ON SOME INEQUALITIES FOR BETA AND GAMMA FUNCTIONS VIA SOME CLASSICAL INEQUALITIES

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We improve several results recently established by Dragomir et al. in (2000) for the Gamma and Beta functions. All we need is some clever applications of classical inequalities.

1. Introduction

Recently, in the survey paper [6] various inequalities for Beta and Gamma functions obtained from some classical inequalities are given. The most common way in which the Gamma function is defined is the following integral representation:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0. \tag{1.1}$$

The integral in (1.1) is uniformly convergent for all $a \le x \le b$, where $0 < a \le b < \infty$, so we also have

$$\Gamma^{(k)}(x) = \int_0^\infty e^{-t} t^{x-1} (\log t)^k dt, \quad x > 0. \tag{1.2}$$

Various well-known formulas for Gamma function are also given in [6]. For example,

$$\Gamma(x) = s^x \int_0^\infty e^{-st} t^{x-1} dt, \quad x, s > 0. \tag{1.3}$$

The Beta function is given by

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x > 0,\ y > 0, \tag{1.4}$$

and its connection to Gamma function is also well known:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.5}$$

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Among others know formulas for Beta function given in [6] is the following one:

$$B(x+1,y)+B(x,y+1)=B(x,y), \quad x,y>0. \tag{1.6}$$

Let us note that (1.6) is a special case of the following formula:

$$\sum_{k=0}^{n}\binom{n}{k}B(x+k,y+n-k)=B(x,y), \quad x,y>0. \tag{1.7}$$

Indeed, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}B(x+k,y+n-k) &= \sum_{k=0}^{n}\binom{n}{k}\int_0^1 t^{x+k-1}(1-t)^{y+n-k-1}dt \\ &= \int_0^1 \sum_{k=1}^{n}\binom{n}{k} t^{x+k-1}(1-t)^{y+n-k-1}dt \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\left[\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}\right]dt \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}dt = B(x,y).\end{aligned} \tag{1.8}$$

For example, the following inequalities are obtained in [6]. If $p,q>1$, $x\in[0,1]$, then

$$\begin{aligned}&\left|B(p,q)-x^{p-1}(1-x)^{q-1}\right| \\ &\quad\le \max\{p-1,q-1\}\frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^2\right],\end{aligned} \tag{1.9}$$

$$\begin{aligned}&\left|B(p,q)-x^{p-1}(1-x)^{q-1}\right| \\ &\quad\le \max\{p-1,q-1\}B(p-1,q-1)\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^2\right].\end{aligned} \tag{1.10}$$

If $s>1$, $p,q>2-1/s>1$, $1/s+1/r=1$, then

$$\begin{aligned}&\left|B(p,q)-x^{p-1}(1-x)^{q-1}\right| \\ &\quad\le \frac{1}{(r+1)^{1/r}}\left[x^{r+1}+(1-x)^{r+1}\right]^{1/r}\max\{p-1,q-1\}\left[B(s(p-2)+1,\ s(q-2)+1)\right]^{1/s}.\end{aligned} \tag{1.11}$$

In this paper, we will give some improvements and generalizations of these and some other results from [6].

2. Inequalities via Chebyshev's inequality

The following result is well known in the literature as Chebyshev's integral inequality for synchronous (asynchronous) mappings (see, e.g., [16, pages 239–293] or [17, pages 197–208]).

Lemma 2.1. *Let $f,g,h : I \subset \mathbf{R} \to \mathbf{R}$ be such that $h(x) \geq 0$ for $x \in I$ and h, hfg, hf and hg are integrable on I. If f and g are synchronous (asynchronous) on I, that is, if it holds*

$$(f(x) - f(y))(g(x) - g(y)) \geq (\leq) 0 \quad \forall x, y \in I, \tag{2.1}$$

then we have the inequality

$$\int_I h(x)dx \int_I h(x)f(x)g(x)dx \geq (\leq) \int_I h(x)f(x)dx \int_I h(x)g(x)dx. \tag{2.2}$$

Theorem 2.2. *Let m, p, and k be real numbers with $m, p > 0$ and $p > k > -m$, and let n be a nonnegative integer,*

$$k(p - m - k) \geq (\leq) 0, \tag{2.3}$$

then we have

$$\Gamma^{(2n)}(p)\Gamma^{(2n)}(m) \geq (\leq)\Gamma^{(2n)}(m+k). \tag{2.4}$$

Proof. Consider the mappings $f,g,h : [0,\infty) \to [0,\infty)$ given by

$$f(x) = x^{p-k-m}, \qquad g(x) = x^k, \qquad h(x) = x^{m-1}e^{-x}(\log x)^{2n}. \tag{2.5}$$

If the condition (2.3) holds, then functions f and g are synchronous (asynchronous) on $(0,\infty)$ and then, by Chebyshev's inequality for $I = (0,\infty)$, we have

$$\begin{aligned} &\int_0^\infty x^{m-1}e^{-x}(\log x)^{2n}dx \int_0^\infty x^{p-k-m}x^k x^{m-1}e^{-x}(\log x)^{2n}dx \\ &\quad \geq (\leq) \int_0^\infty x^{p-k-m}x^{m-1}e^{-x}(\log x)^{2n}dx \int_0^\infty x^k x^{m-1}e^{-x}(\log x)^{2n}dx, \end{aligned} \tag{2.6}$$

that is,

$$\begin{aligned} &\int_0^\infty x^{m-1}e^{-x}(\log x)^{2n}dx \int_0^\infty x^{p-1}e^{-x}(\log x)^{2n}dx \\ &\quad \geq (\leq) \int_0^\infty x^{p-k-1}e^{-x}(\log x)^{2n}dx \int_0^\infty x^{k+m-1}e^{-x}(\log x)^{2n}dx. \end{aligned} \tag{2.7}$$

Hence, (2.4) follows from the integral representation (1.2). □

Remark 2.3. For $n = 0$ ($\Gamma^{(0)} = \Gamma$) the following result follows from [6]:

$$\Gamma(p)\Gamma(m) \geq (\leq)\Gamma(p-k)\Gamma(m+k) \tag{2.8}$$

or, in equivalent form

$$B(p,m) \geq (\leq) B(p-k, m+k). \tag{2.9}$$

Corollary 2.4. *Let $p > 0$ and $q \in \mathbf{R}$ be such that $|q| < p$, and let n be a nonnegative integer. Then*

$$\left[\Gamma^{(2n)}(p)\right]^2 \leq \Gamma^{(2n)}(p-q)\Gamma^{(2n)}(p+q). \tag{2.10}$$

Proof. Choose in Theorem 2.2 $m = p$ and $k = q$. Then

$$k(p - m - k) = -q^2 \leq 0 \tag{2.11}$$

and by (2.4), we get

$$\left[\Gamma^{(2n)}(p)\right]^2 \leq \Gamma^{(2n)}(p-q)\Gamma^{(2n)}(p+q). \tag{2.12}$$

□

Remark 2.5. For $n = 0$ we have inequality from [6]:

$$\Gamma^2(p) \leq \Gamma(p-q)\Gamma(p+q) \tag{2.13}$$

or, equivalently

$$B(p,p) \leq B(p-q, p+q). \tag{2.14}$$

For $m = 2$, $p = a+b$, $k = b-1$, the condition (2.3) becomes

$$(a-1)(b-1) \geq (\leq) 0, \tag{2.15}$$

that is the positive real numbers a and b are similarly (oppositely) unitary (see [6, Definition 1]), and (2.4) becomes

$$\Gamma^{(2n)}(2)\Gamma^{(2n)}(a+b) \geq (\leq) \Gamma^{(2n)}(a+1)\Gamma^{(2n)}(b+1), \tag{2.16}$$

wherefrom, for $n = 0$, we have the following inequality from [6]:

$$\Gamma(a+b) \geq (\leq) \Gamma(a)\Gamma(b) \tag{2.17}$$

or, equivalently

$$B(a,b) \geq (\leq) \frac{1}{ab}. \tag{2.18}$$

As a consequence of (2.10) it was proved in [6] that the mapping $\log\Gamma(x)$ is superadditive for $x > 1$, and the following inequality holds:

$$\Gamma(na) \geq (n-1)!a^{2(n-1)}\left[\Gamma(a)\right]^n \quad (n \in \mathbf{N},\ a > 0). \tag{2.19}$$

For a given real $m > 0$ and nonnegative integer n, consider the mapping $\Gamma_{m,n}(x) = \Gamma^{(2n)}(x+m)/\Gamma^{(2n)}(m)$.

Corollary 2.6. *The mapping $\Gamma_{m,n}(\cdot)$ is supermultiplicative on $[0,\infty)$.*

Proof. For $p = x+y+m$, $k = y$, the condition (2.3) becomes

$$y(x+y+m-m-y) = xy \geq 0 \tag{2.20}$$

since $x, y \in [0,\infty)$. Hence, (2.4) becomes

$$\Gamma^{(2n)}(m)\Gamma^{(2n)}(x+y+m) \geq \Gamma^{(2n)}(x+m)\Gamma^{(2n)}(y+m) \tag{2.21}$$

which is equivalent to

$$\Gamma_{m,n}(x+y) \geq \Gamma_{m,n}(x)\Gamma_{m,n}(y) \tag{2.22}$$

and the corollary is proved. □

3. An inequality via Hölder inequality

The following inequality is a generalization of [6, Theorem 5].

THEOREM 3.1. *Let $a,b \geq 0$ with $a+b=1$ and $x,y>0$ be real numbers and let n be a nonnegative integer. Then*

$$\Gamma^{(2n)}(ax+by) \leq \left[\Gamma^{(2n)}(x)\right]^a\left[\Gamma^{(2n)}(y)\right]^b, \tag{3.1}$$

that is, the mapping $\Gamma^{(2n)}$ is logarithmically convex on $(0,\infty)$.

Proof. We use the following weighted version of Hölder inequality:

$$\left|\int_I f(s)g(s)h(s)ds\right| \leq \left(\int_I |f(s)|^p h(s)ds\right)^{1/p}\left(\int_I |g(s)|^q h(s)ds\right)^{1/q} \tag{3.2}$$

for $p>1$, $1/p+1/q=1$, nonnegative h on I, provided that the other integrals exist and are finite. Choose

$$f(s)=s^{a(x-1)}, \quad g(s)=s^{b(y-1)}, \quad h(s)=e^{-s}(\log s)^{2n}, \quad s\in(0,\infty) \tag{3.3}$$

in (3.2), to get (for $I=(0,\infty)$ and $p=1/a$, $q=1/b$)

$$\begin{aligned}\int_0^\infty & s^{a(x-1)}s^{b(y-1)}e^{-s}(\log s)^{2n}ds\\ &\leq \left(\int_0^\infty s^{a(x-1)\cdot(1/a)}e^{-s}(\log s)^{2n}ds\right)^a\left(\int_0^\infty s^{b(y-1)\cdot(1/b)}e^{-s}(\log s)^{2n}ds\right)^b\end{aligned} \tag{3.4}$$

which is equivalent to

$$\begin{aligned}\int_0^\infty & s^{ax+by-1}e^{-s}(\log s)^{2n}ds\\ &\leq \left(\int_0^\infty s^{x-1}e^{-s}(\log s)^{2n}ds\right)^a\left(\int_0^\infty s^{y-1}e^{-s}(\log s)^{2n}ds\right)^b\end{aligned} \tag{3.5}$$

and the inequality (3.1) is proved. □

4. Inequalities via Grüss' inequality

Let us note that the following interpolation of Grüss' inequality is well known [16, page 295–310].

LEMMA 4.1. *Let f, g and h be integrable functions defined on $[a,b]$ such that*

$$\varphi \le f(x) \le \Phi, \quad \gamma \le g(x) \le \Gamma \quad \forall x \in [a,b], \tag{4.1}$$

where φ, Φ, γ, and Γ are given constants, and h is nonnegative. Then

$$|D(f,g;h)| \le D(f,f;h)^{1/2} D(g,g;h)^{1/2} \le \frac{1}{2}(\Phi-\varphi)(\Gamma-\gamma)\left[\int_a^b h(t)dt\right]^2, \tag{4.2}$$

where

$$D(f,g;h) = \int_a^b h(t)dt \int_a^b h(t)f(t)g(t)dt - \int_a^b h(t)f(t)dt \int_a^b h(t)g(t)dt. \tag{4.3}$$

THEOREM 4.2. *Let $p,q,\alpha,\beta > 0$. Then we have*

$$\begin{aligned} &|B(\alpha,\beta)B(\alpha+p,\beta+q) - B(\alpha+p,\beta)B(\alpha,\beta+q)| \\ &\quad \le \left[B(\alpha,\beta)B(\alpha+2p,\beta) - B^2(\alpha+p,\beta)\right]^{1/2}\left[B(\alpha,\beta)B(\alpha,\beta+2q) - B^2(\alpha,\beta+q)\right]^{1/2} \\ &\quad \le \frac{1}{4}B(\alpha,\beta)^2, \end{aligned} \tag{4.4}$$

where B is Beta function.

Proof. Set in Lemma 4.1: $f(x) = x^p$, $g(x) = (1-x)^q$, $h(x) = x^{\alpha-1}(1-x)^{\beta-1}$, $a = 0$, $b = 1$. Note that we have $\varphi = \gamma = 0$, $\Phi = \Gamma = 1$. □

Remark 4.3. For $\alpha = \beta = 1$ we have the following improvement of inequality [6, (3.32)]:

$$\left| B(p+1,q+1) - \frac{1}{(p+1)(q+1)} \right| \le \frac{pq}{(p+1)(q+1)\sqrt{(2p+1)(2q+1)}} < \frac{1}{4}, \tag{4.5}$$

$(p,q > 0)$.

Note that Theorem 4.2 is also improvement of [6, Proposition 2].

THEOREM 4.4. *Let $n,m,p,q,\alpha,\beta > 0$. Then we have*

$$\begin{aligned}
&|B(\alpha,\beta)B(\alpha+m+p,\beta+n+q) - B(\alpha+m,\beta+n)B(\alpha+p,\beta+q)|\\
&\quad\le \left[B(\alpha,\beta)B(\alpha+2m,\beta+2n) - B^2(\alpha+m,\beta+n)\right]^{1/2}\\
&\quad\quad\times \left[B(\alpha,\beta)B(\alpha+2p,\beta+2q) - B^2(\alpha+p,\beta+q)\right]^{1/2}\\
&\quad\le \frac{1}{4}\cdot\frac{m^m n^n}{(m+n)^{m+n}}\cdot\frac{p^p q^q}{(p+q)^{p+q}}B^2(\alpha,\beta).
\end{aligned} \tag{4.6}$$

Proof. Set in (4.2): $f(x) = x^m(1-x)^n$, $g(x) = x^p(1-x)^q$, $h(x) = x^{\alpha-1}(1-x)^{\beta-1}$, $a = 0$, $b = 1$ and note that (see [6]) minimum of f and g is zero, while maximum of f is $m^m n^n/(m+n)^{m+n}$, and maximum of g is $p^p q^q/(p+q)^{p+q}$. □

Remark 4.5. Theorem 4.4 gives improvement of [6, Theorem 8 and Proposition 1].

THEOREM 4.6. *Let $\alpha,\beta,\gamma,u,v,w > 0$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+\beta+\gamma)\Gamma(\gamma)}{(u+v+w)^{\alpha+\beta+\gamma}w^\gamma} - \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{(u+w)^{\alpha+\gamma}(v+w)^{\beta+\gamma}}\right|\\
&\quad\le \left[\frac{\Gamma(2\alpha+\gamma)\Gamma(\gamma)}{(2u+w)^{2\alpha+\gamma}w^\gamma} - \frac{\Gamma^2(\alpha+\gamma)}{(u+w)^{2(\alpha+\gamma)}}\right]^{1/2}\left[\frac{\Gamma(2\beta+\gamma)\Gamma(\gamma)}{(2v+w)^{2\beta+\gamma}w^\gamma} - \frac{\Gamma^2(\beta+\gamma)}{(v+w)^{2(\beta+\gamma)}}\right]^{1/2}\\
&\quad\le \frac{1}{4}\left(\frac{\alpha}{ue}\right)^\alpha\left(\frac{\beta}{ve}\right)^\beta\frac{\Gamma^2(\gamma)}{w^{2\gamma}}.
\end{aligned} \tag{4.7}$$

Proof. Consider the mapping $f_{\alpha,u}(t) = t^\alpha e^{-ut}$ defined on $(0,\infty)$. Then

$$f'_{\alpha,u}(t) = e^{-ut}t^{\alpha-1}(\alpha-ut) \tag{4.8}$$

which shows that $f_{\alpha,u}$ is increasing on $(0,\alpha/u)$ and decreasing on $(\alpha/u,\infty)$, and the maximal value is $f_{\alpha,u}(\alpha/u) = (\alpha/(ue))^\alpha$. Using (4.2) for $a = 0$, $b \to \infty$, $f(x) = f_{\alpha,u}(x)$, $g(x) = f_{\beta,u}(x)$, $h(x) = f_{\gamma-1,v}(x)$ we will obtain (4.7), using formula (1.3). □

Remark 4.7. For $u = v = w = 1$, we have the following improvement of inequality [6, (3.38)]:

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+\beta+\gamma)\Gamma(\gamma)}{3^{\alpha+\beta+\gamma}} - \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{2^{\alpha+\beta+2\gamma}}\right|\\
&\quad\le \left[\frac{\Gamma(2\alpha+\gamma)\Gamma(\gamma)}{3^{2\alpha+\gamma}} - \frac{\Gamma^2(\alpha+\gamma)}{4^{\alpha+\gamma}}\right]^{1/2}\left[\frac{\Gamma(2\beta+\gamma)\Gamma(\gamma)}{3^{2\beta+\gamma}} - \frac{\Gamma^2(\beta+\gamma)}{4^{\beta+\gamma}}\right]^{1/2}\\
&\quad\le \frac{1}{4}\cdot\frac{\alpha^\alpha}{e^\alpha}\cdot\frac{\beta^\beta}{e^\beta}\cdot\Gamma^2(\gamma).
\end{aligned} \tag{4.9}$$

5. On inequalities via Ostrowski's inequality

The following lemma gives the well-known Ostrowski's inequality (see, e.g., [16, page 469]).

Lemma 5.1. *Let $f:[a,b]\to\mathbf{R}$ be continuous on $[a,b]$, and differentiable on (a,b) with bounded derivative and let $\|f'\|_\infty := \sup_{t\in[a,b]}|f'(t)|<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{(x-(a+b)/2)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{5.1}$$

for all $x\in[a,b]$. The constant is sharp in the sense that it cannot be replaced by a smaller one.

Remark 5.2. Let us note that a generalization of this result involving Lipschitzian function is proved in [6, Theorem 5] and [4]. Moreover, such results are well known (see [16, page 470]).

Theorem 5.3. *Let $p,q>1$ and $x\in[0,1]$. Then*

$$\begin{aligned}|B(p,q)-x^{p-1}(1-x)^{q-1}| &\le K_1\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^2\right]\\ &\le \max\{p-1,q-1\}\frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^2\right],\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}K_1 &= \max_{i=1,2}\left[x_i^{p-2}(1-x_i)^{q-2}(p-1)-(p+q-2)x_i\right],\\ x_{1,2} &= \frac{(p-1)(p+q-3)\pm\sqrt{(p-1)(q-1)(p+q-3)}}{(p+q-2)(p+q-3)}.\end{aligned} \tag{5.3}$$

Proof. Consider Lemma 5.1 for the mapping $l_{a,b}:(0,1)\to\mathbf{R}$, $l_{a,b}(x)=x^a(1-x)^b$. For $p,q>1$, we get

$$\begin{aligned}l'_{p-1,q-1}(t) &= p_{p-2,q-2}(t)\left[(p-1)-(p+q-2)t\right],\quad t\in(0,1),\\ l''_{p-1,q-1}(t) &= p_{p-3,q-3}(t)\left[(p-1)(p-2)(1-t)^2-2(1-p)(1-q)t(1-t)+(q-1)(q-2)t^2\right].\end{aligned} \tag{5.4}$$

Extreme values of $l'_{p-1,q-1}$ we have for $l''_{p-1,q-1}=0$, that is for $x_{1,2}$. From Lemma 4.1 we have the first inequality in (5.2). The second one is a simple consequence of the fact that

$$\begin{aligned}\max_{t\in[0,1]}\left[(p-1)-(p+q-2)t\right] &= \max\{p-1,q-1\},\\ \max_{t\in[0,1]} t^{p-2}(1-t)^{q-2} &= \frac{(p-2)p-2(q-2)^{q-2}}{(p+q-4)^{p+q-4}}.\end{aligned} \tag{5.5}$$

□

Remark 5.4. The above result is an improvement of [6, Theorem 14], that is, of (1.9).

It is well known that Ostrowski's inequality is useful in the estimation of the remainder for a quadrature formula (see [4, 6, 10, 12, 13]).

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,1,\ldots,n-1)$ a sequence of intermediate points for I_n. Consider the Riemann sums

$$R_n(f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i) h_i, \tag{5.6}$$

where $h_i = x_{i+1} - x_i$, $(i = 0,1,\ldots,n-1)$. Then we have the following quadrature formula.

Lemma 5.5. *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b), with bounded derivative on (a,b). Then we have the Riemann quadrature formula*

$$\int_a^b f(x)dx = R_n(f, I_n, \xi) + W_n(f, I_n, \xi), \tag{5.7}$$

where the remainder satisfies the estimate

$$\begin{aligned} |W_n(f, I_n, \xi)| &\le \left[\frac{1}{4}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right] \|f'\|_\infty \\ &\le \frac{1}{2}\|f'\|_\infty \sum_{i=1}^{n-1} h_i^2 \end{aligned} \tag{5.8}$$

for all ξ_i $(i = 0,1,\ldots,n-1)$ as above.

In particular, for $\xi_i = (x_i + x_{i+1})/2$, $(i = 0,1,\ldots,n-1)$, we have the midpoint rule

$$\int_a^b f(x)dx = M_n(f, I_n) + S_n(f, I_n), \tag{5.9}$$

where

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i, \tag{5.10}$$

and the remainder $S_n(f, I_n)$ satisfies the estimation

$$|S_n(f, I_n)| \le \frac{1}{4}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2. \tag{5.11}$$

The following approximation formula for the Beta mapping holds.

Theorem 5.6. *Let $I_n : 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a division of the interval $[0,1]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,1,\ldots,n-1)$ a sequence of intermediate points for I_n and $p, q > 2$. Then* *we have the formula*

$$B(p,q) = \sum_{i=0}^{n-1} \xi_i^{p-1}(1-\xi_i)^{q-1}h_i + T_n(p,q), \tag{5.12}$$

where the remainder $T_n(p,q)$ satisfies the estimation

$$\begin{aligned} |T_n(p,q)| &\le K_1\left[\frac{1}{4}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right] \\ &\le \frac{1}{2}K_1 \sum_{i=0}^{n-1} h_i^2, \end{aligned} \tag{5.13}$$

and K_1 is given by (5.3).

In particular for $\xi_i = (x_i + x_{i+1})/2$ $(i = 0,1,\dots,n-1)$, we get the approximation

$$B(p,q) = \frac{1}{2^{p+q-2}}\sum_{i=0}^{n-1}(x_i + x_{i+1})^{p-1}(2 - x_i - x_{i+1})^{q-1} + V_n(p,q), \tag{5.14}$$

where

$$|V_n(p,q)| \le \frac{1}{4}K_1 \sum_{i=0}^{n-1} h_i^2. \tag{5.15}$$

Remark 5.7. The results above are improvements of those given in [6, Theorem 15] where K_1 is given by

$$\max\{p-1, q-1\}\frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}. \tag{5.16}$$

The following inequality of Ostrowski type is also valid (see [11], [16, page 471]).

Lemma 5.8. *Let f be absolutely continuous on $[a,b]$ with $f' \in L_1(a,b)$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{b-a}\max\{x-a, b-x\}\|f'\|_1. \tag{5.17}$$

Remark 5.9. Let us note that

$$\max\{x-a, b-x\} = \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2}\right|, \tag{5.18}$$

hence (5.17) is the same as result obtained in [8]. An extension of the above result in the case of function of bounded variation was considered in [5, 6].

Theorem 5.10. *Let $p,q > 1$ and $x \in [0,1]$. Then*

$$\begin{aligned} |B(p,q) - x^{p-1}(1-x)^{q-1}| &\le K_2 \max\{x, 1-x\} \\ &\le \max\{p-1, q-1\}B(p-1, q-1)\max\{x, 1-x\}, \end{aligned} \tag{5.19}$$

where

$$K_2 = (p-1)B(p-1,q) + (q-1)B(p,q-1). \tag{5.20}$$

Proof. Let us consider Lemma 5.8 for mapping $l_{p-1,q-1}(t)$. We have

$$\begin{aligned} ||l'_{p-1,q-1}||_1 &= \int_0^1 |l_{p-2,q-2}(t)|\,|[p-1-(p+q-2)t]|\,dt \\ &\le \int_0^1 l_{p-2,q-2}(t)[(q-1)t + (p-1)(1-t)]dt \\ &= (q-1)B(p,q-1) + (p-1)B(p-1,q) \\ &\le \max\{q-1,p-1\}[B(p,q-1) + B(p-1,q)] \\ &= \max\{q-1,p-1\}B(p-1,q-1) \end{aligned} \tag{5.21}$$

by (1.6). □

Remark 5.11. Theorem 5.10 is an improvement of (1.10), that is, [6, Theorem 1.8].

Application of (5.12) in quadrature formulas was considered in [5, 6, 8]. The following result is valid.

Lemma 5.12. *Let f be as in Lemma 5.8 and I_n, ξ_i $(i = 0,1,\dots,n-1)$ as for Lemma 5.5. Then we have the Riemann quadrature formula (5.7) where the remainder satisfies the estimate*

$$\begin{aligned} |W_n(f,I_n,\xi)| &\le \sup_{i=0,1,\dots,n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\|f'\|_1 \\ &\le \left[\frac{1}{2}\nu(h) + \sup_{i=0,1,\dots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]\|f'\|_1 \\ &\le \nu(h)\|f'\|_1, \end{aligned} \tag{5.22}$$

for all ξ_i, $i = 0,1,\dots,n-1$, where $\nu(h) = \max_{i=0,1,\dots,n-1}\{h_i\}$.

In particular, we have the midpoint rule (5.9) and the remainder $S_n(f,I_n)$ satisfies the estimate

$$|S_n(f,I_n)| \le \frac{1}{2}\nu(h)\|f'\|_1. \tag{5.23}$$

Applications of Lemma 5.12 for Beta function gives the following theorem.

Theorem 5.13. *Let the conditions of Theorem 5.6 be fulfilled. The remainder $T_n(p,q)$ in formula (5.12) satisfies the estimation*

$$\begin{aligned} |T_n(p,q)| &\le K_2\left[\frac{1}{2}\nu(h) + \sup_{i=0,1,\dots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \\ &\le K_2\nu(h), \end{aligned} \tag{5.24}$$

where K_2 is given by (5.20).

In particular, for $\xi_i = (x_i + x_{i+1})/2$, $(i = 0,1,\ldots,n-1)$ *the approximation (5.7) is valid, where*

$$|V_n(p,q)| \le \frac{1}{2}K_2\nu(h). \tag{5.25}$$

Remark 5.14. The last theorem gives improvement of [6, Theorem 19].

Fink [11] (see also [15, page 471], [5, 16]) has proved the following result.

Lemma 5.15. *Let* f *be absolutely continuous on* $[a,b]$ *with* $f' \in L_1[a,b]$. *Then for* $1 < s < \infty$, $1/s + 1/r = 1$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{(x-a)^{r+1} + (b-x)^{r+1}}{(r+1)(b-a)^r}\right]^r \|f'\|_s. \tag{5.26}$$

Theorem 5.16. *Let* $s > 1$, $p,q > 2 - 1/s > 1$, $1/s + 1/r = 1$. *Then*

$$\begin{aligned} &|B(p,q) - x^{p-1}(1-x)^{q-1}| \\ &\quad \le K_3\left[\frac{x^{r+1} + (1-x)^{r+1}}{r+1}\right]^{1/r} \\ &\quad \le \max\{p-1,q-1\}\left[B(s(p-2)+1,s(q-2)+1)\right]^{1/s}\left[\frac{x^r + (1-x)^{r+1}}{r+1}\right]^{1/r}, \end{aligned} \tag{5.27}$$

where

$$K_3 = \left[(q-1)^s B(s(p-2)+2,s(q-2)+1) + (p-1)^s B(s(p-2)+1,s(q-2)+2)\right]^{1/s}. \tag{5.28}$$

Proof. Set in Lemma 5.15: $f(t) = l_{p-1,q-1}(t)$, $a = 0$, $b = 1$. It follows

$$\begin{aligned} &\|l'_{p-1,q-1}\|_s \\ &\quad = \left(\int_0^1 l^s_{p-2,q-2}(t)\,|p-1-(p+q-2)t|^s dt\right)^{1/s} \\ &\quad \le \left(\int_0^1 l^s_{p-2,q-2}\left[(q-1)t + (p-1)(1-t)^s\right]dt\right)^{1/s} \\ &\quad \le \left(\int_0^1 l^s_{p-2,q-2}\left[t(q-1)^s + (1-t)(p-1)^s\right]\right)^{1/s} \\ &\quad = \left[(q-1)^s B(s(p-2)+2,s(q-2)+1) + (p-1)^s B(s(p-2)+1,s(q-2)+2)\right]^{1/s} \\ &\quad \le \max\{q-1,p-1\}\left[B(s(p-2)+2,s(q-2)+1) + B(s(p-2)+1,s(q-2)+2)\right]^{1/s} \\ &\quad = \max\{q-1,p-1\}\left[B(s(p-2)+1,s(q-2)+1)\right]^{1+s} \quad \text{(by (1.6))}. \end{aligned} \tag{5.29}$$

□

Remark 5.17. The result above is an improvement of (1.11), that is, [6, Theorem 22].

An application of Lemma 5.15 for quadrature formula was given in [6, 9].

Lemma 5.18. *Let f be as in Lemma 5.15 and I_n, ξ_i $(i = 0,1,\dots,n)$ as for Lemma 5.5. Then the Riemann quadrature formula (5.7) is valid, where the remainder satisfies the estimate*

$$\begin{aligned} |W_n(f,I_n)| &\le \frac{\|f'\|_s}{(r+1)^{1/r}} \left(\sum_{i=0}^{n-1} \left[(\xi_i - x_i)^{r+1} + (x_{i+1} - \xi_i)^{r+1} \right] \right)^{1/r} \\ &\le \frac{\|f'\|_s}{(r+1)^{1/r}} \left(\sum_{i=0}^{n-1} h_i^{r+1} \right)^{1/r}. \end{aligned} \tag{5.30}$$

In particular, for $\xi_i = (x_i + x_{i+1})/2$, $(i = 0,1,\dots,n-1)$, we have the midpoint formula (5.9) ant the remainder $S_n(f,I_n)$ satisfies

$$|S_n(f,I_n)| \le \frac{\|f'\|_s}{(r+1)^{1/r}} \left(\sum_{i=0}^{n-1} h_i^{r+1} \right)^{1/r}. \tag{5.31}$$

This lemma can be used in the proof of the following approximation of the Beta function in terms of Riemann sums.

Theorem 5.19. *Let the conditions of Theorem 5.6 be fulfilled. The remainder $T_n(p,q)$ in formula (5.12) satisfies the estimation*

$$\begin{aligned} |T_n(p,q)| &\le \frac{K_3}{(r+1)^{1/r}} \left(\sum_{i=0}^{n-1} \left[(\xi_i - x_i)^{r+1} + (x_{i+1} - \xi_i)^{r+1} \right] \right)^{1/r} \\ &\le \frac{K_3}{(r+1)^{1/r}} \left(\sum_{i=0}^{n-1} h_i^{r+1} \right)^{1/r}. \end{aligned} \tag{5.32}$$

In particular for $\xi_i = (x_i + x_{i+1})/2$ $(i = 0,1,\dots,n)$, we have the approximation formula (5.7), where

$$|V_n(p,q)| \le \frac{1}{2} \frac{K_3}{(r+1)^{1/r}} \left(\sum_{i=0}^{n-1} h_i^{r+1} \right)^{1/r}. \tag{5.33}$$

Remark 5.20. Theorem 5.19 gives an improvement of [6, Theorem 23].

6. Inequalities via Milovanović-Pečarić-Fink inequality

Milovanović and Pečarić in [14] and Fink in [11] (see also [16, page 470]) have considered generalization of Ostrowski's inequality (5.1) in the form

$$\left| \frac{1}{n} \left[f(x) + \sum_{k=1}^{n-1} F_k(x) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le K(n,s,x,a,b) \| f^{(n)} \|_s, \tag{6.1}$$

where F_k is defined by

$$F_k(x) = \frac{n-k}{k!(b-a)}\left[f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k\right]. \tag{6.2}$$

For $n = 1$ the sum above is defined to be zero.

In fact, Milovanović and Pečarić have proved that [16, page 469]:

$$K(n,\infty,x,a,b) = \frac{(x-a)^{n+1} + (b-x)^{n+1}}{n(n+1)!(b-a)} \tag{6.3}$$

while Fink proved that

$$K(n,s,x,a,b) = \frac{\left[(x-a)^{nr+1} + (b-x)^{nr+1}\right]^{1/r}}{n!(b-a)} B\left((n-1)r+1, r+1\right)^{1/r}, \tag{6.4}$$

where $1 < s \leq \infty$, $1/s + 1/r = 1$, B is the Beta function and

$$K(n,1,x,a,b) = \frac{(n-1)^{n-1}}{n^n n!(b-a)} \max\left\{(x-a)^n, (b-x)^n\right\}, \tag{6.5}$$

where, of course, for $n = 1$ it holds $(n-1)^{n-1} \equiv 1$, for $s = \infty$, $r = 1$ and for $s = 1$, $r = \infty$.

It is clear that Lemmas 5.1, 5.8, and 5.15 are special cases of the results above for $n = 1$. Also, by (5.18) it is clear that (6.5) can be given in equivalent form

$$K(n,1,x,a,b) = \frac{(n-1)^{n-1}}{n^n n!(b-a)}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^n. \tag{6.6}$$

Theorem 6.1. (i) *Let $p,q > n+1-1/s$, $1 \leq s \leq \infty$, $x \in [0,1]$. Then*

$$\left|B(p,q) - \frac{1}{n}x^{p-1}(1-x)^{q-1}\right| \leq K(n,\infty,x,0,1)\left|\left|l^{(n)}_{p-1,q-1}\right|\right|_s, \tag{6.7}$$

where

$$l^{(n)}_{p-1,q-1} = l_{p-n-1,q-n-1}(t)\sum_{i=0}^{n}(-1)^n\binom{n}{k}(p-1)^{(n-k)}(q-1)^{(k)}(1-t)^{n-k}t^k \tag{6.8}$$

and $a^{(k)} = a(a-1)\cdots(a-k+1)$, $a^{(0)} = 1$.

We also have for $s = \infty$, $p,q > n$,

$$\left|\left|l^{(n)}_{p-1,q-1}\right|\right|_\infty \leq \max_{0\leq k\leq n}\left\{(p-1)^{(n-k)}(q-1)^{(k)}\right\}\frac{(p-n-1)^{p-n-1}(q-n-1)^{q-n-1}}{(p+q-2n-2)^{p+q-2n-2}}. \tag{6.9}$$

(ii) *For $s = 1$, $p,q > n$,*

$$\begin{aligned} \left|\left|l^{(n)}_{p-1,q-1}\right|\right|_1 &\leq \sum_{k=0}^{n}\binom{n}{k}(p-1)^{(n-k)}(q-1)^{(k)}B(p-n+k-2,q-k-2) \\ &\leq \max_{0\leq k\leq n}\left\{(p-1)^{(n-k)}(q-1)^{(k)}\right\}B(p-n,q-n). \end{aligned} \tag{6.10}$$

(iii) *For* $1 < s < \infty$, $p+q > n+1-1/s$,

$$\begin{aligned}
&\|l^{(n)}_{p-1,q-1}\|_s \\
&\quad\le \left[\sum_{k=0}^{n}\binom{n}{k}\left[(p-1)^{(n-k)}(q-1)^{(k)}\right]^s B\big(s(p-n-1)+k-1,s(q-n-1)+n-k-1\big)\right]^{1/s} \\
&\quad\le \max_{0\le k\le n}\left\{(p-1)^{(n-k)}(q-1)^{(k)}\right\}\left[B\big(s(p-n-1)+1,s(q-n-1)+1\big)\right]^{1/s}.
\end{aligned} \tag{6.11}$$

Proof. Let us consider Milovanović-Pečarić-Fink inequality (6.1) for the function

$$f(t) = l_{p-1,q-1}(t). \tag{6.12}$$

We have

$$f^{(n)}(t) = l^{(n)}_{p-1,q-1}(t) = l_{p-n-1,q-n-1}(t)\sum_{k=0}^{n}(-1)^k\binom{n}{k}(p-1)^{(n-k)}(q-1)^{(k)}(1-t)^{n-k}t^k. \tag{6.13}$$

Since $p,q > n$, we have for $k = 1,\dots,n-1$

$$f^{(k)}(0) = l^{(k)}_{p-1,q-1}(0) = 0, \qquad f^{(k)}(1) = l^{(k)}_{p-1,q-1}(1) = 0 \tag{6.14}$$

that is $F_k(x) = 0$. So we get (6.7).

Also we have

$$\left|l^{(n)}_{p-1,q-1}(t)\right| = l_{p-n-1,q-n-1}(t)\left|\sum_{k=0}^{n}(-1)^k\binom{n}{k}(p-1)^{(n-k)}(q-1)^{(k)}(1-t)^{n-k}t^k\right|, \tag{6.15}$$

that is,

$$\left|l^{(n)}_{p-1,q-1}(t)\right| = l_{p-n-1,q-n-1}(t)\sum_{k=0}^{n}\binom{n}{k}(p-1)^{(n-k)}(q-1)^{(k)}(1-t)^{n-k}t^k. \tag{6.16}$$

So we have

$$\begin{aligned}
\|l^{(n)}_{p-1,q-1}\|_\infty &= \max_{t\in[0,1]}\left|l^{(n)}_{p-1,q-1}(t)\right| \\
&\le \max_{t\in[0,1]} l_{p-n-1,q-n-1}(t)\max_{t\in[0,1]}\sum_{k=0}^{n}\binom{n}{k}(p-1)^{(n-k)}(q-1)^{(k)}(1-t)^{n-k}t^k \\
&\le \max_{t\in[0,1]} l_{p-n-1,q-n-q}(t)\max_{k=0,1,\dots,n}\left\{(p-1)^{(n-k)}(q-1)^{(k)}\right\}\sum_{k=1}^{n}\binom{n}{k}(1-t)^{n-k}t^k \\
&= \max_{t\in[0,1]} l_{p-n-1,q-n-1}(t)\max_{k=0,1,\dots,n}\left\{(p-1)^{(n-k)}(q-1)^{(k)}\right\}.
\end{aligned} \tag{6.17}$$

Using (6.16) we also have

$$\begin{aligned} \|l^{(n)}_{p-1,q-1}\|_1 &\le \int_0^1 l_{p-n-1,q-n-1}(t)\sum_{k=0}^{n}\binom{n}{k}(p-1)^{(n-k)}(q-1)^{(k)}(1-t)^{n-k}t^k\,dt \\ &= \sum_{k=0}^{n}\binom{n}{k}(p-1)^{(n-k)}(q-1)^{(k)}B(p-n+k,q-k) \\ &\le \max_{k=0,1,\ldots,n}\{(p-1)^{(n-k)}(q-1)^{(k)}\}\sum_{k=0}^{n}\binom{n}{k}B(p-n-k,q-k) \\ &= \max_{k=0,1,\ldots,n}\{(p-1)^{(n-k)}(q-1)^{(k)}\}B(p-n,q-n) \quad \text{(by (1.6))}. \end{aligned} \tag{6.18}$$

So (6.1) gives (6.9).

Similarly we have (6.11) since by (6.16) it follows, using Jensen's inequality:

$$\begin{aligned} &\|l^{(n)}_{p-1,q-1}\|_s \\ &\le \left\{\int_0^1 l^s_{p-n-1,q-n-1}(t)\left[\sum_{k=0}^{n}\binom{n}{k}(p-1)^{(n-k)}(q-1)^{(k)}(1-t)^{n-k}t^k\right]^s dt\right\}^{1/s} \\ &\le \left\{\int_0^1 l^s_{p-n-1,q-n-1}(t)\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}t^k\left[(p-1)^{(n-k)}(q-1)^{(k)}\right]^s\right\}^{1/s} \\ &= \left\{\sum_{k=0}^{n}\binom{n}{k}B\big(s(p-n-1)+k+1,s(q-n-1)+n-k+1\big)\left[(p-1)^{(n-k)}(q-1)^{(k)}\right]^s\right\}^{1/s} \\ &\le \max_{k=0,1,\ldots,n}\{(p-1)^{(n-k)}(q-1)^{(k)}\}\left[\sum_{k=0}^{n}\binom{n}{k}B\big(s(p-n-1)+k+1,s(q-n-1)+n-k+1\big)\right]^{1/s} \\ &= \max_{k=0,1,\ldots,n}\{(p-1)^{(n-k)}(q-1)^{(k)}\}\left[B\big(s(p-n-1)+1,s(q-n-1)+1\big)\right]^{1/s}. \end{aligned} \tag{6.19}$$

□

7. On some inequalities of the Ostrowski type in probability theory and applications for the Beta function

Let X be a random variable with the probability density function $f:[a,b]\subset\mathbf{R}\to\mathbf{R}_+$ and with cumulative distribution function $F(x)=\Pr(X\le x)$.

The following result was proved by Barnett and Dragomir [2].

THEOREM 7.1. *Let $f \in L_\infty[a,b]$ and put $\|f\|_\infty = \sup_{t\in[a,b]}|f(t)| < \infty$. Then we have the inequality*

$$\left|\Pr(X \le x) - \frac{b - E(X)}{b-a}\right| \le \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2}\right](b-a)\|f\|_\infty \tag{7.1}$$

for all $x \in [a,b]$. The constant $1/4$ in (7.1) is sharp.

A Beta random variable X with parameters (p,q) has the probability density function

$$f(x;p,q) := \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}, \quad 0 < x < 1, \tag{7.2}$$

where B is the Beta function.

THEOREM 7.2 [2]. *Let X be a Beta random variable with the parameters (p,q), $p,q > 1$. Then*

$$\left|\Pr(X \le x) - \frac{q}{p+q}\right| \le \left[\frac{1}{4} + \left(x - \frac{1}{2}\right)^2\right]\frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}, \tag{7.3}$$

where $x \in [0,1]$.

In particular, we have

$$\left|\Pr\left(X \le \frac{1}{2}\right) - \frac{q}{p+q}\right| \le \frac{1}{4}\cdot\frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}. \tag{7.4}$$

Some related results based on Ostrowski type inequality for functions from $L_1[a,b]$ are obtained in [1, 6]. For example, the following result is valid.

THEOREM 7.3. *Let X be a Beta random variable with parameters (p,q), $p,q > 0$. Then*

$$\left|\Pr(X \le x) - \frac{q}{p+q}\right| \le \frac{1}{2} + \left|x - \frac{1}{2}\right|, \tag{7.5}$$

for all $x \in [0,1]$ and, in particularly

$$\left|\Pr\left(X \le \frac{1}{2}\right) - \frac{q}{p+q}\right| \le \frac{1}{2}. \tag{7.6}$$

Dragomir, Barnett, and Wang [7] (see also [6]) are proved.

Theorem 7.4. *Let X be a random variable with the probability density function $f:[a,b]\subset \mathbf{R}\to\mathbf{R}_+$. If $f\in L_s[a,b]$, $s>1$, then we have the inequality*

$$\left|\Pr(X\le x)-\frac{b-E(X)}{b-a}\right| \le \frac{r}{r+1}\|f\|_s(b-a)^{1/r}\left[\left(\frac{x-a}{b-a}\right)^{(1+r)/r}+\left(\frac{b-x}{b-a}\right)^{(1+r)/r}\right] \le \frac{r}{r+1}\|f\|_s(b-a)^{1/r} \tag{7.7}$$

for all $x\in[a,b]$, where $1/s+1/r=1$.

An improvement of Theorem 7.4 was obtained in [3].

Theorem 7.5. *Let the assumptions of Theorem 7.4 be fulfilled. Then*

$$\left|\Pr(X\le x)-\frac{b-E(X)}{b-a}\right| \le \left[\frac{(x-a)^{r+1}+(b-x)^{r+1}}{(r+1)(b-a)^r}\right]^{1/r}\|f\|_s. \tag{7.8}$$

The following result from [3] is also improvement of result proved in [6, 7].

Theorem 7.6. *Let $s>1$ and X be a Beta random variable with parameters (p,q), $p>1-1/s$, $q>1-1/s$. Then we have the inequality*

$$\left|\Pr(X\le x)-\frac{q}{p+q}\right| \le \left[\frac{x^{1+r}+(1-x)^{1+r}}{1+r}\right]^{1/r}\frac{B\big(s(p-1)+1,s(q-1)+1\big)^{1/s}}{B(p,q)} \tag{7.9}$$

for all $x\in[0,1]$.

In particular, we have

$$\left|\Pr\left(X\le\frac{1}{2}\right)-\frac{q}{p+q}\right| \le \frac{B\big(s(p-1)+1,s(q-1)+1\big)^{1/s}}{2(1+r)^{1/r}B(p,q)}. \tag{7.10}$$

Now, we will give some extension of previous results.

Theorem 7.7. *Let X be a random variable with the probability density function $f:[a,b]\subset\mathbf{R}\to\mathbf{R}_+$ and with cummulative distribution function $F(x)=\Pr(X\le x)$. If $f^{(n-1)}\in L_s[a,b]$, $n\ge 1$, $s>1$, and $F^{(i)}(0)=F^{(i)}(1)=0$, $i=1,\dots,n-2$ (if $n\ge 3$), then*

$$\left|\frac{1}{n}\left[\Pr(X\le x)+\frac{(n-1)(b-x)}{b-a}\right]-\frac{b-E(X)}{b-a}\right| \le K(n,s,x,a,b)\|f^{(n-1)}\|_s, \tag{7.11}$$

where $K(n,s,x,a,b)$ are given by (6.3), (6.4), and (6.5) or (6.6).

Proof. Set in (6.1): $f(x)=F(x)$ and note that $F(a)=0$, $F(b)=1$, $\int_a^b F(t)dt=b-E(x)$, $F_k(x)=0$, $k=2,\dots,n-1$, while $F_1(x)=(n-1)((b-x)/(b-a))$. □

Theorem 7.8. *Let $1 \le s \le \infty$ and X be a Beta random variable with parameters (p,q), $p > n - 1/s$, $q > n - 1/s$. Then*

$$\left| \frac{1}{n}\left[\Pr(X \le x) + (n-1)(1-x) \right] - \frac{q}{p+q} \right| \le K(n,s,x,0,1) \| f^{(n-1)} \|_s, \tag{7.12}$$

where

$$f^{(n-1)}(t) = \frac{l_{p-n,q-n}(t)}{B(p,q)} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (p-1)^{(n-k-1)} (q-1)^{(k)} (1-t)^{n-k-1} t^k. \tag{7.13}$$

Further, we have

(i) *For $s = \infty$, $p,q > n$*

$$\| f^{(n-1)} \|_\infty \le \max_{k=0,1,\dots,n-1} \{ (p-1)^{(n-k-1)} (q-1)^{(k)} \} \frac{(p-n)^{p-n} (q-n)^{q-n}}{B(p,q)(p+q-2n)^{p+q-2n}}. \tag{7.14}$$

(ii) *For $s = 1$, $p,q > n-1$*

$$\begin{aligned} \| f^{(n-1)} \|_1 &\le \frac{1}{B(p,q)} \sum_{k=0}^{n-1} \binom{n-1}{k} (p-1)^{(n-k-1)} (q-1)^{(k)} B(p-n+k+1, q-k) \\ &\le \frac{1}{B(p,q)} \max_{0 \le k \le n-1} \{ (p-1)^{(n-k-1)} (q-1)^{(k)} \} B(p-n+1, q-n+1). \end{aligned} \tag{7.15}$$

(iii) *For $0 < s < \infty$, $p,q > n - 1/s$*

$$\begin{aligned} &\| f^{(n-1)} \|_s \\ &\le \frac{1}{B(p,q)} \left\{ \sum_{k=0}^{n-1} \binom{n-1}{k} \left[(p-1)^{(n-k-1)} (q-1)^{(k)} \right]^s B(s(p-n)+k+1, s(q-n)+n-k) \right\}^{1/s} \\ &\le \frac{1}{B(p,q)} \max_{0 \le k \le n-1} \{ (p-1)^{(n-k-1)} (q-1)^{(k)} \} \left[B(s(p-n)+1, s(q-n)+1) \right]^{1/s}. \end{aligned} \tag{7.16}$$

Proof. A Beta random variable X with parameters (p,q) has the probability density function

$$f(x;p,q) := \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}, \quad 0 < x < 1. \tag{7.17}$$

We have

$$E(X) = \frac{1}{B(p,q)} \int_0^1 x \cdot x^{p-1}(1-x)^{q-1} dx = \frac{B(p+1,q)}{B(p,q)} = \frac{p}{p+q}. \tag{7.18}$$

So from Theorem 7.7 we have (7.12). Proof of the rest of the theorem is similar to that of Theorem 6.1. □

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