. On the other hand, results (2.7) and (2.8) give refinements of the Poincaré inequality.

3. Proof of Theorem 2.1

In this section, we prove Theorem 2.1. For $\varphi \in C_0^\infty(\mathbb{R}^d)$, we set

$$\eta(t,x) = P_t\varphi(x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^d. \tag{3.1}$$

First, we give two lemmas.

Lemma 3.1. *Assume (2.1). If $\varphi \in C_0^\infty(\mathbb{R}^d)$, then*

$$(D\eta)_t,\ D^\beta\eta\ (\beta = 0,1,2,3) \text{ are continuous on } [0,\infty)\times\mathbb{R}^d, \tag{3.2}$$

$$(D\eta)_t = D\eta_t \quad \text{on } [0,\infty)\times\mathbb{R}^d. \tag{3.3}$$

Proof. Since $F \in C^4(\mathbb{R}^d;\mathbb{R}^d)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$, it follows from the theory in [1, Chapter 1] that

$$D^\beta\eta \text{ is continuous on } [0,\infty)\times\mathbb{R}^d \quad \text{for } \beta = 0,1,2,3. \tag{3.4}$$

Since η of (3.1) satisfies the Kolmogorov equation

$$\eta_t = \frac{1}{2}\Delta\eta + F\cdot D\eta \quad \text{on } [0,\infty)\times\mathbb{R}^d, \tag{3.5}$$

we have, for any $R,T > 0$,

$$D\eta(t+h,x) - D\eta(t,x) = \int_t^{t+h} DL\eta(s,x)ds, \quad 0 \le t \le T,\ |x| < R, \tag{3.6}$$

where $h \in \mathbb{R}$ is chosen such that $t+h \ge 0$. By (3.4) and (3.6), we conclude that $D\eta(t,x)$ is differentiable with respect to t for $|x| < R$ and

$$(D\eta)_t(t,x) = DL\eta(t,x) = D\eta_t(t,x), \quad 0 \le t \le T,\ |x| < R. \tag{3.7}$$

Since $R,T > 0$ are arbitrary, (3.3) follows. By (3.4) and (3.7), $(D\eta)_t$ is continuous on $[0,\infty)\times\mathbb{R}^d$. The proof is complete. □

Lemma 3.2. *Assume that (2.1) holds and $\varphi \in C_0^\infty(\mathbb{R}^d)$. Let*

$$\chi(t,x) = \left|D\eta(t,x)\right|^2 = \sum_{j=1}^d \left|D_j\eta(t,x)\right|^2, \quad (t,x) \in [0,\infty)\times\mathbb{R}^d. \tag{3.8}$$

Then,

$$\chi_t,\ D^\beta\chi\ (\beta = 0,1,2) \text{ are continuous on } [0,\infty)\times\mathbb{R}^d, \tag{3.9}$$

$$|D^2\eta|^2 + \chi_t \le L\chi - 2\alpha\chi \quad \text{on } [0,\infty)\times\mathbb{R}^d. \tag{3.10}$$

When $F(x) = -\alpha x$, inequality (3.10) is reduced to an equality.

Proof. We obtain (3.9) from (3.2). Differentiating equation (3.5) with respect to x_j, we have, by (3.3),

$$(D_j\eta)_t = \frac{1}{2}\Delta(D_j\eta) + \sum_{i=1}^d F_i[D_i(D_j\eta)] + \sum_{i=1}^d (D_jF_i)(D_i\eta) \quad \text{on } [0,\infty)\times\mathbb{R}^d. \tag{3.11}$$

On the other hand, we note that

$$\frac{1}{2}D_i\chi = \sum_{j=1}^{d}(D_j\eta)[D_i(D_j\eta)], \quad 1 \le i \le d, \tag{3.12}$$

$$\frac{1}{2}\Delta\chi = |D^2\eta|^2 + \sum_{j=1}^{d}(D_j\eta)[\Delta(D_j\eta)], \tag{3.13}$$

$$\sum_{i,j=1}^{d}(D_jF_i)(D_i\eta)(D_j\eta) \le -\alpha\chi. \tag{3.14}$$

Here we used (2.1) in (3.14). Inequality (3.14) is reduced to an equality when $F(x) = -\alpha x$. Then, by (3.11)–(3.14), we obtain on $[0,\infty)\times\mathbb{R}^d$

$$\begin{aligned}\frac{1}{2}\chi_t &= \sum_{j=1}^{d}(D_j\eta)(D_j\eta)_t \\ &= \frac{1}{2}\sum_{j=1}^{d}(D_j\eta)[\Delta(D_k\eta)] \\ &\quad + \sum_{i,j=1}^{d}F_i(D_j\eta)[D_i(D_j\eta)] + \sum_{i,j=1}^{d}(D_jF_i)(D_i\eta)(D_j\eta) \\ &\le \frac{1}{2}\left(\frac{1}{2}\Delta\chi - |D^2\eta|^2\right) + \frac{1}{2}F\cdot D\chi - \alpha\chi.\end{aligned} \tag{3.15}$$

Thus, (3.10) follows. It is easy to see that inequality (3.10) is reduced to an equality when $F(x) = -\alpha x$. The proof is complete. □

Now, we prove Theorem 2.1.

Proof of Theorem 2.1.

Step 1. In this step, we will show Theorem 2.1 under the assumption that $\varphi \in C_0^\infty(\mathbb{R}^d)$. We choose $0 < T < +\infty$ arbitrarily. Integrating (3.10) over $[0,T]\times\mathbb{R}^d$ with respect to $dt \times d\nu$, we have

$$\begin{aligned}&\int_0^T dt\int_{\mathbb{R}^d}|D^2\eta(t,\cdot)|^2 d\nu + \int_{\mathbb{R}^d}\left[|D\eta(T,\cdot)|^2 - |D\varphi(\cdot)|^2\right]d\nu \\ &\quad\le \int_0^T dt\int_{\mathbb{R}^d}L\chi(t,\cdot)d\nu - 2\alpha\int_0^T dt\int_{\mathbb{R}^d}|D\eta(t,\cdot)|^2 d\nu.\end{aligned} \tag{3.16}$$

By Lemma 3.2, inequality (3.16) is reduced to an equality when $F(x) = -\alpha x$. Since ν is the invariant probability measure for $\{P_t\}$ as in (2.4), we have

$$\int_{\mathbb{R}^d}L\chi(t,\cdot)d\nu = 0, \quad t \ge 0. \tag{3.17}$$

On the other hand, by [2, Corollary 3.6], we have

$$\int_0^T dt \int_{\mathbb{R}^d} \left|DP_t\varphi\right|^2 d\nu = \int_{\mathbb{R}^d} [|\varphi|^2 - |P_T\varphi|^2] d\nu. \tag{3.18}$$

Thus, we obtain by (3.16)–(3.18)

$$\begin{aligned} \int_0^T dt \int_{\mathbb{R}^d} \left|D^2P_t\varphi\right|^2 d\nu + 2\alpha \int_{\mathbb{R}^d} \left[|\varphi|^2 - \left|P_T\varphi\right|^2\right] d\nu \\ \le \int_{\mathbb{R}^d} |D\varphi|^2 d\nu - \int_{\mathbb{R}^d} \left|DP_T\varphi\right|^2 d\nu, \quad T > 0. \end{aligned} \tag{3.19}$$

Now, let T tend to positive infinity in (3.19). Using (1.7) and the ergodic property

$$\lim_{T\to\infty} P_T\varphi(x) = \overline{\varphi}, \quad x \in \mathbb{R}^d \tag{3.20}$$

(cf. [2, (3.11)]), we have obtained (2.7). Then, by (1.5) and (3.18), we have (2.5) and (2.6). Since inequality (3.19) is reduced to the equality when $F(x) = -\alpha x$, it is not difficult to see (2.8).

Step 2. In this step, we conclude Theorem 2.1. Let $\varphi \in H^1(\nu)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $H^1(\nu)$, we can choose $\{\varphi_n\} \subset C_0^\infty(\mathbb{R}^d)$ such that $\varphi_n \to \varphi$ in $H^1(\nu)$. By Step 1, we see that

$$\int_0^\infty dt \int_{\mathbb{R}^d} \left|D^2P_t\varphi_m - D^2P_t\varphi_n\right|^2 d\nu \le \int_{\mathbb{R}^d} \left|D\varphi_m - D\varphi_n\right|^2 d\nu. \tag{3.21}$$

Thus, $\{D^2P_t\varphi_n\}$ is a Cauchy sequence in $L^2((0,\infty)\times\mathbb{R}^d, dt\times d\nu; \mathbb{R}^{d^2})$. Hence, we find an element $f \in L^2((0,\infty)\times\mathbb{R}^d, dt\times d\nu; \mathbb{R}^{d^2})$ such that

$$D^2P_\cdot\varphi_n(\cdot) \longrightarrow f(\cdot,\cdot) \quad \text{in } L^2\left((0,\infty)\times\mathbb{R}^d, dt\times d\nu; \mathbb{R}^{d^2}\right). \tag{3.22}$$

By the Fubini theorem, we see that $f(t,\cdot) \in L^2(\mathbb{R}^d, \nu; \mathbb{R}^{d^2})$, t-a.e. On the other hand, by (3.22), we find a subsequence $\{n_j\}$ such that

$$\int_{\mathbb{R}^d} \left|D^2P_t\varphi_{n_j}(\cdot) \longrightarrow f(t,\cdot)\right|^2 d\nu \longrightarrow 0, \quad t\text{-a.e.} \tag{3.23}$$

This means that

$$D^2P_t\varphi_{n_j}(\cdot) \longrightarrow f(t,\cdot) \quad \text{in } L^2\left(\mathbb{R}^d, \nu; \mathbb{R}^{d^2}\right), \quad t\text{-a.e.} \tag{3.24}$$

Since $P_t\varphi_{n_j} \in H^2(\nu)\,(= \mathrm{dom}_2(D^2))$ and D^2 is a closed operator in $L^2(\nu)$, we obtain

$$P_t\varphi \in H^2(\nu), \quad f(t,\cdot) = D^2P_t\varphi(\cdot), \quad t\text{-a.e.} \tag{3.25}$$

Then we obtain (2.5). Next, by (3.24), (3.25), Step 1, and Fatou's lemma, we have

$$\begin{aligned}\int_0^\infty dt \int_{\mathbb{R}^d} |D^2 P_t \varphi|^2 d\nu &\leq \liminf_{n\to\infty} \int_0^\infty dt \int_{\mathbb{R}^d} |D^2 P_t \varphi_{n_j}|^2 d\nu \\ &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} |D\varphi_{n_j}|^2 d\nu \\ &= \int_{\mathbb{R}^d} |D\varphi|^2 d\nu.\end{aligned} \tag{3.26}$$

Hence, by (1.5) and (3.18), we obtain (2.6). Finally, by (3.22) and (3.25), we conclude that

$$D^2 P_\cdot \varphi_n(\cdot) \longrightarrow D^2 P_\cdot \varphi(\cdot) \quad \text{in } L^2\big((0,\infty)\times\mathbb{R}^d, dt\times d\nu; \mathbb{R}^{d^2}\big). \tag{3.27}$$

Therefore, (2.7) follows from Step 1. By (3.27) and Step 1, it is easy to see (2.8). The proof is complete. □

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