LOCAL SOLUTION OF CARRIER'S EQUATION IN A NONCYLINDRICAL DOMAIN

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We study Carrier's equation in a noncylindrical domain. We use the penalty method combined with Faedo-Galerkin and compactness arguments. We obtain results of the existence and uniqueness of the local solution.

1. Introduction

The wave equation of the type

$$\frac{\partial^2 u}{\partial t^2} - \left[P_0 + P_1 \int_\alpha^\beta \left(\frac{\partial u}{\partial x}(t)\right)^2 dx\right] \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.1}$$

is a model for small vibrations of an elastic string fixed at α, β with P_0 and P_1 constants, and was investigated by Kirchhoff [4] and Carrier [1]. By approximations on the above equation, Carrier obtained the model

$$\frac{\partial^2 u}{\partial t^2} - \left[P_0 + P_1 \int_\alpha^\beta (u(t))^2 dx\right] \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which is known in the literature as Carrier's equation. Medeiros et al. [7, 8] studied the problem of small vibrations of an elastic string with moving boundary.

In this work, we study the following generalization of (1.2):

$$\frac{\partial^2 u}{\partial t^2} - \widehat{M}\left(x, t, \int_{\alpha(t)}^{\beta(t)} (u(t))^2 dx\right) \frac{\partial^2 u}{\partial x^2} = f \quad \text{in } \widehat{Q}, \tag{1.3a}$$

$$u = 0 \quad \text{on } \widehat{\sum}, \tag{1.3b}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } (\alpha(0), \beta(0)), \tag{1.3c}$$

where

(i) $\widehat{Q} = \bigcup_{0<t<T}]\alpha(t), \beta(t)[\times \{t\}$ is a noncylindrical domain,

(ii) $\widehat{\sum} = \bigcup_{0<t<T} \{(\alpha(t), t), (\beta(t), t)\}$ is the lateral boundary of $\widehat{Q}$.

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We use the penalty method as in Ebihara [2] combined with the Faedo-Galerkin method and compactness arguments. We obtain results of the existence and uniqueness of the local solution. This method was used by several authors, for example, Ebihara et al. [3], Pereira et al. [9]. We use some ideas contained in [7, 8] to obtain the necessary estimates.

2. Notations, assumptions, and main results

We use the notations contained in Lions and Magenes [6] and Lions [5] for the spaces $L^p(\Omega)$, $W^{m,p}(\Omega)$, $1 \le p \le \infty$, $m \in \mathbb{N}$. In particular, for $p = 2$, $m = 1$, and $\Omega = (0,1)$, we consider $L^2(0,1)$ and $H_0^1(0,1)$ with the scalar product and norms represented by $(\cdot,\cdot)$, $|\cdot|$, $((\cdot,\cdot))$, and $\|\cdot\|$, respectively, and given by

$$
\begin{aligned}
(u,v) &= \int_0^1 u(x)v(x)dx, \qquad |u|^2 = \int_0^1 (u(x))^2 dx, \\
((u,v)) &= \int_0^1 \frac{du}{dx}(x)\frac{dv}{dx}(x)dx, \qquad \|u\|^2 = \int_0^1 \left|\frac{du}{dx}(x)\right|^2 dx.
\end{aligned} \tag{2.1}
$$

We consider the following assumptions about $\widehat{M}$:

(M1) $\widehat{M} \in L^\infty_{\text{loc}}([0,\infty), W^{1,\infty}(\widehat{Q}))$ and for each $L > 0$, $\partial\widehat{M}/\partial\lambda \in L^\infty(\widehat{Q} \times (0,L))$;

(M2) $\widehat{M}$, $\partial\widehat{M}/\partial x$, $\partial\widehat{M}/\partial\lambda$, $\partial\widehat{M}/\partial t$ are continuous with respect to λ, a.e. $(x,t) \in \widehat{Q}$;

(M3) there exists a real number $m_0 > 0$ such that $\widehat{M}(x,t,\lambda) \ge m_0$ for all $(x,t) \in \widehat{Q}$ and $\lambda \ge 0$.

With respect to the noncylindrical domain $\widehat{Q}$ we consider the following assumptions:

(H1) $\alpha, \beta \in C^3([0,T])$, $\alpha(t) < \beta(t)$, for all $t \in [0,T]$;

(H2) $\alpha'(t) < 0$, $\beta'(t) > 0$, for all $t \in (0,T]$, and $\alpha'(0) = 0 = \beta'(0)$;

(H3) $\gamma'(t) \le \sqrt{m_0/2}$, for all $t \in [0,T]$, where $\gamma(t) = \beta(t) - \alpha(t)$;

(H4) $|\alpha''(t) + \gamma''(t)y| \le |\alpha'(t) + \gamma'(t)y|^2/\gamma(t)$, for all $t \in [0,T]$, $y \in [0,1]$.

We observe that when (x,t) varies in $\widehat{Q}$ the point (y,t) of $\mathbb{R}^2$, with $y = (x - \alpha(t))/\gamma(t)$, varies in $Q = (0,1) \times (0,T)$, so we define the diffeomorphism $\tau : \widehat{Q} \to Q$ by

$$
\tau(x,t) = (y,t), \quad \text{where } y = \frac{x - \alpha(t)}{\gamma(t)}. \tag{2.2}
$$

We also consider the diffeomorphism $\hat{\tau} : \widehat{Q} \times [0,\infty) \to Q \times [0,\infty)$ defined by $\hat{\tau}(x,t,\lambda) = (y,t,\lambda)$ with y as above.

Denoting

$$
\begin{aligned}
v(y,t) &= (u \circ \tau^{-1})(y,t), \\
M(y,t,\lambda) &= \left(\widehat{M} \circ (\hat{\tau})^{-1}\right)(y,t,\lambda), \\
g(y,t) &= (f \circ \tau^{-1})(y,t),
\end{aligned} \tag{2.3}
$$

$v_0(y) = u_0(\gamma_0 y + \alpha_0)$, and $v_1 = (\gamma_0 y + \alpha_0)$, where $\gamma_0 = \gamma(0)$, $\alpha_0 = \alpha(0)$, and defining

$$
\begin{aligned}
b(y,t) &= -2\frac{\alpha'(t) + \gamma'(t)y}{\gamma(t)}, \\
c(y,t) &= -\left[\frac{\alpha''(t) + \gamma''(t)y}{\gamma(t)} - 2\gamma'(t)\frac{\alpha'(t) + \gamma'(t)y}{\gamma^2(t)}\right],
\end{aligned}
\tag{2.4}
$$

problem (1.3) is transformed into the following cylindrical problem:

$$
\begin{aligned}
v''(y,t) - \left[\frac{1}{\gamma^2(t)}M\Big(y,t,\gamma(t)\,|\,v(t)\,|^2\Big) - \frac{b^2(y,t)}{4}\right]\frac{\partial^2 v}{\partial y^2}(y,t) \\
+ b(y,t)\frac{\partial v'}{\partial y}(y,t) + c(y,t)\frac{\partial v}{\partial y}(y,t) = g(y,t), \quad \text{in } Q,
\end{aligned}
\tag{2.5a}
$$

$$v = 0 \quad \text{on } \textstyle\sum, \tag{2.5b}$$

$$v(y,0) = v_0(y), \quad v'(y,0) = v_1(y), \quad \text{in } (0,1), \tag{2.5c}$$

where $'$ denotes the derivative with respect to t.

If $\Omega_t = (\alpha(t),\beta(t))$ and $\Omega_0 = (\alpha_0,\beta_0)$, where $\alpha_0 = \alpha(0)$ and $\beta_0 = \beta(0)$, the main result of this work is given by the following theorem.

Theorem 2.1. *Let $\widehat{M}$ satisfy (M1)–(M3) and let $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy (H1)–(H4). Given $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$, and $f \in L^2(0,T,H_0^1(\Omega_t))$ with $f' \in L^2(0,T,L^2(\Omega_t))$, there exist $T_0 > 0$ and a unique u satisfying*

$$
\begin{aligned}
u &\in L^\infty(0,T_0,H_0^1(\Omega_t) \cap H^2(\Omega_t)), \\
u' &\in L^\infty(0,T_0,H^1(\Omega_t)), \\
u'' &\in L^2(0,T_0,L^2(\Omega_t)),
\end{aligned}
\tag{2.6}
$$

and u is the solution of (1.3).

First of all we prove the equivalent result in a cylindrical domain.

Theorem 2.2. *Let $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ be as in (H1)–(H4) and let M satisfy*

(i) $M \in L^\infty_{\text{loc}}([0,\infty),W^{1,\infty}(Q))$ and for each $L > 0$, $\partial M/\partial\lambda \in L^\infty(Q \times (0,L))$,

(ii) $M,\partial M/\partial\lambda$, $\partial M/\partial t$ are continuous with respect to λ, a.e. $(y,t) \in Q$.

There exists a real number $m_0 > 0$ such that

$$M(y,t,\lambda) \geq m_0, \quad \forall(y,t) \in Q,\ \lambda \geq 0. \tag{2.7}$$

Given $v_0 \in H_0^1(0,1) \cap H^2(0,1)$, $v_1 \in H_0^1(0,1)$, and $g \in L^2([0,T],H_0^1(0,1))$ with $g' \in L^2([0,T],L^2(0,1))$, there exist $T_0 > 0$ and a unique v satisfying

$$v \in L^\infty(0,T_0,H_0^1(0,1) \cap H^2(0,1)), \tag{2.8a}$$

$$v' \in L^\infty(0,T_0,H_0^1(0,1)), \tag{2.8b}$$

$$v'' \in L^\infty(0,T_0,L^2(0,1)), \tag{2.8c}$$

and v is the solution of (2.5).

3. Proof of results

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be the orthonormal complete set of $L^2(0,1)$ given by the eigenvectors of the operator $-d^2/dx^2$, that is,

$$-\frac{d^2}{dx^2}w_\nu = \lambda_\nu w_\nu, \quad w_\nu = 0 \quad \text{on } \Gamma. \tag{3.1}$$

We represent by $V_m = [w_1,\dots,w_m]$ the subspace of $H_0^1(0,1)\cap H^2(0,1)$ generated by the m first vectors w_ν.

We consider $F:(0,\infty)\to\mathbb{R}$ a function satisfying

$$F\in C^1(0,\infty), \qquad F'(\xi)<0, \quad \forall\xi>0, \qquad F(\xi)=1, \quad \forall\xi\geq 1. \tag{3.2}$$

$$\text{There exist } \mu_0,\ \eta_0,\ \delta>0 \text{ such that } F(\xi)\geq\frac{\mu_0}{\xi^{\eta_0}}, \quad \forall\xi\in(0,\delta]. \tag{3.3}$$

Let $K>0$ such that

$$|v_1|^2<K. \tag{3.4}$$

For each $\varepsilon>0$, we consider the following penalized problem: for each $m\in\mathbb{N}$, let $v_{\varepsilon m}(t)\in V_m$ satisfy

$$\begin{aligned}&\left(v''_{\varepsilon m}(t)-\left[\frac{1}{\gamma^2(t)}M\left(y,t,\gamma(t)|v_{\varepsilon m}(t)|^2\right)-\frac{b^2(y,t)}{4}\right]\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t),w\right)\\&\quad+\left(c(y,t)\frac{\partial v_{\varepsilon m}}{\partial y}(t)+b(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(t)+\varepsilon F\left(\frac{K-|v'_{\varepsilon m}(t)|^2}{\varepsilon}\right)v'_{\varepsilon m}(t),w\right)=\left(g(t),w\right),\end{aligned} \tag{3.5}$$

for all $w\in V_m$, with the initial conditions

$$v_{\varepsilon m}(0)=v_{0m}\longrightarrow v_0 \quad \text{in } H_0^1(0,1)\cap H^2(0,1), \tag{3.6a}$$

$$v'_{\varepsilon m}(0)=v_{1m}\longrightarrow v_1 \quad \text{in } H_0^1(0,1). \tag{3.6b}$$

As in [9] we prove that we can apply Carathéodory's theorem to (3.5)–(3.6b) obtaining the existence of $T_{\varepsilon m}>0$ and $v_{\varepsilon m}:[0,T_{\varepsilon m})\to V_{\varepsilon m}$ as a solution of (3.5) satisfying

$$v_{\varepsilon m}\in C^2([0,T_{\varepsilon m})),\ |v'_{\varepsilon m}(t)|^2<K, \quad \forall m\geq m_1,\ t\in[0,T_{\varepsilon m}),\ \text{for each } \varepsilon>0. \tag{3.7}$$

LEMMA 3.1. *There exist constants $C_0 > 0$ and $\varepsilon_0 > 0$ such that*

$$|v''_{\varepsilon m}(0)| \leq C_0, \tag{3.8}$$

for all $m \geq m_1$, $0 < \varepsilon < \varepsilon_0$.

Proof. From (3.6) the existence of a constant $N > 0$ follows, such that

$$\|v'_{\varepsilon m}(0)\| \leq N, \quad \|v_{\varepsilon m}(0)\| \leq N, \quad |v_{\varepsilon m}(0)| \leq N, \quad |\triangle v_{\varepsilon m}(0)| \leq N, \quad \forall m \geq m_1. \tag{3.9}$$

From (3.4) there exists $\rho > 0$ such that

$$|v_1|^2 < \rho < K, \tag{3.10}$$

so

$$|v_{1m}|^2 < \rho < K, \quad \forall m \geq m_1. \tag{3.11}$$

Then, for each $0 < \varepsilon < \min\{1, K - \rho\} = \varepsilon_0$, we have

$$\frac{K - |v_{1m}|^2}{\varepsilon} > 1, \quad \forall m \geq m_1, \tag{3.12}$$

thus, from (3.2),

$$F\left(\frac{K - |v_{1m}|^2}{\varepsilon}\right) = 1, \quad \forall m \geq m_1. \tag{3.13}$$

Let

$$M_0 = \max_{[0,1]\times[0,\gamma_0 N^2]} |M(y,0,\lambda)|. \tag{3.14}$$

We supposed that $\alpha'(0) = \beta'(0) = 0$, then it follows from (H4) that $b(y,0) = c(y,0) = 0$, so by considering $t = 0$ and $w = v''_{\varepsilon m}(0)$ in (3.5), we obtain, from (3.13),

$$|v''_{\varepsilon m}(0)| \leq \frac{1}{\gamma_0^2}[M_0 + \gamma_0^2]N + |g(0)|, \tag{3.15}$$

and the proof of the lemma is complete. □

Lemma 3.2. *Fix* $\varepsilon > 0$ *and* $m \geq m_1$. *Then*

(i) *there exists* $\lim_{t\to T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2$ *such that* $\lim_{t\to T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2 < K$.

Proof. Taking the derivate with respect to t of (3.5) we obtain

$$\begin{aligned}\Bigg(v'''_{\varepsilon m}(t) + \bigg[\frac{2\gamma'(t)}{\gamma^3(t)} M\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big) - \frac{1}{\gamma^2(t)} M'\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big) \\ - \frac{\gamma'(t)}{\gamma^2(t)}\frac{\partial M}{\partial\lambda}\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big)\,|\,v_{\varepsilon m}(t)\,|^2 - \frac{2}{\gamma(t)}\frac{\partial M}{\partial\lambda}\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big) \\ \times (v_{\varepsilon m}(t), v'_{\varepsilon m}(t)) + \frac{1}{4}(b^2(y,t))'\bigg]\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t) \\ + \bigg[\frac{b^2(y,t)}{4} - \frac{1}{\gamma^2(t)} M\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big)\bigg]\frac{\partial^2 v'_{\varepsilon m}}{\partial y^2}(t) + c'(y,t)\frac{\partial v_{\varepsilon m}}{\partial y}(t) \\ + [c(y,t) + (b(y,t))']\frac{\partial v'_{\varepsilon m}}{\partial y}(t) + b(y,t)\frac{\partial v''_{\varepsilon m}}{\partial y}(t) - 2F'\left(\frac{K - |\,v'_{\varepsilon m}(t)\,|^2}{\varepsilon}\right) \\ \times (v''_{\varepsilon m}(t),\, v'_{\varepsilon m}(t))\, v'_{\varepsilon m}(t) + \varepsilon F\left(\frac{K - |\,v'_{\varepsilon m}(t)\,|^2}{\varepsilon}\right) v''_{\varepsilon m}(t), w\Bigg) = (g'(t),\, w).\end{aligned} \tag{3.16}$$

From (3.7) we obtain that

$$|\,v_{\varepsilon m}(t)\,|^2 < K_{\varepsilon m}, \quad \forall t \in [0, T_{\varepsilon m}). \tag{3.17}$$

Let

$$C_{1\varepsilon m} = \operatorname*{sup\,ess}_{[0,1]\times[0,T]\times[0,\gamma(T)K_{\varepsilon m}]} \left\{ |\,M(y,t,\lambda)\,|,\ |\,M'(y,t,\lambda)\,|,\ \left|\frac{\partial M}{\partial\lambda}(y,t,\lambda)\right| \right\}. \tag{3.18}$$

We remember that

$$\begin{aligned}|\,\alpha'(t) + \gamma'(t)y\,| &\leq \gamma'(t), \\ |\,\alpha''(t) + \gamma''(t)y\,| &\leq \frac{|\,\alpha'(t) + \gamma'(t)y\,|^2}{\gamma(t)}, \quad t \in [0,T],\ y \in [0,1],\end{aligned} \tag{3.19}$$

so we obtain from (H3)

$$\begin{aligned}
&\left|\frac{1}{4}(b^2(y,t))'\right| \le 4\left(\frac{\gamma'(t)}{\gamma(t)}\right)^3 \le \frac{m_0\sqrt{2m_0}}{\gamma_0^3},\\
&\left|2\frac{\gamma'(t)}{\gamma^3(t)}M\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big)\right| \le \frac{\sqrt{2m_0}}{\gamma_0^3}C_{1\varepsilon m},\\
&\left|\frac{1}{\gamma^2(t)}M\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big)\right| \le \frac{1}{\gamma_0^2}C_{1\varepsilon m},\\
&\left|\frac{1}{\gamma^2(t)}\frac{\partial M}{\partial\lambda}\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big)\Big[\gamma'(t)\,|\,v_{\varepsilon m}(t)\,|^2+2\gamma(t)(v_{\varepsilon m}(t),\,v'_{\varepsilon m}(t))\Big]\right|\\
&\qquad\le K_{\varepsilon m}C_{1\varepsilon m}\left[\frac{1}{\gamma_0^2}\sqrt{\frac{m_0}{2}}+\frac{2}{\gamma_0}K\right],\\
&\left|\frac{b^2(y,t)}{4}-\frac{1}{\gamma^2(t)}M\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big)\right| \le \left(\frac{\gamma'(t)}{\gamma(t)}\right)^2+\frac{1}{\gamma^2(t)}C_{1\varepsilon m} \le \frac{1}{\gamma_0^2}\left[\frac{m_0}{2}+C_{1\varepsilon m}\right],\\
&|\,c(y,t)+b'(y,t)\,| \le \frac{7m_0}{2\gamma_0^2}.
\end{aligned}\tag{3.20}$$

Taking $w = v''_{\varepsilon m}(t)$ in (3.16) it follows from (3.20) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\,|\,v''_{\varepsilon m}(t)\,|^2+\varepsilon F\left(\frac{K-|\,v'_{\varepsilon m}(t)\,|^2}{\varepsilon}\right)|\,v''_{\varepsilon m}(t)\,|^2-2F'\left(\frac{K-|\,v'_{\varepsilon m}(t)\,|^2}{\varepsilon}\right)(v''_{\varepsilon m}(t),\,v'_{\varepsilon m}(t))^2\\
&\quad\le |\,v''_{\varepsilon m}(t)\,|\left\{\left[C_{1\varepsilon m}\left(\frac{\sqrt{2m_0}}{\gamma_0^3}+\frac{1}{\gamma_0^2}+\frac{1}{\gamma_0^2}\sqrt{\frac{m_0}{2}}K_{\varepsilon m}+\frac{2}{\gamma_0}K_{\varepsilon m}K\right)+\frac{m_0\sqrt{2m_0}}{\gamma_0^3}\right]\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|\right.\\
&\qquad\left.+\left[\frac{m_0}{2\gamma_0^2}+\frac{1}{\gamma_0^2}C_{1\varepsilon m}\right]\left|\frac{\partial^2 v'_{\varepsilon m}}{\partial y^2}(t)\right|+C_2||v_{\varepsilon m}(t)||+\left(\frac{3m_0}{2\gamma_0^2}+\frac{2m_0}{\gamma_0^2}\right)||v'_{\varepsilon m}(t)||\right\}\\
&\quad+|\,g'(t)\,|+\frac{\sqrt{2m_0}}{\gamma_0}\,|\,v''_{\varepsilon m}(t)\,|\,||v''_{\varepsilon m}(t)||,
\end{aligned}\tag{3.21}$$

where $C_2 = \max_{[0,1]\times[0,T]}|c'(y,t)|$. But in V_m the norms are equivalent, so by using (3.7) and (3.17) we obtain that there exist constants $C_{2\varepsilon m} > 0$ and $C_{3\varepsilon m} > 0$ depending on ε and m such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\,|\,v''_{\varepsilon m}(t)\,|^2+\varepsilon F\left(\frac{K-|\,v'_{\varepsilon m}(t)\,|^2}{\varepsilon}\right)|\,v''_{\varepsilon m}(t)\,|^2\\
&\qquad-2F'\left(\frac{K-|\,v'_{\varepsilon m}(t)\,|^2}{\varepsilon}\right)(v''_{\varepsilon m}(t),\,v'_{\varepsilon m}(t))^2 \le C_{2\varepsilon m}\Big[\,|\,v''_{\varepsilon m}(t)\,|^2+|\,g'(t)\,|^2\Big]+C_{3\varepsilon m}.
\end{aligned}\tag{3.22}$$

Using the hypothesis on F and F', Lemma 3.1, and Gronwall's inequality in (3.22) it follows that

$$|v''_{\varepsilon m}(t)|^2 \le C_{4\varepsilon m}, \tag{3.23}$$

where $C_{4\varepsilon m} > 0$ is a constant depending on $\varepsilon > 0$ and m.

It follows from (3.7) and (3.23) that $\lim_{t\to T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2$ exists and

$$\lim_{t\to T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2 \le K. \tag{3.24}$$

Suppose that

$$\lim_{t\to T_{\varepsilon m}} |v'_{\varepsilon m}(t)|^2 = K. \tag{3.25}$$

Taking $w = v'_{\varepsilon m}(t)$ in (3.5), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v'_{\varepsilon m}(t)|^2 + \varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon}\right)|v'_{\varepsilon m}(t)|^2 \\
&\quad + \int_0^1 \left[\frac{b^2(y,t)}{4} - \frac{1}{\gamma^2(t)}M\left(y,t,\gamma(t)|v_{\varepsilon m}(t)|^2\right)\right]\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)v'_{\varepsilon m}(y,t)dy \\
&\quad + \int_0^1 c(y,t)\frac{\partial v_{\varepsilon m}}{\partial y}(y,t)v'_{\varepsilon m}(y,t)dy + \int_0^1 b(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)v'_{\varepsilon m}(y,t)dy \\
&= \int_0^1 g(y,t)v'_{\varepsilon m}(y,t)dy,
\end{aligned} \tag{3.26}$$

but

$$\begin{aligned}
\left|\frac{b^2(y,t)}{4} - \frac{1}{\gamma^2(t)}M\left(y,t,\gamma(t)|v_{\varepsilon m}(t)|^2\right)\right| &\le \frac{1}{\gamma_0^2}\left[\frac{m_0}{2} + C_{1\varepsilon m}\right], \\
|c(y,t)| &\le \frac{3}{\gamma_0^2}\frac{m_0}{2}, \\
|b(y,t)| &\le \frac{\sqrt{2m_0}}{\gamma_0},
\end{aligned} \tag{3.27}$$

so we have

$$\begin{aligned}
&\varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon}\right)|v'_{\varepsilon m}(t)|^2 \\
&\quad \le |v''_{\varepsilon m}(t)||v'_{\varepsilon m}(t)| + \frac{1}{\gamma_0^2}\left[\frac{m_0}{2} + C_{1\varepsilon m}\right]\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\right||v'_{\varepsilon m}(y,t)| \\
&\qquad + \frac{3}{\gamma_0^2}\frac{m_0}{2}\|v_{\varepsilon m}(t)\||v'_{\varepsilon m}(t)| + \frac{\sqrt{2m_0}}{\gamma_0}\|v'_{\varepsilon m}(t)\||v'_{\varepsilon m}(t)| + |g(t)||v'_{\varepsilon m}(t)|,
\end{aligned} \tag{3.28}$$

then from (3.23), (3.17), and (3.7) and the equivalence of the norms in V_m there exists a constant $C_{5\varepsilon m} > 0$ such that

$$\varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon}\right) v'_{\varepsilon m}(t)|^2 \le C_{5\varepsilon m}. \tag{3.29}$$

We conclude from (3.25) and (3.29) that there exists $\delta_1 > 0$ such that

$$\varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon}\right) \le C_{5\varepsilon m}\sqrt{\frac{2}{K}}, \quad \text{for } t \in (T_{\varepsilon m} - \delta_1, T_{\varepsilon m}). \tag{3.30}$$

But by (3.25) and the hypothesis on F we have

$$\lim_{t\to T_{\varepsilon m}} \varepsilon F\left(\frac{K - |v'_{\varepsilon m}(t)|^2}{\varepsilon}\right) = \infty, \tag{3.31}$$

which is a contradiction to (3.30). Then the proof is completed. □

From (3.7) and Lemma 3.2(i) we can extend the solution to $[0,T]$ and

$$|v_{\varepsilon m}(t)|^2 \le K_1 \text{ e } |v'_{\varepsilon m}(t)|^2 \le K, \quad \forall t \in [0,T], \tag{3.32}$$

where K and K_1 are independent of $\varepsilon > 0$ and $m \ge m_1$.

Lemma 3.3. *There exists a constant $K_2 > 0$ independent of $\varepsilon > 0$ and $m \ge m_1$ such that*

$$||v'_{\varepsilon m}(t)||^2 + \left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|^2 \le K_2, \quad \forall t \in [0,T]. \tag{3.33}$$

Proof. We observe that

$$\begin{aligned}
\int_0^1 c(y,t)\frac{\partial v_{\varepsilon m}(y,t)}{\partial y}\frac{\partial^2 v'_{\varepsilon m}}{\partial y^2}(y,t)dy &= c(1,t)\frac{\partial v_{\varepsilon m}}{\partial y}(1,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(1,t) \\
&\quad -c(0,t)\frac{\partial v_{\varepsilon m}}{\partial y}(0,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(0,t) - \int_0^1 c(y,t)\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)dy \\
&\quad -\int_0^1 \frac{\partial c}{\partial y}(y,t)\frac{\partial v_{\varepsilon m}}{\partial y}(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)dy, \\
\int_0^1 b(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)\frac{\partial^2 v'_{\varepsilon m}}{\partial y^2}(y,t)dy &= \frac{1}{2}\int_0^1 b(y,t)\frac{\partial}{\partial y}\left(\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)\right)^2 dy \\
&= \frac{1}{2}b(1,t)\left(\frac{\partial v'_{\varepsilon m}}{\partial y}(1,t)\right)^2 - \frac{1}{2}b(0,t)\left(\frac{\partial v'_{\varepsilon m}}{\partial y}(0,t)\right)^2 \\
&\quad -\frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}(y,t)\left(\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)\right)^2 dy,
\end{aligned} \tag{3.34}$$

so if we take $w = -(\partial^2 v'_{\varepsilon m}/\partial y^2)(y,t)$ in (3.5) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{|v'_{\varepsilon m}(t)|^2+\int_0^1\left[\frac{1}{\gamma^2(t)}M\Big(y,t,\gamma(t)|v_{\varepsilon m}(t)|^2\Big)-\frac{b^2(y,t)}{4}\right]\left(\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\right)^2dy\right\}\\
&\quad+\frac{1}{2}\int_0^1\frac{d}{dt}\left[\frac{b^2(y,t)}{4}-\frac{1}{\gamma^2(t)}M\Big(y,t,\gamma(t)|v_{\varepsilon m}(t)|^2\Big)\right]\left(\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\right)^2dy\\
&\quad+c(0,t)\frac{\partial v_{\varepsilon m}}{\partial y}(0,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(0,t)-c(1,t)\frac{\partial v_{\varepsilon m}}{\partial y}(1,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(1,t)\\
&\quad+\int_0^1 c(y,t)\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)dy+\int_0^1\frac{\partial c}{\partial y}(y,t)\frac{\partial v_{\varepsilon m}}{\partial y}(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)dy\\
&\quad+\frac{1}{2}b(0,t)\left(\frac{\partial v'_{\varepsilon m}}{\partial y}(0,t)\right)^2-\frac{1}{2}b(1,t)\left(\frac{\partial v'_{\varepsilon m}}{\partial y}(1,t)\right)^2+\frac{1}{2}\int_0^1\frac{\partial b}{\partial y}(y,t)\left(\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)\right)^2dy\\
&\leq \|g(t)\|\,\|v'_{\varepsilon m}(t)\|.
\end{aligned}\tag{3.35}$$

It follows from (H3) and (H4) that

$$\begin{aligned}
&\left|c(1,t)\frac{\partial v_{\varepsilon m}}{\partial y}(1,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(1,t)\right|\\
&\qquad\leq\left[2\frac{\gamma'(t)\beta'(t)}{\gamma^2(t)}+\left(\frac{\beta'(t)}{\gamma(t)}\right)^2\right]\left|\frac{\partial v_{\varepsilon m}}{\partial y}(1,t)\right|\left|\frac{\partial v'_{\varepsilon m}}{\partial y}(1,t)\right|\\
&\qquad\leq\frac{\beta'(t)}{\gamma(t)}\left|\frac{\partial v'_{\varepsilon m}}{\partial y}(1,t)\right|^2+\frac{4(\gamma'(t))^2\beta'(t)+(\beta'(t))^3}{2\gamma^3(t)}\left|\frac{\partial v_{\varepsilon m}}{\partial y}(1,t)\right|^2\\
&\qquad\leq\frac{\beta'(t)}{\gamma(t)}\left|\frac{\partial v'_{\varepsilon m}}{\partial y}(1,t)\right|^2+\frac{5}{2\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left(\frac{\pi+1}{\pi}\right)^2\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|^2,\\
&\left|c(0,t)\frac{\partial v_{\varepsilon m}}{\partial y}(0,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(0,t)\right|\\
&\qquad\leq\frac{-\alpha'(t)}{\gamma(t)}\left|\frac{\partial v'_{\varepsilon m}}{\partial y}(0,t)\right|^2+\frac{5}{2\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left(\frac{\pi+1}{\pi}\right)^2\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|^2,\\
&\int_0^1\frac{d}{dt}\left[\frac{1}{\gamma^2(t)}M\Big(y,t,\gamma(t)|v_{\varepsilon m}(t)|^2\Big)-\frac{b^2(y,t)}{4}\right]\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\right|^2dy\\
&\qquad=\int_0^1\left\{-\frac{2\gamma'(t)}{\gamma^3(t)}M\Big(y,t,\gamma(t)|v_{\varepsilon m}(t)|^2\Big)+\frac{1}{\gamma^2(t)}\frac{\partial M}{\partial t}\Big(y,t,\gamma(t)|v_{\varepsilon m}(t)|^2\Big)\right.\\
&\qquad\quad+\frac{1}{\gamma^2(t)}\frac{\partial M}{\partial\lambda}\Big(y,t,\gamma(t)|v_{\varepsilon m}(t)|^2\Big)\Big[\gamma'(t)|v_{\varepsilon m}(t)|^2+2\gamma(t)(v_{\varepsilon m}(t),\,v'_{\varepsilon m}(t))\Big]\\
&\qquad\quad+\frac{2}{\gamma^3(t)}(\alpha'(t)+\gamma'(t)y)\big[(\alpha''(t)+\gamma''(t)y)\gamma(t)-(\alpha'(t)+\gamma'(t)y)\gamma'(t)\big]\\
&\qquad\quad\left.\times\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\right|^2dy\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\{M_1\left[\frac{2}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{1/2}+\frac{1}{\gamma_0^2}\left(\frac{m_0}{2}\right)^{1/2}K_1+\frac{2}{\gamma_0}KK_1\right]+\frac{4}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\right\}\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|^2,\\
&\int_0^1 \frac{\partial c}{\partial y}(y,t)\frac{\partial v_{\varepsilon m}}{\partial y}(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)dy \leq \frac{2}{\gamma_0^2}m_0\|v_{\varepsilon m}(t)\|\,\|v'_{\varepsilon m}(t)\|,\\
&\int_0^1 \frac{\partial b}{\partial y}(y,t)\left(\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)\right)^2 dy \leq \frac{2}{\gamma_0}\left(\frac{m_0}{2}\right)^{1/2}\|v'_{\varepsilon m}(t)\|^2,\\
&\int_0^1 c(y,t)\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)dy \leq \frac{3}{\gamma_0^2}\frac{m_0}{2}\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|\|v'_{\varepsilon m}(t)\|.
\end{aligned}
\tag{3.36}
$$

By substituting the above inequalities in (3.35) we obtain

$$
\begin{aligned}
&\frac{1}{2}\left\{\frac{d}{dt}\|v'_{\varepsilon m}(t)\|^2+\int_0^1\left[\frac{1}{\gamma^2(t)}M\Big(y,t,\gamma(t)\,|\,v_{\varepsilon m}(t)\,|^2\Big)-\frac{b^2(y,t)}{4}\right]\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\right|^2 dy\right\}\\
&\leq \left[M_1\left(\frac{\sqrt{2m_0}}{\gamma_0^3}+\frac{1}{\gamma_0^2}+\frac{1}{\gamma_0^2}\sqrt{\frac{m_0}{2}}K_1+\frac{2}{\gamma_0}\sqrt{K_1K}\right)+\frac{m_0\sqrt{2m_0}}{\gamma_0^3}\right]\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|^2\\
&\quad+\frac{3}{\gamma_0^2}\frac{m_0}{2}\|v'_{\varepsilon m}(t)\|\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|+\frac{2m_0}{\gamma_0^2}\|v_{\varepsilon m}(t)\|\,\|v'_{\varepsilon m}(t)\|+\frac{2}{\gamma_0}\sqrt{\frac{m_0}{2}}\|v'_{\varepsilon m}(t)\|^2\\
&\quad+\frac{5}{\gamma_0^3}\left(\frac{m_0}{2}\right)^{3/2}\left(\frac{\pi+1}{\pi}\right)^2\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|^2+\|g(t)\|\,\|v'_{\varepsilon m}(t)\|\\
&\leq C_3\left[\|v'_{\varepsilon m}(t)\|^2+\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(t)\right|^2\right]+\frac{1}{2}\|g(t)\|^2,
\end{aligned}
\tag{3.37}
$$

where

$$
M_1=\max \operatorname*{sup\,ess}_{[0,T]\times[0,1]\times[0,\gamma_0K_1]}\left\{M(y,t,\lambda),\ |M'(t,y,\lambda)|,\ \left|\frac{\partial}{\partial y}M(y,t,\lambda)\right|,\ \left|\frac{\partial}{\partial\lambda}M(y,t,\lambda)\right|\right\}.
\tag{3.38}
$$

Integrating inequality (3.37) from 0 to t, the result follows from the convergence of the initial data, Gronwall's lemma, (H6), and (2.7). □

Lemma 3.4. *There exists a constant $K_3>0$ independent of $\varepsilon>0$ and $m\geq m_1$ such that*

$$
|\,v''_{\varepsilon m}(t)\,|\leq K_3,\quad \forall t\in[0,T].
\tag{3.39}
$$

Proof. Considering $w=v''_{\varepsilon m}(t)$ in (3.16), using Lemmas 3.2 and 3.3, and by the same arguments used in the proof of Lemma 3.2, it follows that there exists a constant $C_4>0$

independent of $\varepsilon > 0$ and $m \geq m_1$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|v''_{\varepsilon m}(t)\right|^2 + \varepsilon F\left(\frac{K-\left|v'_{\varepsilon m}(t)\right|^2}{\varepsilon}\right)\left|v''_{\varepsilon m}(t)\right|^2 - 2F'\left(\frac{K-\left|v'_{\varepsilon m}(t)\right|^2}{\varepsilon}\right)\left(v''_{\varepsilon m}(t), v'_{\varepsilon m}(t)\right)^2 \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1\left[\frac{1}{\gamma^2(t)}M\left(y,t,\gamma(t)\left|v_{\varepsilon m}(t)\right|^2\right) - \frac{b^2(y,t)}{4}\right]\left|\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)\right|^2 dy \\
&\leq C_4 + \frac{1}{2}\left|v''_{\varepsilon m}(t)\right|^2 + \frac{1}{2}\left|g'(t)\right|^2.
\end{aligned} \tag{3.40}$$

The result follows from this inequality, the convergence of the initial data, and from Gronwall's lemma. □

3.1. Proof of Theorem 2.2. We observe that

(i) if $|v'_{\varepsilon m}(t)|^2 \leq K/2$, then $(K - |v'_{\varepsilon m}(t)|^2)/\varepsilon \geq K/2\varepsilon$, so from the hypothesis on F we have

$$F\left(\frac{K-\left|v'_{\varepsilon m}(t)\right|^2}{\varepsilon}\right) = 1, \quad \text{for } \varepsilon > 0 \text{ such that } 2\varepsilon < K; \tag{3.41}$$

(ii) if $|v'_{\varepsilon m}(t)|^2 > K/2$, taking $w = v'_{\varepsilon m}(t)$ in (3.5), we have

$$\begin{aligned}
&\varepsilon F\left(\frac{K-\left|v'_{\varepsilon m}(t)\right|^2}{\varepsilon}\right) \\
&\quad \leq \frac{2}{K}\left\{\left|v''_{\varepsilon m}(t)\right|\left|v'_{\varepsilon m}(t)\right| + \int_0^1\left[\frac{1}{\gamma^2(t)}M\left(y,t,\gamma(t)\left|v_{\varepsilon m}(t)\right|^2\right) - \frac{b^2(y,t)}{4}\right]\left|\frac{\partial^2 v_{\varepsilon m}}{\partial y^2}(y,t)\right|\right. \\
&\qquad \times \left|v'_{\varepsilon m}(t)\right| dy + \int_0^1\left|c(y,t)\frac{\partial v_{\varepsilon m}}{\partial y}(y,t)v'_{\varepsilon m}(y,t)\right| dy \\
&\qquad \left. + \int_0^1\left|b(y,t)\frac{\partial v'_{\varepsilon m}}{\partial y}(y,t)v'_{\varepsilon m}(y,t)\right| dy + \int_0^1\left|g(y,t)v'_{\varepsilon m}(y,t)\right| dy\right\}.
\end{aligned} \tag{3.42}$$

Using the estimates obtained in the above inequality, there exists a constant $K_4 > 0$ such that

$$\varepsilon F\left(\frac{K-\left|v'_{\varepsilon m}(t)\right|^2}{\varepsilon}\right) \leq K_4, \quad \text{for } \varepsilon > 0. \tag{3.43}$$

From the above inequalities, there exists a constant $K_5 > 0$ independent of $\varepsilon > 0$ and $m \geq m_1$ such that

$$\varepsilon F\left(\frac{K-\left|v'_{\varepsilon m}(t)\right|^2}{\varepsilon}\right) \leq K_4, \quad \forall t \in [0,T],\ m \geq m_1,\ 0 < \varepsilon < \frac{K}{2}. \tag{3.44}$$

From the lemmas and (3.44) it follows that there exists a subsequence of $(v_{\varepsilon m})$, still denoted by $(v_{\varepsilon m})$, such that

$$v_{\varepsilon m} \longrightarrow v, \quad \text{weakly* in } L^\infty(0,T,H_0^1(0,1)\cap H^2(0,1)), \tag{3.45a}$$

$$v'_{\varepsilon m} \longrightarrow v', \quad \text{weakly* in } L^\infty(0,T,H_0^1(0,1)), \tag{3.45b}$$

$$v''_{\varepsilon m} \longrightarrow v'', \quad \text{weakly* in } L^\infty(0,T,L^2(0,1)), \tag{3.45c}$$

$$\varepsilon F\left(\frac{K-|v'_{\varepsilon m}(t)|^2}{\varepsilon}\right) \longrightarrow \chi, \quad \text{weakly* in } L^\infty(0,T). \tag{3.45d}$$

From (3.45a)–(3.45c) we conclude that $(v_{\varepsilon m}(t))$ and $(v'_{\varepsilon m}(t))$ are equicontinuous with respect to the norms on $H_0^1(0,1)$ and $L^2(0,1)$, respectively. So there exists a subsequence of $(v_{\varepsilon m})$, still denoted by $(v_{\varepsilon m})$, such that

$$v_{\varepsilon m} \longrightarrow v, \quad \text{strongly in } H_0^1(0,1), \text{ uniformly in } [0,T], \tag{3.46}$$

$$v'_{\varepsilon m} \longrightarrow v', \quad \text{strongly in } L^2(0,1), \text{ uniformly in } [0,T]. \tag{3.47}$$

But $M \in L^\infty_{\text{loc}}([0,\infty), W^{1,\infty}(Q))$ and for each $L>0$, $(\partial M/\partial\lambda) \in L^\infty(Q\times(0,L))$, so

$$M\left(t,\, \gamma(t)\,|v_{\varepsilon m}(t)|^2\right) \longrightarrow M\left(t,\, \gamma(t)\,|v(t)|^2\right) \quad \text{strongly in } L^2(0,1), \text{ uniformly in } [0,T]. \tag{3.48}$$

Passing to the limit, when $m\to\infty$ and $\varepsilon\to 0$, in (3.5), we obtain

$$\begin{gathered} v''(t)+\left[-\frac{1}{\gamma^2(t)}M\left(y,t,\gamma(t)\,|v(t)|^2\right)+\frac{b^2(y,t)}{4}\right]\frac{\partial^2 v}{\partial y^2}(t)+c(y,t)\frac{\partial v}{\partial y}(t) \\ +b(y,t)\frac{\partial v'}{\partial y}(t)+\chi(t)v'(t)=g(t) \quad \text{in } L^2(0,T,L^2(0,1)), \end{gathered} \tag{3.49}$$

and $v(0)=v_0$ and $v'(0)=v_1$.

We have $v'\in C^0([0,T],L^2(0,1))$ and $|v'(0)|^2=|v_1|^2<K$, so there exists $T_0>0$ such that

$$|v'(t)|^2<K, \quad \forall t\in[0,T_0]. \tag{3.50}$$

By considering $\beta_0=\max_{[0,T_0]}|v'(t)|^2$ and $\rho_0=K-\beta_0>0$, we have

$$|v'(t)|^2\le\beta_0=K-\rho_0<K-\frac{\rho_0}{2}, \quad \forall t\in[0,T_0]. \tag{3.51}$$

From (3.47) there exists $m_2\in\mathbb{N}$ such that

$$|v'_{\varepsilon m}(t)|^2<\frac{\rho_0}{4}+|v'(t)|^2<K-\frac{\rho_0}{4}, \quad \forall t\in[0,T_0]. \tag{3.52}$$

Then, for $0<\varepsilon<(\rho_0/4)$, we have

$$\frac{K-|v'_{\varepsilon m}(t)|^2}{\varepsilon}\ge 1, \quad m\ge m_2,\ t\in[0,T_0], \tag{3.53}$$

so

$$\varepsilon F\left(\frac{K-|v'_{\varepsilon m}(t)|^2}{\varepsilon}\right)=\varepsilon, \quad t\in[0,T_0]. \tag{3.54}$$

Thus by the uniqueness of the limit it follows that $\chi=0$ in $[0,T_0]$.

For the uniqueness we consider v and $\widehat{v}$ solutions in $[0,T_0]$, and $w=v-\widehat{v}$. So we have

$$\begin{aligned} w''(t)&+\left[-\frac{1}{\gamma^2(t)}M\big(y,t,\gamma(t)|v(t)|^2\big)+\frac{b^2(y,t)}{4}\right]\frac{\partial^2 w}{\partial y^2}(t)\\ &+\left[\frac{1}{\gamma^2(t)}M\big(y,t,\gamma(t)|\widehat{v}(t)|^2\big)-\frac{1}{\gamma^2(t)}M\big(y,t,\gamma(t)|v(t)|^2\big)\right]\frac{\partial^2\widehat{v}}{\partial y^2}(t)\\ &+c(y,t)\frac{\partial w}{\partial y}(t)+b(y,t)\frac{\partial w'}{\partial y}(t)=0 \quad \text{in } L^2(0,T,L^2(0,1)). \end{aligned} \tag{3.55}$$

By defining

$$\Psi(t)=\frac{1}{2}|w'(t)|^2+\frac{1}{2}\int_0^1\left[\frac{1}{\gamma^2(t)}M\big(y,t,\gamma(t)|v(t)|^2\big)-\frac{b^2(y,t)}{4}\right]\left(\frac{\partial w}{\partial y}(t)\right)^2dy \tag{3.56}$$

and multiplying the equation in (3.55) by w', we obtain

$$\Psi'(t)-C\Psi(t)\le 0, \tag{3.57}$$

for some constant $C>0$, and since $\Psi(0)=0$ we conclude that $w=0$ in $[0,T_0]$.

3.2. Proof of Theorem 2.1. In this section, we study the noncylindrical problem (1.3). We represent by Ω_t and Ω_0 the intervals $(\alpha(t),\beta(t))$ and (α_0,β_0), respectively, and we denote by $|\cdot|_0$, $\|\cdot\|_0$, $|\cdot|_t$, $\|\cdot\|_t$ the norms in $L^2(\Omega_0)$, $H_0^1(\Omega_0)$, $L^2(\Omega_t)$, $H_0^1(\Omega_t)$, respectively.

We consider the change of variables $x=\alpha(t)+\gamma(t)y$, and we define

$$\begin{gathered} M(y,t,\lambda)=\widehat{M}(\alpha(t)+\gamma(t)y,t,\lambda),\\ v_0(y)=u_0(\alpha_0+\gamma_0 y)\ \text{e}\ v_1(y)=u_1(\alpha_0+\gamma_0 y),\\ g(y,t)=f(\alpha(t)+\gamma(t)y,t). \end{gathered} \tag{3.58}$$

We can see that we are in the conditions of Theorem 2.2. It follows that there exist $T_0>0$ and a unique solution v of problem (2.5) satisfying conditions (2.8). By considering $u(x,t)=v(y,t)$ with $x=\alpha(t)+\gamma(t)y$ we verify that u is the solution of Theorem 2.1.

The regularity and uniqueness of u claimed in Theorem 2.1 are obtained by the regularity and uniqueness of the solution v of Theorem 2.2.

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