INEQUALITIES FOR DUAL AFFINE QUERMASSINTEGRALS

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For star bodies, the dual affine quermassintegrals were introduced and studied in several papers. The aim of this paper is to study them further. In this paper, some inequalities for dual affine quermassintegrals are established, such as the Minkowski inequality, the dual Brunn-Minkowski inequality, and the Blaschke-Santaló inequality.

1. Introduction

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) and $\mathcal{K}_o^n$ denote the subset of $\mathcal{K}^n$ that consists of convex bodies with the origin in their interiors. Denote by $\text{vol}_i(K \mid \xi)$ the i-dimensional volume of the orthogonal projection of K onto an i-dimensional subspace $\xi \subset \mathbb{R}^n$. Affine quermassintegrals are important geometric invariants related to the projection of convex body. These quermassintegrals were introduced by Lutwak [7], and can be defined by letting $\Phi_0(K) = V(K)$, $\Phi_n(K) = k_n$, and for $0 < i < n$,

$$\Phi_i(K) = k_n \left(\int_{G(n,n-i)} \left[\frac{\text{vol}_{n-i}(K \mid \xi)}{k_{n-i}} \right]^{-n} d\xi \right)^{-1/n}, \tag{1.1}$$

where the Grassmann manifold $G(n, i)$ is endowed with the normalized Haar measure, and k_n is the volume of the unit ball B_n in $\mathbb{R}^n$.

Furthermore, in [6], Lutwak introduced the dual affine quermassintegrals of a star body L containing the origin in its interior, $\widetilde{\Phi}_i(L)$, by letting $\widetilde{\Phi}_0(L) = V(L)$, $\widetilde{\Phi}_n(L) = k_n$, and for $0 < i < n$,

$$\widetilde{\Phi}_i(L) = k_n \left(\int_{G(n,n-i)} \left[\frac{\text{vol}_{n-i}(L \cap \xi)}{k_{n-i}} \right]^n d\xi \right)^{1/n}, \tag{1.2}$$

where $\mathrm{vol}_i(L \cap \xi)$ denotes the i-dimensional volume of intersection of L with an i-dimensional subspace $\xi \subset \mathbb{R}^n$.

Grinberg [4] proved that both the affine quermassintegrals and the dual affine quermassintegrals are invariant under volume-preserving affine transformations.

For star bodies, the dual affine quermassintegrals were studied in [3, 4, 7, 10]. The aim of this paper is to study them further. For reader's convenience, we try to make the paper self-contained. This paper, except for the introduction, is divided into three sections. In Section 2 we recall some basics about convex bodies, star bodies, and dual mixed volume.

In Section 3, we introduce the concept of the mixed p-dual affine quermassintegrals and establish the Minkowski inequality for them (Theorem 3.1). As an application, the dual Brunn-Minkowski inequality for the dual affine quermassintegrals is obtained.

In Section 4, we establish a connection between the affine quermassintegrals and the dual affine quermassintegrals for a given convex body.

2. Notation and preliminary works

As usual, S^{n-1} denotes the unit sphere, B_n the unit ball, and o the origin in Euclidean n-space $\mathbb{R}^n$.

Let K be a nonempty compact convex body in $\mathbb{R}^n$, the support function h_K of K is defined by

$$h_K(u) = \max\{u \cdot x : x \in K\}, \quad u \in S^{n-1}, \tag{2.1}$$

where $u \cdot x$ denotes the usual inner product of u and x in $\mathbb{R}^n$.

If K is a convex body that contains the origin in its interior, the polar body K^* of K, with respect to the origin, is defined by

$$K^* = \{x \in \mathbb{R}^n \mid x \cdot y \le 1, y \in K\}. \tag{2.2}$$

For a compact subset L of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we will use $\rho(L, \cdot)$ to denote its radial function; that is, for $u \in S^{n-1}$,

$$\rho(L, u) = \rho_L(u) = \max\{\lambda > 0 : \lambda u \in L\}. \tag{2.3}$$

If $\rho(L, \cdot)$ is continuous and positive, L will be called a star body.

Let $\mathcal{S}_o^n$ denote the set of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. Two star bodies $K, L \in \mathcal{S}_o^n$ are said to be dilatate (of one another) if $\rho(K, u)/\rho(L, u)$ is independent of $u \in S^{n-1}$.

Let $L_j \in \mathcal{S}_o^n \, (1 \le j \le n)$. The dual mixed volume $\widetilde{V}(L_1, \ldots, L_n)$ is defined by

$$\widetilde{V}(L_1, \ldots, L_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{L_1}(u) \rho_{L_2}(u) \cdots \rho_{L_n}(u) du. \tag{2.4}$$

We use the notation $\widetilde{V}(L_1, i_1; \ldots; L_n, i_n)$ for the dual mixed volume in which L_j appears i_j times.

If $x_i \in \mathbb{R}^n$, $1 \le i \le m$, then $x_1 \widetilde{+} \cdots \widetilde{+} x_m$ is defined to be the usual vector sum of the points x_i, if all of them belong to a line through o, and 0 otherwise.

Let $L_i \in \mathscr{S}_o^n$ and $t_i \geq 0$, $1 \leq i \leq m$, then

$$t_1 L_1 \tilde{+} \cdots \tilde{+} t_m L_m = \{t_1 x_1 \tilde{+} \cdots \tilde{+} t_m x_m : x_i \in L_i\} \tag{2.5}$$

is called a radial linear combination.

The following elementary property of dual mixed volumes will be used later. For $K, L, L_j \in \mathscr{S}_o^n$ $(1 \leq j \leq n-1)$,

$$\tilde{V}(L_1, \ldots, L_{n-1}, K \tilde{+} L) = \tilde{V}(L_1, \ldots, L_{n-1}, K) + \tilde{V}(L_1, \ldots, L_{n-1}, L). \tag{2.6}$$

For $K, L \in \mathscr{S}_o^n$, the Minkowski inequality for dual mixed volumes [3, page 373] states

$$\tilde{V}(K, n-p; L, p)^n \leq V(K)^{n-p} V(L)^p, \tag{2.7}$$

with equality if and only if K is a dilate of L.

The above elementary results (and definitions) are from the theory of convex bodies. The reader may consult the standard works on the subject [1, 3, 5, 9, 10] for reference.

3. The dual Brunn-Minkowski inequalities for dual affine quermassintegrals

In this section, we will prove the dual Brunn-Minkowski inequality for the dual harmonic quermassintegrals. At first, we introduce the concept of mixed p-dual affine quermassintegrals.

Let $K, L \in \mathscr{S}_o^n$, $\xi \in G(n, i)$ and $0 \leq p \leq i$. We define mixed p-dual affine quermassintegrals, $\tilde{\Phi}_{p,i}(K, L)$. Let first $\tilde{V}_{p,i}(K, L; \xi)$ by

$$\tilde{V}_{p,i}(K, L; \xi) = \tilde{V}(K \cap \xi, i-p; L \cap \xi, p). \tag{3.1}$$

It is easy to verify that $\tilde{V}_{p,i}(K, K; \xi) = \text{vol}_i(K \cap \xi)$, for all $0 \leq p \leq n-i$, and $\tilde{V}_{i,i}(K, L) = \text{vol}_i(L \cap \xi)$, for all K.

Now we define the mixed p-dual affine quermassintegrals $\tilde{\Phi}_{p,i}(K, L)$ by

$$\tilde{\Phi}_{p,i}(K, L) = k_n \left(\int_{G(n, n-i)} \left[\frac{\tilde{V}_{p, n-i}(K, L; \xi)}{k_{n-i}} \right]^n d\xi \right)^{1/n}. \tag{3.2}$$

If $p = 1$, we will write $\tilde{\Phi}_i(K, L)$, rather than $\tilde{\Phi}_{1,i}(K, L)$. It follows that $\tilde{\Phi}_{p,i}(K, K) = \tilde{\Phi}_i(K)$, for all $0 \leq p \leq n-i$ and $\tilde{\Phi}_{n-i,i}(K, L) = \tilde{\Phi}_i(L)$, for all K.

For the mixed p-dual affine quermassintegrals, we have the following Minkowski inequality.

THEOREM 3.1. *Let $K, L \in \mathscr{S}_o^n$ and $0 \leq i < n$. If $0 \leq p \leq i$, then*

$$\tilde{\Phi}_{p,i}(K, L)^{n-i} \leq \tilde{\Phi}_i(K)^{n-i-p} \tilde{\Phi}_i(L)^p, \tag{3.3}$$

with equality if and only if K is a dilate of L.

Proof. Let $\xi \in G(n, n-i)$. By (2.7), we get

$$
\begin{aligned}
\tilde{V}_{p,n-i}(K,L;\xi) &= \tilde{V}(K \cap \xi, n-i-p; L \cap \xi, p) \\
&\leq \mathrm{vol}_{n-i}(K \cap \xi)^{(n-i-p)/(n-i)} \, \mathrm{vol}_{n-i}(L \cap \xi)^{p/(n-i)}.
\end{aligned}
\tag{3.4}
$$

According to (3.4) and the Hölder integral inequality, we have

$$
\begin{aligned}
\tilde{\Phi}_{p,i}(K,L) &= k_n \left(\int_{G(n,n-i)} \left[\frac{\tilde{V}_{p,n-i}(K,L;\xi)}{k_{n-i}} \right]^n d\mu_i(\xi) \right)^{1/n} \\
&\leq k_n \left(\int_{G(n,n-i)} \left[\frac{\mathrm{vol}_{n-i}(K \cap \xi)}{k_{n-i}} \right]^{n(n-i-p)/(n-i)} \left[\frac{\mathrm{vol}_{n-i}(L \cap \xi)}{k_{n-i}} \right]^{np/(n-i)} d\mu_i(\xi) \right)^{1/n} \\
&\leq \tilde{\Phi}_i(K)^{(n-i-p)/(n-i)} \tilde{\Phi}_i(L)^{p/(n-i)}.
\end{aligned}
\tag{3.5}
$$

By the equality conditions of Hölder integral inequality and the Minkowski inequality for dual mixed volumes, the equality of (3.3) holds if and only if K is a dilatate of L. $\square$

As an application of **Theorem 3.1**, we have the following dual Brunn-Minkowski inequality for the dual affine quermassintegrals.

THEOREM 3.2. *Let $K, L \in \mathscr{S}_o^n$ and $0 \leq i \leq n-1$. Then*

$$
\tilde{\Phi}_i(K \tilde{+} L)^{1/(n-i)} \leq \tilde{\Phi}_i(K)^{1/(n-i)} + \tilde{\Phi}_i(L)^{1/(n-i)},
\tag{3.6}
$$

with equality if and only if K is a dilatate of L.

Proof. Let $\xi \in G(n, i)$ and $K, L \in \mathscr{S}_o^n$, it is easy to prove that

$$
(K \tilde{+} L) \cap \xi = (K \cap \xi) \tilde{+} (L \cap \xi).
\tag{3.7}
$$

In fact, for $u \in S^{n-1} \cap \xi$, we have

$$
\rho_{(K\tilde{+}L)\cap\xi}(u) = \rho_{K\tilde{+}L}(u) = \rho_K(u) + \rho_L(u) = \rho_{K\cap\xi}(u) + \rho_{L\cap\xi}(u) = \rho_{K\cap\xi\tilde{+}L\cap\xi}(u).
\tag{3.8}
$$

By (2.6), (3.7), for $M \in \mathscr{S}_o^n$, we have

$$
\begin{aligned}
\tilde{V}_{1,i}(M, K\tilde{+}L; \xi) &= \tilde{V}(M \cap \xi, i-1; (K\tilde{+}L) \cap \xi) \\
&= \tilde{V}(M \cap \xi, i-1; (K \cap \xi)\tilde{+}(L \cap \xi)) \\
&= \tilde{V}(M \cap \xi, i-1; K \cap \xi) + \tilde{V}(M \cap \xi, i-1; L \cap \xi) \\
&= \tilde{V}_{1,i}(M, K; \xi) + \tilde{V}_{1,i}(M, L; \xi).
\end{aligned}
\tag{3.9}
$$

According to (3.2) and Minkowski integral inequality, we have

$$
\widetilde{\Phi}_i(M, K \widetilde{+} L) = k_n \left(\int_{G(n,n-i)} \left[\frac{\widetilde{V}_{1,n-i}(M, K \widetilde{+} L; \xi)}{k_{n-i}} \right]^n d\mu_{n-i}(\xi) \right)^{1/n}
$$

$$
= k_n \left(\int_{G(n,n-i)} \left[\frac{\widetilde{V}_{1,n-i}(M, K; \xi) + \widetilde{V}_{1,n-i}(M, L; \xi)}{k_{n-i}} \right]^n d\mu_{n-i}(\xi) \right)^{1/n}
$$

$$
\leq \widetilde{\Phi}_i(M, K) + \widetilde{\Phi}_i(M, L) \leq \widetilde{\Phi}_i(M)^{(n-i-1)/(n-i)} \left(\widetilde{\Phi}_i(K)^{1/(n-i)} + \widetilde{\Phi}_i(L)^{1/(n-i)} \right),
$$
$$(3.10)$$

with equality if and only if K and L are dilatate of M. Now we take $K \widetilde{+} L$ for M, and recall that $\widetilde{\Phi}_i(K, K) = \widetilde{\Phi}_i(K)$; then Theorem 3.2 follows. $\qquad\square$

Remark 3.3. Theorem 3.2 is a dual of Lutwak's inequality for affine quermassintegrals, which was proved in [7]: *let K and L be convex bodies in $\mathbb{R}^n$ and $0 \leq i \leq n-1$, then*

$$
\Phi_i(K + L)^{1/(n-i)} \geq \Phi_i(K)^{1/(n-i)} + \Phi_i(L)^{1/(n-i)},
$$
$$(3.11)$$

with equality if and only if K and L are homothetic.

4. More about the dual affine quermassintegrals

Let K be a convex body of constant width, K^* is the polar body of K. We proved that among convex bodies of constant width, precisely the ball attains the minimal value of $\widetilde{\Phi}_{n-1}(K^*)$.

Theorem 4.1. *Let $K \in \mathcal{K}_o^n$. If*

$$
\mathrm{vol}_1\left(K \mid \xi\right) = \mathrm{vol}_1\left(B_n \mid \xi\right),
$$
$$(4.1)$$

for all $\xi \in G(n, 1)$, then

$$
\widetilde{\Phi}_{n-1}(K^*) \geq \widetilde{\Phi}_{n-1}(B_n^*),
$$
$$(4.2)$$

with equality if and only if $K = B_n$.

Proof. For all $u \in S^{n-1}$, (4.1) is equivalent to

$$
h(K, u) + h(K, -u) = 2,
$$
$$(4.3)$$

and the chord length of K^* in direction u satisfies

$$
\rho(K^*, u) + \rho(K^*, -u) \geq \frac{4}{h(K, u) + h(K, -u)} = 2,
$$
$$(4.4)$$

where we have used the inequality between arthmetic and harmonic means.

Notice that if $\xi \in G(n, 1)$, then $\mathrm{vol}_1(K^* \cap \xi)$ is just the chord length of K^* along ξ. By (1.2), we have

$$
\begin{aligned}
\tilde{\Phi}_{n-1}(K^*) &= k_n \left(\int_{G(n,1)} \left[\frac{\mathrm{vol}_1(K^* \cap \xi)}{2} \right]^n d\xi \right)^{1/n} \\
&= k_n \left(\frac{1}{nk_n} \int_{S^{n-1}} \left[\frac{\rho(K^*, u) + \rho(K^*, -u)}{2} \right]^n du \right)^{1/n} \geq k_n = \tilde{\Phi}_{n-1}(B_n^*).
\end{aligned}
\tag{4.5}
$$

Equality holds if and only if $h(K, u) = h(K, -u) = 1$, which implies K is a unit ball centered at the origin. □

The following theorem which establishes a connection between the affine quermassintegrals and the dual affin equermassintegrals generalizes the Blaschke-Santaló inequality.

Theorem 4.2. *Let K be a centered convex body and $0 \leq i < n$. Then*

$$
\tilde{\Phi}_i(K^*)\Phi_i(K) \leq k_n^2,
\tag{4.6}
$$

with equality if and only if K is an ellipsoid.

To prove the inequality (4.6), the following lemma will be needed.

Lemma 4.3 [8]. *Let $K \in \mathcal{K}_o^n$ and $\xi \in G(n, i)$. Then*

$$
K^* \cap \xi = (K \mid \xi)^*.
\tag{4.7}
$$

Proof of Theorem 4.2. Let $s = n - i$, and $\xi \in G(n, s)$. By the Blaschke-Santaló inequality, for the body $K \mid \xi$ in ξ, we have

$$
\mathrm{vol}_s \left[(K \mid \xi)^* \right] \mathrm{vol}_s (K \mid \xi) \leq k_s^2,
\tag{4.8}
$$

with equality if and only if $K \mid \xi$ is an ellipsoid in ξ.

According to Lemma 4.3, we obtain

$$
\left[\frac{V_s(K^* \cap \xi)}{k_s} \right]^n \leq \left[\frac{V_s(K \mid \xi)}{k_s} \right]^{-n},
\tag{4.9}
$$

with equality if and only if $K \mid \xi$ is an ellipsoid in ξ. We integrate both sides of inequality (4.9) over $G(n, s)$ and get

$$
\left(\frac{\tilde{\Phi}_i(K^*)}{k_n} \right)^n \leq \left(\frac{\Phi_i(K)}{k_n} \right)^{-n}.
\tag{4.10}
$$

This is the desired inequality

$$
\tilde{\Phi}_i(K^*)\Phi_i(K) \leq k_n^2,
\tag{4.11}
$$

with equality if and only if K is an ellipsoid. The equality condition follows from the fact that, for $s > 1$, ellipsoid is the only body all of whose s-dimensional projections are s-dimensional ellipsoids (see [3, page 95]). □

Remark 4.4. The case $i = 0$ of (4.6) is the well-known Blaschke-Santaló inequality.

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