WEIGHTED INEQUALITIES FOR THE SAWYER TWO-DIMENSIONAL HARDY OPERATOR AND ITS LIMITING GEOMETRIC MEAN OPERATOR

ANNA WEDESTIG

Received 3 November 2003

We consider $Tf = \int_0^{x_1}\int_0^{x_2} f(t_1,t_2)dt_1dt_2$ and a corresponding geometric mean operator $Gf = \exp(1/x_1x_2)\int_0^{x_1}\int_0^{x_2}\log f(t_1,t_2)dt_1dt_2$. E. T. Sawyer showed that the Hardy-type inequality $\|Tf\|_{L_u^q} \le C\|f\|_{L_v^p}$ could be characterized by three independent conditions on the weights. We give a simple proof of the fact that if the weight v is of product type, then in fact only one condition is needed. Moreover, by using this information and by performing a limiting procedure we can derive a weight characterization of the corresponding two-dimensional Pólya-Knopp inequality with the geometric mean operator G involved.

1. Introduction

The following remarkable result was proved by Sawyer in [3, Theorem 1].

Theorem 1.1. *Let $1 < p \le q < \infty$ and let u and v be weight functions on $\mathbb{R}_+^2$. Then*

$$\begin{aligned}\left(\int_0^\infty\int_0^\infty\left(\int_0^{x_1}\int_0^{x_2} f(t_1,t_2)dt_1dt_2\right)^q u(x_1,x_2)dx_1dx_2\right)^{1/q}&\\ \le C\left(\int_0^\infty\int_0^\infty f(x_1,x_2)^p v(x_1,x_2)dx_1dx_2\right)^{1/p}&\end{aligned} \tag{1.1}$$

holds for all positive and measurable functions f on $\mathbb{R}_+^2$ if and only if

$$\sup_{y_1,y_2>0}\left(\int_{y_1}^\infty\int_{y_2}^\infty u(x_1,x_2)dx_1dx_2\right)^{1/q}\left(\int_0^{y_1}\int_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{1/p'} = A_1 < \infty, \tag{1.2}$$

$$\sup_{y_1,y_2>0}\frac{\left(\int_0^{y_1}\int_0^{y_2}\left(\int_0^{x_1}\int_0^{x_2} v(t_1,t_2)^{1-p'}dt_1dt_2\right)^q u(x_1,x_2)dx_1dx_2\right)^{1/q}}{\left(\int_0^{y_1}\int_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{1/p}} = A_2 < \infty, \tag{1.3}$$

$$\sup_{y_1,y_2>0}\frac{\left(\int_{y_1}^\infty\int_{y_2}^\infty\left(\int_{x_1}^\infty\int_{x_2}^\infty u(t_1,t_2)dt_1dt_2\right)^{p'} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{1/p'}}{\left(\int_{y_1}^\infty\int_{y_2}^\infty u(x_1,x_2)dx_1dx_2\right)^{1/q'}} = A_3 < \infty. \tag{1.4}$$

Journal of Inequalities and Applications 2005:4 (2005) 387–394
DOI: 10.1155/JIA.2005.387

However in [4] it was proved that to characterize the two-dimensional Pólya-Knopp inequality

$$\begin{aligned}\left(\int_0^\infty\int_0^\infty\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log f(t_1,t_2)\,dt_1dt_2\right)\right]^q u(x_1,x_2)\,dx_1dx_2\right)^{1/q}\\ \leq C\left(\int_0^\infty\int_0^\infty f^p(x_1,x_2)v(x_1,x_2)\,dx_1dx_2\right)^{1/p}\end{aligned}\tag{1.5}$$

for $0 < p \leq q < \infty$, only one condition was needed. An interesting observation is that this inequality can be characterized by just using *one* integral condition even if the inequality seems to be a natural limiting inequality of the Sawyer result mentioned above.

The aim of this paper is to find a two-dimensional weight characterization that allow us to perform a limiting procedure (as in [2, 4]), and receive a weight characterization of the corresponding two-dimensional Pólya-Knopp inequality (1.5). From the corresponding result in one dimension (see [2, 4]), we know that this requires special homogeneity properties of the conditions that for instance the condition (1.2) doesn't have. On the other hand the fact that (1.5) is equivalent to a one-weighted Pólya-Knopp inequality makes it possible for us to use an Hardy inequality where we allow one weight to be of product type and thus characterize the Hardy inequality with only one condition and with the special homogeneity properties (see Section 2). In Section 3 we will also show that with that condition and the corresponding estimates of the best constant we will, by performing a limiting procedure (as in [2, 4]), receive exactly the same condition and estimate of the best constant C for the weighted two dimensional Pólya-Knopp inequality (1.5) as in [4].

2. A two-dimensional Hardy-type inequality

Our main result reads.

Theorem 2.1. *Let $1 < p \leq q < \infty$, $s_1,s_2 \in (1,p)$, let u be a weight function on $\mathbb{R}_+^2$ and let v_1 and v_2 be weight functions on $\mathbb{R}_+$. Then the inequality*

$$\begin{aligned}\left(\int_0^\infty\int_0^\infty\left(\int_0^{x_1}\int_0^{x_2}f(t_1,t_2)\,dt_1dt_2\right)^q u(x_1,x_2)\,dx_1dx_2\right)^{1/q}\\ \leq C\left(\int_0^\infty\int_0^\infty f^p(x_1,x_2)v_1(x_1)v_2(x_2)\,dx_1dx_2\right)^{1/p}\end{aligned}\tag{2.1}$$

holds for all measurable functions $f \geq 0$ if and only if

$$\begin{aligned}A_W(s_1,s_2) &= \sup_{t_1,t_2>0} V_1(t_1)^{(s_1-1)/p}V_2(t_2)^{(s_2-1)/p}\\ &\quad\times\left(\int_{t_1}^\infty\int_{t_2}^\infty u(x_1,x_2)V_1(x_1)^{q((p-s_1)/p)}V_2(x_2)^{q((p-s_2)/p)}\,dx_1dx_2\right)^{1/q} < \infty,\end{aligned}\tag{2.2}$$

where $V_1(t_1) = \int_0^{t_1} v_1(x_1)^{1-p'}dx_1$ and $V_2(t_2) = \int_0^{t_2} v_2(x_2)^{1-p'}dx_2$.

Moreover, if C is the best possible constant in (2.1), then

$$\sup_{1<s_1,s_2<p}\left(\frac{(p/(p-s_1))^p}{(p/(p-s_1))^p+1/(s_1-1)}\right)^{1/p}\left(\frac{(p/(p-s_2))^p}{(p/(p-s_2))^p+1/(s_2-1)}\right)^{1/p}A_W(s_1,s_2)$$
$$\leq C\leq \inf_{1<s_1,s_2<p}A_W(s_1,s_2)\left(\frac{p-1}{p-s_1}\right)^{1/p'}\left(\frac{p-1}{p-s_2}\right)^{1/p'}. \tag{2.3}$$

For the proof of Theorem 2.1 we need the following Minkowski inequality (see [4]).

Lemma 2.2. *Let $r>1$, $-\infty\leq a_1<b_1\leq\infty$, $-\infty\leq a_2<b_2\leq\infty$ and let Φ and Ψ be positive measurable functions on $[a_1,b_1]\times[a_2,b_2]$. Then*

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Phi(x_1,x_2)\left(\int_{a_1}^{x_1}\int_{a_2}^{x_2}\Psi(t_1,t_2)dt_1dt_2\right)^r dx_1dx_2\right)^{1/r}\\ &\qquad\leq \int_{a_1}^{b_1}\int_{a_1}^{b_2}\Psi(t_1,t_2)\left(\int_{t_1}^{b_1}\int_{t_1}^{b_2}\Phi(x_1,x_2)dx_1d_2\right)^{1/r}dt_1dt_2.\end{aligned} \tag{2.4}$$

Proof of Theorem 2.1. Let $f^p(x_1,x_2)v_1(x)v_2(x_2)=g(x_1,x_2)$ in (2.1). Then (2.1) is equivalent to the inequality

$$\begin{aligned}&\left(\int_0^\infty\int_0^\infty\left(\int_0^{x_1}\int_0^{x_1}g(t_1,t_2)^{1/p}v_1(t_1)^{-1/p}v_2(t_2)^{-1/p}dt_1dt_2\right)^q u(x_1,x_2)dx_1dx_2\right)^{1/q}\\ &\qquad\leq C\left(\int_0^\infty\int_0^\infty g(x_1,x_2)dx_1dx_2\right)^{1/p}.\end{aligned} \tag{2.5}$$

Assume that (2.2) holds. By applying Hölder's inequality, the fact that $(d/dt_1)V_1(t_1)=v_1(t_1)^{1-p'}=v_1(t_1)^{-p'/p}$, $(d/dt_2)V_2(t_2)=v_2(t_2)^{1-p'}=v_2(t_2)^{-p'/p}$ and Lemma 2.2 we have

$$\begin{aligned}&\left(\int_0^\infty\int_0^\infty\left(\int_0^{x_1}\int_0^{x_2}g(t_1,t_2)^{1/p}v_1(t_1)^{-1/p}v_2(t_2)^{-1/p}dt_1dt_2\right)^q u(x_1,x_2)dx_1dx_2\right)^{1/q}\\
&=\left(\int_0^\infty\int_0^\infty\left(\int_0^{x_1}\int_0^{x_2}g(t_1,t_2)^{1/p}V_1(t_1)^{(s_1-1)/p}V_2(t_2)^{(s_2-1)/p}V_1(t_1)^{-(s_1-1)/p}v_1(t_1)^{-1/p}\right.\right.\\
&\qquad\left.\left.\times V_2(t_2)^{-(s_2-1)/p}v_2(t_2)^{-1/p}dt_1dt_2\right)^q u(x_1,x_2)dx_1dx_2\right)^{1/q}\\
&\leq\left(\int_0^\infty\int_0^\infty\left(\int_0^{x_1}\int_0^{x_2}g(t_1,t_2)V_1(t_1)^{s_1-1}V_2(t_2)^{s_2-1}dt_1dt_2\right)^{q/p}\right.\\
&\quad\times\left(\int_0^{x_1}V_1(t_1)^{-(s_1-1)p'/p}v_1(t_1)^{-p'/p}dt_1\right)^{q/p'}\\
&\quad\left.\times\left(\int_0^{x_2}V_2(t_2)^{-(s_2-1)p'/p}v_2(t_2)^{-p'/p}dt_2\right)^{q/p'}u(x_1,x_2)dx_1dx_2\right)^{1/q}\\
&=\left(\frac{p-1}{p-s_1}\right)^{1/p'}\left(\frac{p-1}{p-s_2}\right)^{1/p'}\left(\int_0^\infty\int_0^\infty\left(\int_0^{x_1}\int_0^{x_2}g(t_1,t_2)V_1(t_1)^{s_1-1}V_2(t_2)^{s_2-1}dt_1dt_2\right)^{q/p}\right.\\
&\qquad\left.\times V_1(x_1)^{q((p-s_1)/p)}V_2(x_2)^{q((p-s_2)/p)}u(x_1,x_2)dx_1dx_2\right)^{1/q}\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{p-1}{p-s_1}\right)^{1/p'}\left(\frac{p-1}{p-s_2}\right)^{1/p'}\left(\int_0^\infty\int_0^\infty g(t_1,t_2)V_1(t_1)^{s_1-1}V_2(t_2)^{s_2-1}\right.\\
&\qquad\left.\times\left(\int_{t_1}^\infty\int_{t_2}^\infty V_1(x_1)^{q((p-s_1)/p)}V_2(x_2)^{q((p-s_2)/p)}u(x_1,x_2)\,dx_1dx_2\right)^{p/q}dt_1dt_2\right)^{1/p}\\
&\leq \left(\frac{p-1}{p-s_1}\right)^{1/p'}\left(\frac{p-1}{p-s_2}\right)^{1/p'}A_W(s_1,s_2)\left(\int_0^\infty\int_0^\infty g(t_1,t_2)\,dt_1dt_2\right)^{1/p}.
\end{aligned}
\tag{2.6}
$$

Hence (2.5) and, thus, (2.1) holds with a constant satisfying the right-hand side inequality in (2.3).

Now we assume that (2.1) and, thus, (2.5) holds and choose the test function

$$
\begin{aligned}
g(x_1,x_2) &= \left(\frac{p}{p-s_1}\right)^p\left(\frac{p}{p-s_2}\right)^p\\
&\quad\times V_1(t_1)^{-s_1}v_1(x_1)^{1-p'}V_2(t_2)^{-s_2}v_2(x_2)^{1-p'}\chi_{(0,t_1)}(x_1)\chi_{(0,t_2)}(x_2)\\
&\quad+\left(\frac{p}{p-s_1}\right)^p V_1(t_1)^{-s_1}v_1(x_1)^{1-p'}V_2(x_2)^{-s_2}v_2(x_2)^{1-p'}\chi_{(0,t_1)}(x_1)\chi_{(t_2,\infty)}(x_2)\\
&\quad+\left(\frac{p}{p-s_2}\right)^p V_1(x_1)^{-s_1}v_1(x_1)^{1-p'}V_2(t_2)^{-s_2}v_2(x_2)^{1-p'}\chi_{(t_1,\infty)}(x_1)\chi_{(0,t_2)}(x_2)\\
&\quad+V_1(x_1)^{-s_1}v_1(x_1)^{1-p'}V_2(x_2)^{-s_2}v_2(x_2)^{1-p'}\chi_{(t_1,\infty)}(x_1)\chi_{(t_2,\infty)}(x_2),
\end{aligned}
\tag{2.7}
$$

where t_1, t_2 are fixed numbers >0. Then the integral on right-hand side of (2.5) can be estimated as follows:

$$
\begin{aligned}
&\left(\int_0^\infty\int_0^\infty g(x_1,x_2)\,dx_1dx_2\right)^{1/p}\\
&\quad=\left(\int_0^{t_1}\left(\frac{p}{p-s_1}\right)^p V_1(t_1)^{-s_1}v_1(x_1)^{1-p'}dx_1\int_0^{t_2}\left(\frac{p}{p-s_2}\right)^p V_2(t_2)^{-s_2}v_2(x_2)^{1-p'}dx_2\right.\\
&\qquad+\int_0^{t_1}\left(\frac{p}{p-s_1}\right)^p V_1(t_1)^{-s_1}v_1(x_1)^{1-p'}dx_1\int_{t_2}^\infty V_2(x_2)^{-s_{21}}v_2(x_2)^{1-p'}dx_2\\
&\qquad+\int_{t_1}^\infty V_1(x_1)^{-s_1}v_1(x_1)^{1-p'}dx_1\int_0^{t_2}\left(\frac{p}{p-s_2}\right)^p V_2(t_2)^{-s_2}v_2(x_2)^{1-p'}dx_2\\
&\qquad\left.+\int_{t_1}^\infty V_1(x_1)^{-s_1}v_1(x_1)^{1-p'}dx_1\int_{t_2}^\infty V_2(x_2)^{-s_2}v_2(x_2)^{1-p'}dx_2\right)^{1/p}\\
&\quad\leq\left(\left(\frac{p}{p-s_1}\right)^p+\frac{1}{s_1-1}\right)^{1/p}\left(\left(\frac{p}{p-s_2}\right)^p+\frac{1}{s_2-1}\right)^{1/p}V_1(t_1)^{(1-s_1)/p}V_2(t_2)^{(1-s_2)/p}.
\end{aligned}
\tag{2.8}
$$

Moreover, the left-hand side of (2.5) is greater than

$$
\begin{aligned}
&\bigg(\int_{t_1}^{\infty}\int_{t_2}^{\infty}\bigg[\bigg(\int_0^{t_1}\frac{p}{p-s_1}V_1(t_1)^{-s_1/p}v_1(y_1)^{1-p'}dy_1\bigg)\bigg(\int_0^{t_2}\frac{p}{p-s_2}V_2(t_2)^{-s_2/p}v_2(y_2)^{1-p'}dy_2\bigg)\\
&\qquad+\bigg(\int_0^{t_1}\frac{p}{p-s_1}V_1(t_1)^{-s_1/p}v_1(y_1)^{1-p'}dy_1\bigg)\bigg(\int_{t_2}^{x_2}V_2(y_1)^{-s_2/p}v_2(y_2)^{1-p'}dy_2\bigg)\\
&\qquad+\bigg(\int_{t_1}^{x_1}V_1(y_1)^{-s_1/p}v_1(y_1)^{1-p'}dy_1\bigg)\bigg(\int_0^{t_2}\frac{p}{p-s_2}V_2(t_2)^{-s_2/p}v_2(y_2)^{1-p'}dy_2\bigg)\\
&\qquad+\bigg(\int_{t_1}^{x_1}V_1(y_1)^{-s_1/p}v_1(y_1)^{1-p'}dy_1\bigg)\\
&\qquad\times\bigg(\int_{t_2}^{x_2}V_2(y_1)^{-s_2/p}v_2(y_2)^{1-p'}dy_2\bigg)\bigg]^q u(x_1,x_2)\,dx_1dx_2\bigg)^{1/q}\\
&=\cdots\\
&=\frac{p}{p-s_1}\frac{p}{p-s_2}\bigg(\int_{t_1}^{\infty}\int_{t_2}^{\infty}u(x_1,x_2)V_1(x_1)^{q((p-s_1)/p)}V_2(x_2)^{q((p-s_2)/p)}dx_1dx_2\bigg)^{1/q}.
\end{aligned}
\tag{2.9}
$$

Hence, (2.5) implies that

$$
\begin{aligned}
&\frac{p}{p-s_1}\frac{p}{p-s_2}\bigg(\int_{t_1}^{\infty}\int_{t_2}^{\infty}u(x_1,x_2)V_1(x_1)^{q((p-s_1)/p)}V_2(x_2)^{q((p-s_2)/p)}dx_1dx_2\bigg)^{1/q}\\
&\qquad\le C\bigg(\Big(\frac{p}{p-s_1}\Big)^p+\frac{1}{s_1-1}\bigg)^{1/p}\bigg(\Big(\frac{p}{p-s_2}\Big)^p+\frac{1}{s_2-1}\bigg)^{1/p}V_1(t_1)^{(1-s_1)/p}V_2(t_2)^{(1-s_2)/p},
\end{aligned}
\tag{2.10}
$$

that is, that

$$
\begin{aligned}
&\bigg(\frac{(p/(p-s_1))^p}{(p/(p-s_1))^p+1/(s_1-1)}\bigg)^{1/p}\bigg(\frac{(p/(p-s_2))^p}{(p/(p-s_2))^p+1/(s_2-1)}\bigg)^{1/p}V_1(t_1)^{(s_1-1)/p}V_2(t_2)^{(s_2-1)/p}\\
&\quad\times\bigg(\int_{t_1}^{\infty}\int_{t_2}^{\infty}u(x_1,x_2)V_1(x_1)^{q((p-s_1)/p)}V(x_2)^{q((p-s_2)/p)}dx_1dx_2\bigg)^{1/q}\le C.
\end{aligned}
\tag{2.11}
$$

We conclude that (2.2) and the left-hand side of the estimate of (2.3) hold. The proof is complete. □

3. A two-dimensional Pólya-Knopp inequality

Here, we will give another proof of two-dimensional Pólya-Knopp inequality (1.5) proved in [4] by proving that this theorem is just the natural limit result of our theorem (Theorem 2.1).

Theorem 3.1 [4]. *The inequality (1.5) holds for all positive and measurable functions on $\mathbb{R}_+^2$ if and only if*

$$D_W(s_1,s_2) := \sup_{\substack{y_1>0\\ y_2>0}} y_1^{(s_1-1)/p} y_2^{(s_2-1)/p} \left(\int_{y_1}^{\infty} \int_{y_2}^{\infty} x_1^{-s_1 q/p} x_2^{-s_2 q/p} w(x_1,x_2)\, dx_1 dx_2 \right)^{1/q} < \infty, \tag{3.1}$$

where $s_1,s_2 > 1$ and

$$w(x_1,x_2) = \left[\exp\left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \log \frac{1}{v(t_1,t_2)}\, dt_1 dt_2 \right) \right]^{q/p} u(x_1,x_2) \tag{3.2}$$

and the best possible constant C in (1.5) can be estimated in the following way:

$$\begin{aligned} &\sup_{s_1,s_2>1} \left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1} \right)^{1/p} \left(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1} \right)^{1/p} D_W(s_1,s_2) \\ &\quad \le C \le \inf_{s_1,s_2>1} e^{(s_1+s_2-2)/p} D_W(s_1,s_2). \end{aligned} \tag{3.3}$$

Remark 3.2. For the case $p = q = 1$, a similar result was recently proved by Heinig, Kerman and Krbec [1] but without the estimates of the operator norm (= the best constant C in (1.5)) pointed out in (3.3) here.

Proof of Theorem 3.1. If we in the inequality (1.5) replace $f^p(x_1,x_2)v(x_1,x_2)$ with $f^p(x_1,x_2)$ and let $w(x_1,x_2)$ be defined as in (3.2), then (1.5) is equivalent to

$$\begin{aligned} &\left(\int_0^{\infty} \int_0^{\infty} \left[\exp\left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \log f(y_1,y_2)\, dy_1 dy_2 \right) \right]^q w(x_1,x_2)\, dx_1 dx_2 \right)^{1/q} \\ &\quad \le C \left(\int_0^{\infty} \int_0^{\infty} f^p(x_1,x_2)\, dx_1 dx_2 \right)^{1/p}. \end{aligned} \tag{3.4}$$

Further, by using Theorem 2.1 with the special weights $u(x_1,x_2) = w(x_1,x_2)x_1^{-q}x_2^{-q}$ and $v_1(x_1) = v_2(x_2) = 1$ we have that

$$\begin{aligned} &\left(\int_0^{\infty} \int_0^{\infty} \left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} f(t_1,t_2)\, dt_1 dt_2 \right)^q w(x_1,x_2)\, dx_1 dx_2 \right)^{1/q} \\ &\quad \le C \left(\int_0^{\infty} \int_0^{\infty} f^p(x_1,x_2)\, dx_1 dx_2 \right)^{1/p} \end{aligned} \tag{3.5}$$

holds for all $f \ge 0$ if and only if

$$A_W(s_1,s_2) = \sup_{t_1,t_2>0} t_1^{(s_1-1)/p} t_2^{(s_2-1)/p} \left(\int_{t_1}^{\infty} \int_{t_2}^{\infty} w(x_1,x_2) x_1^{-s_1 q/p} x_2^{-s_2 q/p}\, dx_1 dx_2 \right)^{1/q} < \infty, \tag{3.6}$$

where $s_1,s_2 \in (1,p)$. We note that $A_W(s_1,s_2)$ coincides with the constant $D_W(s_1,s_2) = D_W(s_1,s_2,q,p)$ defined by (3.1). Moreover, if C is the best possible constant in (3.5), then

$$\begin{aligned}\sup_{1<s_1,s_2<p}&\left(\frac{(p/(p-s_1))^p}{(p/(p-s_1))^p+1/(s_1-1)}\right)^{1/p}\left(\frac{(p/(p-s_2))^p}{(p/(p-s_2))^p+1/(s_2-1)}\right)^{1/p}D_W(s_1,s_2)\\ &\le C \le \inf_{1<s_1,s_2<p} D_W(s_1,s_2)\left(\frac{p-1}{p-s_1}\right)^{1/p'}\left(\frac{p-1}{p-s_2}\right)^{1/p'}.\end{aligned}\tag{3.7}$$

Now, if we replace f in (3.5) with f^α, $0<\alpha<p$ and after that replace p with p/α and q with q/α in (3.5), (3.6), and (3.7), then we find that, for $1<s_1,s_2<p/\alpha$,

$$\begin{aligned}&\left(\int_0^\infty\int_0^\infty\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2} f^\alpha(t_1,t_2)\,dt_1\,dt_2\right)^{q/\alpha} w(x_1,x_2)\,dx_1\,dx_2\right)^{1/q}\\ &\qquad\le C_\alpha\left(\int_0^\infty\int_0^\infty f^p(x_1,x_2)\,dx_1\,dx_2\right)^{1/p}\end{aligned}\tag{3.8}$$

holds for all $f \ge 0$ if and only if $D_W(s_1,s_2,q/\alpha,p/\alpha) = D_W^\alpha(s_1,s_2,q,p) < \infty$. Moreover, if C_α is the best possible constant in (3.8), then

$$\begin{aligned}\sup_{1<s_1,s_2<p/\alpha}&\left(\frac{(p/(p-\alpha s_1))^{p/\alpha}}{(p/(p-\alpha s_1))^p+1/(s_1-1)}\right)^{1/p}\left(\frac{(p/(p-\alpha s_2))^{p/\alpha}}{(p/(p-\alpha s_2))^p+1/(s_2-1)}\right)^{1/p}D_W^\alpha(s_1,s_2,q,p)\\ &\le C_\alpha \le \inf_{1<s_1,s_2<p/\alpha} D_W^\alpha(s_1,s_2,q,p)\left(\frac{p-\alpha}{p-\alpha s_1}\right)^{(p-\alpha)/\alpha p}\left(\frac{p-\alpha}{p-\alpha s_2}\right)^{(p-\alpha)/\alpha p}.\end{aligned}\tag{3.9}$$

We also note that

$$\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2} f^\alpha(t_1,t_2)\,dt_1\,dt_2\right)^{1/\alpha} \downarrow \exp\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\ln f(t_1,t_2)\,dt_1\,dt_2, \quad \text{as } \alpha \longrightarrow 0_+.\tag{3.10}$$

We conclude that (3.1) holds exactly when $\limsup_{\alpha\to 0_+} C_\alpha < \infty$ and this holds, according to (3.9), exactly when (3.6) holds. Moreover, when $\alpha \to 0_+$ (3.9) implies that the upper estimate in (3.3) holds. For the lower estimate we apply the following testfunction (c.f. [4]): For fixed t_1 and t_2, $t_1,t_2>0$, let

$$\begin{aligned}g(x_1,x_2) = g_0(x_1,x_2) &= t_1^{-1}t_2^{-1}\chi_{(0,t_1)}(x_1)\chi_{(0,t_2)}(x_2)\\ &\quad+ t_1^{-1}\chi_{(0,t_1)}(x_1)\frac{e^{-s_2}t_2^{s_2-1}}{x_2^{s_2}}\chi_{(t_2,\infty)}(x_2)\\ &\quad+\frac{e^{-s_1}t_1^{s_1-1}}{x_1^{s_1}}\chi_{(t_1,\infty)}(x_1)t_2^{-1}\chi_{(0,t_2)}(x_2)\\ &\quad+\frac{e^{-(s_1+s_2)}t_1^{s_1-1}t_2^{s_2-1}}{x_1^{s_1}x_2^{s_2}}\chi_{(t_1,\infty)}(x_1)\chi_{(t_2,\infty)}(x_2).\end{aligned}\tag{3.11}$$

The proof is complete. □

Remark 3.3. This proof shows that the Pólya-Knopp inequality (1.5) characterized in Theorem 3.1 may be regarded as a natural limiting inequality of the (Sawyer type) Hardy inequality characterized in Theorem 2.1.

Acknowledgment

I thank Professor L. E. Persson for some valuable suggestions which have improved the final version of this paper.

References

[1] H. P. Heinig, R. Kerman, and M. Krbec, *Weighted exponential inequalities*, Georgian Math. J. **8** (2001), no. 1, 69–86.

[2] L.-E. Persson and V. D. Stepanov, *Weighted integral inequalities with the geometric mean operator*, J. Inequal. Appl. **7** (2002), no. 5, 727–746, An abbreviated version can also be found in Russian Akad. Sci. Dokl. Math. 63 (2001), 201-202.

[3] E. Sawyer, *Weighted inequalities for the two-dimensional Hardy operator*, Studia Math. **82** (1985), no. 1, 1–16.

[4] A. Wedestig, *Some new Hardy type inequalities and their limiting inequalities*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 3, 1–15, Article 61.

Anna Wedestig: Department of Mathematics, Luleå University, 97 187 Luleå, Sweden
E-mail address: annaw@sm.luth.se